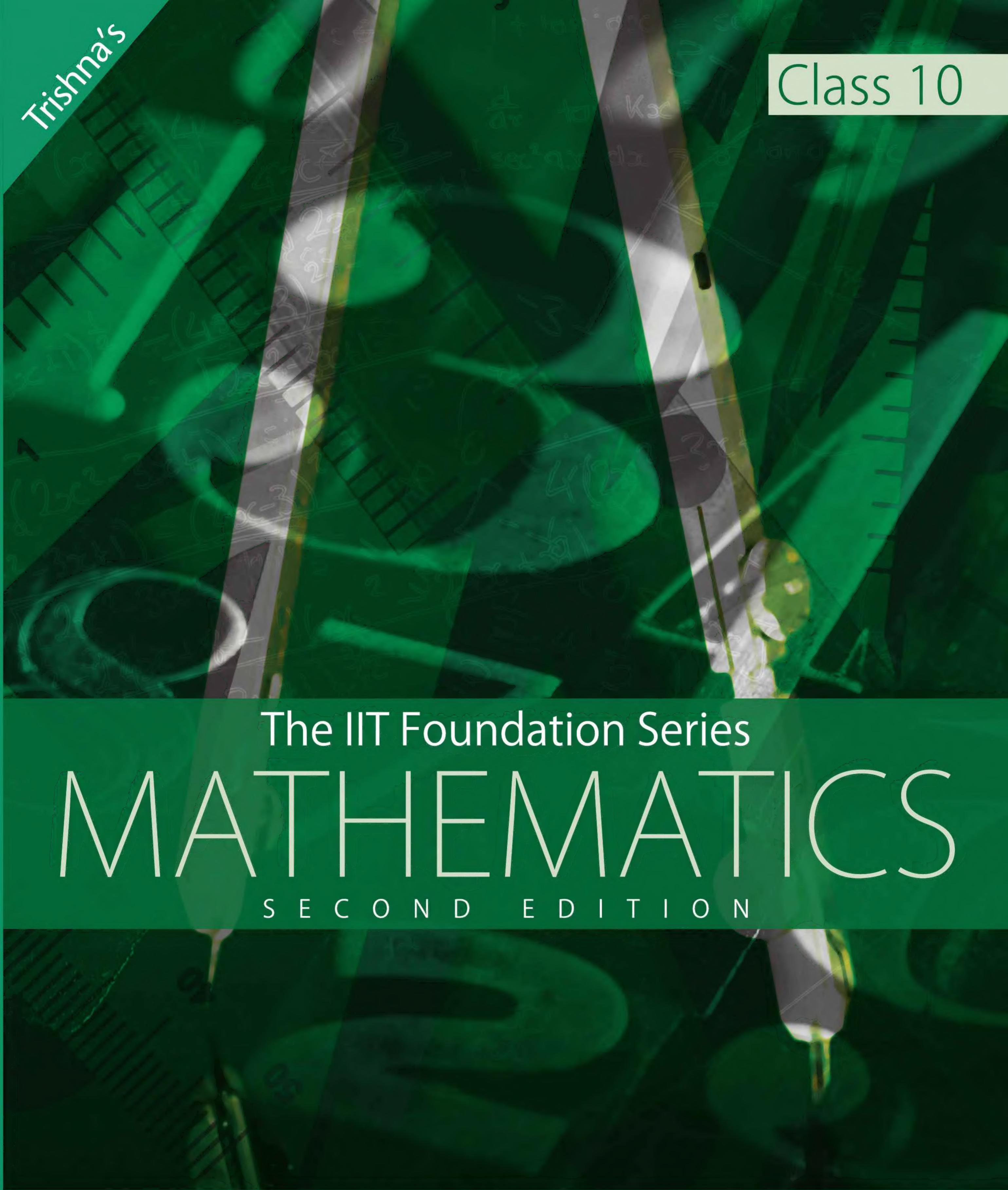

Trishna's

Class 10

The IIT Foundation Series

MATHEMATICS

SECOND EDITION

ALWAYS LEARNING

PEARSON

MATHEMATICS

CLASS 10

The IIT Foundation Series

(Second Edition)

Trishna Knowledge Systems
A division of
Triumphant Institute of Management Education Pvt. Ltd.

PEARSON
Chandigarh • Delhi • Chennai

ISBN 9788131761328
eISBN 9789332510944

Head Office: A-8(A), Sector 62, Knowledge Boulevard, 7th Floor, NOIDA 201 309, India
Registered Office: 11 Local Shopping Centre, Panchsheel Park, New Delhi 110 017, India

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Preface to the Second Edition

It is with enthusiasm that we present the second edition of the IIT Foundation series of books. The success of our first edition indicates that there exists a large group of students who wish to delve into Mathematics, Physics and Chemistry, beyond the restrictions of their school syllabus. We take this opportunity to thank all those who have contributed to the success and to reaffirm our commitment to inculcating scientific temper among the student community.

Over the last two years we have received feedback from many students and teachers who have used our books, and it has given us great satisfaction to know that they have benefitted immensely. The teachers have taught from these books and have appreciated the approach adopted in the presentation of concepts. Their feedback has encouraged us to extend the IIT-Foundation series of books to include books for students of Class 7 as well. In most school curricula, the fundamentals of Mathematics and Science that students study in Classes 8 to 10 are introduced in Class 7. While a proper understanding of these fundamentals is essential for good performance in higher classes, it is not possible for Class 7 students to develop indepth understanding of concepts unless these are presented lucidly.

With this edition, therefore, we present books for Class 7. We are confident that these will help students develop conceptual clarity at their early age, and that the student community will take full advantage of this new inclusion.

As with all our publications, the objective of this series is to provide students with a comprehensive understanding of fundamental concepts, to teach them the application of these concepts, and to help hone their problem-solving skills. We trust all our young students will find these books relevant and enlightening.

Preface

As the old adage goes, "nothing succeeds like success." The truth in this maxim cannot be overstated in today's competitive world. The present-day student is under immense pressure to thrive and emerge triumphantly in examinations. Students aspire to get into pre-eminent educational institutes to pursue the best courses–be it in engineering, medicine, arts or sciences–to enable them to prepare for careers at the global level. Their performance in entrance examinations are often the cornerstones that determine if they would be admitted into these hallowed halls of learning. With most of these exams being designed to challenge the innate talent and ingenuity of students, it is only natural that they find these tests most demanding and that they find themselves competing with the country's best minds for those few coveted seats. Only those students with a thorough understanding of the fundamental concepts and exceptional problem-solving skills pass out with flying colours in these tests.

The "IIT Foundation Series" books are designed to provide students with a comprehensive understanding of the fundamental concepts, to teach them the application of these concepts and to hone their problem-solving skills.

The objective of the IIT Foundation Series books is to ensure that students are able to delve beyond the restrictions of their regular school syllabus and get a fundamental understanding of Mathematics, Physics and Chemistry. The books are designed to kindle student interest in these subjects and to encourage them to ask questions that lead to a firm grip on the principles governing each concept.

Irrespective of the field of study that the student may choose to take up later, it is imperative that he or she develops a sound understanding of Mathematics and Science, since it forms the basis for most modern-day activities. Lack of a firm background in these subjects may not only limit the capacity of the student to solve complex problems but also lessen his or her chances to make it into top-notch institutes that provide quality education.

This book is intended to serve as the backbone of the student's preparation for a range of competitive exams, going beyond the realms of the usual school curriculum to provide that extra edge so essential in tackling a typical question paper.

A distinctive feature of this book is that it has been written by a team of well-qualified teachers experienced in imparting the fundamentals of Mathematics and Science, and their applications to active learners at T.I.M.E. (Triumphant Institute of Management Education Pvt. Ltd). They have also been instrumental in developing high-quality study material for IIT Foundation courses for Classes 7 to 10. We are sure that you will find this book, prepared by

About the IIT Foundation Series

This book is a perfect companion for the students of 7th to 10th Grade. It will help them achieve the much-needed conceptual clarity in the topics which form the basis for their higher study.

Some of the important features of the book are listed below:

- Builds skills that will help students succeed in school and various competitive examinations.
- The methodology is aimed at helping students thoroughly understand the concepts in Mathematics, Physics and Chemistry.
- Helps develop a logical approach to Mathematics, Physics and Chemistry, thereby enabling more effective learning.
- Lays stress on questions asked by board/school examinations as well as application of concepts.
- The concepts are explained in a well structured and lucid manner, using simple language. This aids learning.
- A large number of examples have been included to help reinforce the concepts involved.
- Different levels of practice exercises have been provided which help students develop the necessary application and problem-solving skills.
- The exercises have been designed keeping in mind the various board/school examinations and competitive examinations, such as the NTSE, NLSTSE, Science Olympiad and Cyber Olympiad.
- The book will not only help the students in better understanding of what is taught in regular school classes (and hence enable them to do well in board examinations) but will also help in developing the acumen, resulting in a distinctive edge over their peers.
- Given below are a few examples that demonstrate how the course will help students in understanding the fundamentals:

How does a kingfisher catch fish?

The kingfisher flies vertically over the position of the fish, then plunges into the water at a 90^0 angle. The concept here is that the normally incident rays do not undergo refraction, hence the fish lies exactly where it appears to be. At any other angle, the apparent location of the fish would be different from its real location.

Why do we normally swing our arms while walking, and why not when we carry a load in our hands?

The center of gravity of a body depends on the distribution of mass in the body. As we walk, the movement of the legs tends to cause a shift in the centre of gravity. To compensate for this shift we swing our arms. When we are carrying a load in the hands, however, the effective C.G is lower, making it easier to maintain balance.

Why does salt become damp when kept exposed during the rainy season and not when kept exposed during summer?

In the rainy season humidity in the atmosphere is very high, i.e., there is a lot of moisture in the atmosphere. Thus, calcium chloride, which is the impurity present in common salt, absorbs this moisture and makes the salt damp. In summer, however, as the temperature is high, calcium chloride tends to loose moisture through the process of evaporation, and the salt is left free-flowing.

Structure of the IIT Foundation Series

The IIT Foundation Series is available in Mathematics, Physics and Chemistry. Each chapter in the book is divided into three parts, namely, theory, test your concepts and concept application.

Theory:

The theory part deals with the various concepts in Physics/Chemistry/Mathematics, which is a part of the syllabus prescribed by major boards for Class X. The concepts are explained in a lucid manner, and diagrams have been provided, wherever necessary, to illustrate these concepts.

Test your Concepts:

This exercise is provided at the end of the theory section of each chapter. These exercises are a collection of very short answer, short answer and essay type of descriptive questions. It is intended to provide students with model questions that they may face in the board examination.

Students are expected to prepare for these questions before they attempt any examination based on that particular chapter. Towards the end of the book, the students will find key points for selected questions of the exercise. These key points provide students with an idea of the points that should be a part of an answer for such a question.

Concept Application:

This is a collection of exercises in four different classes: Class 7, Class 8, Class 9 and Class 10.

Class 8 consists of basic objective questions. These questions test the basic knowledge of students and enable them to gauge their understanding of concepts when they start solving this exercise. The key for this exercise is provided at the end of the respective chapter.

Classes 9 and 10 consist of descriptive questions of a higher level of difficulty. These questions help students to *apply the concepts* that they have learnt. Key points for selected questions of these exercises have been provided at the end of each chapter in order to help students solve these questions.

These books are available for 7th, 8th, 9th and 10th classes separately for Mathematics, Physics and Chemistry.

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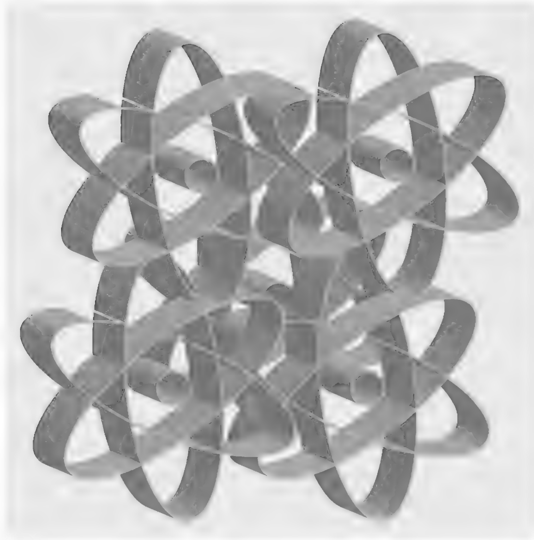
CHAPTER 1

Number Systems



INTRODUCTION

Earlier we have learnt about number line, irrational numbers, how to represent irrational numbers on the number line, real numbers and their decimal representation, representing real numbers on the number line and operations on the number line. We shall now present a detailed study of Euclid's division algorithm and fundamental theorem of Arithmetic. Further we continue the revision of irrational numbers and the decimal expansion of rational numbers.

We are very familiar with division rule.

That is, dividend = (divisor × quotient) + remainder.

Euclid's division lemma is based on this rule. We use this result to obtain the HCF of two numbers. And also we know that, every composite number can be expressed as the product of primes in a unique way. This is the fundamental theorem of Arithmetic. This result is used to prove the irrationality of a number.

In the previous class we have studied about rational numbers. In this chapter we shall explore when exactly the decimal expansion of a rational number say $\frac{p}{q}$ $(q \neq 0)$, is terminating and when it is non-terminating repeating.

Euclid's division lemma

For any two positive integers, say x and y, there exist unique integers say q and r satisfying $x = yq + r$ where $0 \leq r < y$.

Example Consider the integers 9 and 19.

$19 = 9 \times 2 + 1$

Example Consider the integers 6 and 24.

$24 = 6 \times 4 + 0$

Note:

1. Euclid's division algorithm is used for finding the greatest common divisor of two numbers.
2. An algorithm is a process of solving particular problems.

Example

Find the HCF of 250 and 30.

Solution

By using Euclid's division lemma, we get
$250 = 30 \times 8 + 10$
Now consider, the divisor and remainder.
Again by using Euclid's division lemma, we get
$30 = 10 \times 3 + 0$
Here, we notice the remainder is zero and we cannot proceed further.
The divisor at this stage is 10.
The HCF of 250 and 30 is 10.
It can be verified by listing out all the factors of 250 and 30.

Note: Euclid's Division Algorithm is stated for only positive integers, it can be extended for all negative integers.

Euclid's Division Algorithm has several applications. The following examples give the idea of the applications.

Example

Show that every positive even integer is of the form 2n and every positive odd integer is of the form 2n + 1.

Solution

For any integer x and $y = 2, x = 2n + r$ where $n \geq 0$
But $0 \leq r < 2$
$\Rightarrow r = 0$ or 1
When $r = 0, x = 2n$.
$\Rightarrow$ x is a positive even integer.
When $r = 1, x = 2n + 1$.
$\Rightarrow$ x is a positive odd integer.

Example

A trader has 612 Dettol soaps and 342 Pears soaps. He packs them in boxes and each box contains exactly one type of soap. If every box contains the same number of soaps, then find the number of soaps in each box such that the number of boxes is the least.

Solution

The required number is HCF of 612 and 342.

This number gives the maximum number of soaps in each box and the number of boxes with them be the least.

By using Euclid's division algorithm, we have
$612 = 342 \times 1 + 270$

$342 = 270 \times 1 + 72$
$270 = 72 \times 3 + 54$
$72 = 54 \times 1 + 18$
$54 = 18 \times 3 + 0$
Here we notice that the remainder is zero, and the divisor at this stage is 18.
$\therefore$ HCF of 612 and 342 is 18.
So, the trader can pack 18 soaps per box.

Fundamental theorem of arithmetic

Every composite number can be expressed as the product of prime factors uniquely.
Note: In general $a = p_1\, p_2\, p_3 \ldots . p_n$ where $p_1, p_2, p_3, \ldots\ldots p_n$ are primes in ascending order.

☛ *Example*

Write 1800 as product of prime factors.

Solution

$$\begin{array}{r|l} 2 & \underline{1800} \\ 2 & \underline{900} \\ 2 & \underline{450} \\ 3 & \underline{225} \\ 3 & \underline{75} \\ 5 & \underline{25} \\ & 5 \end{array}$$

$\therefore$ $1800 = 2^3 \times 3^2 \times 5^2$.
Let us see the applications of fundamental theorem.

☛ *Example*

Check whether there is any value of x for which 6^x ends with 5.

Solution

If 6^x ends with 5, then 6^x would contain the prime 5.
But, $6^x = (2 \times 3)^x = 2^x \times 3^x$
$\Rightarrow$ The prime numbers in the factorization of 6^x are 2 and 3.
By uniqueness of fundamental theorem, there are no primes other than 2 and 3 in 6^x.
$\therefore$ 6^x never ends with 5.

☛ *Example*

Show that $5 \times 3 \times 2 + 3$ is a composite number.

Solution

$5 \times 3 \times 2 + 3 = 3(5 \times 2 + 1) = 3(11) = 3 \times 11$
$\therefore$ The given number is a composite number.

☛ *Example*

Find the HCF and LCM of 48 and 56 by prime factorization method.

Solution

$48 = 2^4 \times 3^1$
$56 = 2^3 \times 7^1$
HCF $= 2^3$ (The product of common prime factors with lesser index)
LCM $= 2^4 \times 3^1 \times 7^1$ (product of common prime factors with greater index).

Example

Find the HCF and LCM of 36, 48 and 60 by prime factorization method.

Solution

$36 = 2^2 \times 3^2$
$48 = 2^4 \times 3^1$
$60 = 2^2 \times 3^1 \times 5^1$
HCF $= 2^2 \times 3^1 = 12$
LCM $= 2^4 \times 3^2 \times 5^1 = 720$

Example

Two bells toll at intervals of 24 minutes and 36 minutes respectively. If they toll together at 9 am, after how many minutes do they toll together again, at the earliest?

Solution

The required time is LCM of 24 and 36.

$24 = 2^3 \times 3^1$

$36 = 2^2 \times 3^2$

$\therefore$ LCM of 24 and 36 is $2^3 \times 3^2 = 72$
So, they will toll together after 72 minutes.

Irrational numbers

A number which cannot be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Example

$\sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}$, 0.123421635....etc.

Theorem 1
If p divides x^3, then p divides x, where x is a positive integer and p is a prime number.

Proof
Let $x = p_1 p_2 \ldots\ldots p_n$, where $p_1, p_2, p_3, \ldots\ldots p_n$ are primes, not necessarily distinct.
$\Rightarrow x^3 = p_1^3\ p_2^3 \ldots\ldots p_n^3$
Given that p divides x^3.
By fundamental theorem, p is one of the primes of x^3.
By the uniqueness of fundamental theorem, the distinct primes of x^3 are same as the distinct primes of x.
$\Rightarrow$ p divides x.

Hence proved.
Similarly, if p divides x^2 then p divides x, where p is a prime number and x is a positive integer.

Theorem 2
Prove that $\sqrt{2}$ is irrational.

Proof
Let us assume that, $\sqrt{2}$ is not irrational.

So, $\sqrt{2}$ is rational.

$\Rightarrow \sqrt{2} = \frac{x}{y}$ where x, y are integers and $y \neq 0$
Let x and y be co-primes.
Taking squares on both sides,
$\Rightarrow 2 = \frac{x^2}{y^2}$
$\Rightarrow 2y^2 = x^2$ .... (1)
$\Rightarrow$ 2 divides x^2
$\Rightarrow$2 divides x.
$\therefore$ for some integer z,
$x = 2z$ .... (2)
from (1) and (2)
$2y^2 = 4z^2$
$\Rightarrow y^2 = 2z^2$
$\Rightarrow$ 2 divides y^2
$\Rightarrow$ 2 divides y
$\therefore$ x and y have at least 2 as a common factor.
But it contradicts the fact that x and y are co-primes.
$\therefore$ $\sqrt{2}$ is irrational.

☛ *Example*

Prove that $\sqrt{5}$ is irrational.

Solution

Let us assume that $\sqrt{5}$ is not irrational.
$\therefore$ $\sqrt{5}$ is rational.
$\Rightarrow \sqrt{5} = \frac{p}{q}$
Let p, q be co-primes.
Taking squares on both the sides,
$\Rightarrow 5 = \frac{p^2}{q^2}$
$\Rightarrow p^2 = 5q^2$ .... (1)
$\Rightarrow$ 5 divides p^2
$\Rightarrow$ 5 divides p.
$\therefore$ For some integer r,

$p = 5r$ .... (2)

From (1) and (2),

$25r^2 = 5q^2$

$\Rightarrow q^2 = 5r^2$

$\Rightarrow$ 5 divides q^2

$\Rightarrow$ 5 divides q.

$\therefore$ p and q have at least 5 as a common factor.

But it contradicts the fact that p and q are co-primes.

$\therefore$ $\sqrt{5}$ is irrational.

Example

Show that $3 + \sqrt{2}$ is irrational.

Solution

Let us assume that $3 + \sqrt{2}$ is rational.

$\therefore$ $3 + \sqrt{2} = \dfrac{p}{q}$ where p and q are integers.

$\Rightarrow \sqrt{2} = \dfrac{p}{q} - 3$

$\Rightarrow \sqrt{2} = \dfrac{p - 3q}{q}$

Since p and q are integers, $\dfrac{p - 3q}{q}$ is rational.

But $\sqrt{2}$ is irrational.

It contradicts our assumption that $3 + \sqrt{2}$ is rational.

$\therefore$ Our assumption is wrong.

Hence $3 + \sqrt{2}$ is irrational.

Rational numbers

Numbers which can be written in the form of $\dfrac{p}{q}$ $(q \neq 0)$, where p, q are integers, are called Rational numbers.

NOTE: Every terminating decimal and non-terminating repeating decimal can be expressed in the form $\dfrac{p}{q}$ $(q \neq 0)$.

Example

(i) $0.27 = \dfrac{27}{100} = \dfrac{27}{2^2 \times 5^2}$

(ii) $2.356 = \dfrac{2^2 \times 589}{2^3 \times 5^3} = \dfrac{589}{2 \times 5^3}$

(iii) $2.0325 = \dfrac{20325}{10000} = \dfrac{3 \times 5^2 \times 271}{2^4 \times 5^4} = \dfrac{3 \times 271}{2^4 \times 5^2}$

From the above examples, we notice that every terminating decimal can be written in the form of $\dfrac{p}{q}$ $(q \neq 0)$, where p and q are co-primes and q is of the form $2^m . 5^n$ (m and n are non-negative integers).

Let us write this result formally.

Theorem 3

Let a be a terminating decimal. Then a can be expressed as $\frac{p}{q}$ $(q \neq 0)$, where p and q are co-primes, and the prime factorization of q is of the form $2^m . 5^n$.

Let us observe the following examples.

1. $\frac{1}{4} = \frac{1}{2^2} = \frac{1 \times 5^2}{2^2 \times 5^2} = \frac{25}{100} = 0.25$

2. $\frac{7}{25} = \frac{7}{5^2} = \frac{7 \times 2^2}{5^2 \times 2^2} = \frac{28}{100} = 0.28$

3. $\frac{23}{125} = \frac{23}{5^3} = \frac{23 \times 2^3}{5^3 \times 2^3} = \frac{184}{1000} = 0.184$

4. $\frac{147}{50} = \frac{147}{2 \times 5^2} = \frac{147 \times 2}{2^2 \times 5^2} = \frac{294}{100} = 2.94$

From the above examples, we notice that every rational number of the form $\frac{p}{q}$ $(q \neq 0)$, where q is of the form $2^m . 5^n$ can be written as $\frac{x}{y}$, where y is of the form 10^k, k being a natural number.

Let us write this result formally.

Theorem 4

If $\frac{p}{q}$ is a rational number where q is of the form $2^m . 5^m$ $(m \in W)$, then $\frac{p}{q}$ has a terminating decimal expansion.

Let us observe the following examples.

Example

(i) $\frac{5}{3} = 1.66666... = 1.\overline{6}$

(ii) $\frac{8}{7} = 1.\overline{142857}$

(iii) $\frac{1}{11} = 0.\overline{09}$

Theorem 5

If $\frac{p}{q}$ is a rational number and q is not of the form $2^m . 5^n$ (m and n $\in W$), then $\frac{p}{q}$ has a non-terminating repeating decimal expansion.

Example

Which of the following rational numbers are terminating decimals?

1. $\frac{17}{2^3 \times 5^2}$ 2. $\frac{25}{3^2 \times 2^3}$ 3. $\frac{68}{2^2 \times 5^2 \times 7^2}$ 4. $\frac{125}{3^3 \times 7^2}$

Solution

Clearly $\frac{17}{2^3 \times 5^2}$ is the only terminating decimal and the remaining are non-terminating decimals.

test your concepts

Very short answer type questions

1. $\frac{7^3}{5^4}$ is a non terminating repeating decimal. (True/False)
2. $0.\overline{5} - 0.\overline{49} =$ ________.
3. If $360 = 2^x \times 3^y \times 5^z$, then $x + y + z =$ ________.
4. If $a = 2+\sqrt{3}$ and $b = \sqrt{2}-\sqrt{3}$, then a + b is ________. (rational/irrational)
5. If 19 divides a^3 (where a is a positive integer), then 19 divides a. (True/False)
6. If $0.\overline{7} = \frac{p}{q}$, then $p + q =$ ________.
7. Product of two irrational numbers is an irrational number. (True/False)
8. $2 \times 3 \times 15 + 7$ is a ________. (prime number/composite number)
9. Product of LCM and HCF of 25 and 625 is ________.
10. For what values of x, $2^x \times 5^x$ ends in 5.

Short answer type questions

11. Find the HCF of 72 and 264 by using Euclid's division algorithm.
12. Show that $7 \times 5 \times 3 \times 2 + 7$ is a composite number.
13. Find the HCF and LCM of 108 and 360 by prime factorization method.
14. Prove that $\sqrt{6}$ is irrational.
15. Without performing the actual division, check whether the following rational numbers are terminating or non-terminating.

    (i) $\frac{23}{175}$ (ii) $\frac{125}{325}$ (iii) $\frac{73}{40}$ (iv) $\frac{157}{125}$

Essay type questions

16. A fruit vendor has 732 apples and 942 oranges. He distributes these fruits among the students of an orphanage, such that each of them gets either only apples or only oranges in equal number. Find the least possible number of students.
17. Write 75600 as the product of prime factors.
18. Aloukya and Manoghna run around a circular track and they take 180 seconds and 150 seconds respectively to complete one revolution. If they start together at 9 am from the same point, how long it would take for them to meet again for the first time at the starting point?
19. Prove that $5 - \sqrt{5}$ is irrational.
20. Express $23.3\overline{24}$ in $\frac{p}{q}$ form, where p and q are integers.

CONCEPT APPLICATION

Concept Application Level—1

1. If n is a natural number, then $9^{2n} - 4^{2n}$ is always divisible by ________.

(1) 5 (2) 13
(3) both (1) and (2) (4) neither (1) nor (2)

2. N is a natural number such that when N^3 is divided by 9, it leaves remainder *a*. It can be concluded that

(1) *a* is a perfect square (2) *a* is a perfect cube
(3) both (1) and (2) (4) neither (1) nor (2)

3. The remainder of any perfect square divided by 3 is ________.

(1) 0 (2) 1
(3) either (1) or (2) (4) neither (1) nor (2)

4. Find the HCF of 432 and 504 using prime factorization method.

(1) 36 (2) 72 (3) 96 (4) 108

5. If n is any natural number, then $6^n - 5^n$ always ends with _________.

(1) 1 (2) 3 (3) 5 (4) 7

6. The LCM of two numbers is 1200. Which of the following cannot be their HCF?

(1) 600 (2) 500 (3) 200 (4) 400

7. Which of the following is always true?

(1) The rationalising factor of a number is unique
(2) The sum of two distinct irrational numbers is rational
(3) The product of two distinct irrational numbers is irrational
(4) None of these

8. Find the remainder when the square of any number is divided by 4.

(1) 0 (2) 1
(3) Either (1) or (2) (4) Neither (1) nor (2)

9. Ashok has two vessels which contain 720 ml and 405 ml of milk respectively. Milk in each vessel is poured into glasses of equal capacity to their brim. Find the minimum number of glasses which can be filled with milk.

(1) 45 (2) 35 (3) 25 (4) 30

10. If n is an odd natural number, $3^{2n} + 2^{2n}$ is always divisible by

(1) 13 (2) 5 (3) 17 (4) 19

Concept Application Level—2

11. Given that the units digits of A^3 and A are the same where A is a single digit natural number. How many possibilities can A assume?

(1) 6 (2) 5 (3) 4 (4) 3

12. If the product of two irrational numbers is rational, then which of the following can be concluded?

(1) The ratio of the greater and the smaller numbers is an integer

(2) The sum of the numbers must be rational

(3) The excess of the greater irrational number over the smaller irrational number must be rational

(4) None of the above

13. The LCM and HCF of two numbers are equal, then the numbers must be _______.

(1) prime (2) co-prime (3) composite (4) equal

14. Which of the following is/are always true?

(1) Every irrational number is a surd.

(2) Any surd of the form $\sqrt[n]{a} + \sqrt[n]{b}$ can be rationalised by a surd of the form $\sqrt[n]{a} - \sqrt[n]{b}$, where $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are surds.

(3) Both (1) and (2)

(4) Neither (1) nor (2)

15. The sum of LCM and HCF of two numbers is 1260. If their LCM is 900 more than their HCF, find the product of two numbers.

(1) 203400 (2) 194400 (3) 198400 (4) 205400

Concept Application Level—3

16. Find the remainder when the square of any prime number greater than 3 is divided by 6.

(1) 1 (2) 3 (3) 2 (4) 4

17. If HCF (72, q) = 12 then how many values can q take? (Assume q be a product of a power of 2 and a power of 3 only)

(1) 1 (2) 2 (3) 3 (4) 4

18. Find the HCF of 120 and 156 using Euclid's division algorithm.

(1) 18 (2) 12 (3) 6 (4) 24

19. The HCF of the polynomials $(x^2 - 4x + 4)(x + 3)$ and $(x^2 + 2x - 3)(x - 2)$ is

(1) $x + 3$ (2) $x - 2$ (3) $(x + 3)(x - 2)$ (4) $(x + 3)(x - 2)^2$

20. $\sqrt{3+\sqrt{5}} =$

(1) $\sqrt{2} + 1$ (2) $\sqrt{\frac{5}{2}} + \sqrt{\frac{1}{2}}$ (3) $\sqrt{\frac{7}{2}} - \sqrt{\frac{1}{2}}$ (4) $\sqrt{\frac{9}{2}} - \sqrt{\frac{3}{2}}$

KEY

Very short answer type questions

1. False
2. $0.\overline{06}$
3. 6
4. irrational
5. True
6. 16
7. False
8. prime number
9. 15625
10. No value of x

Short answer type questions

11. 24

13. 1080

15. (i), (ii) non-terminating
(iii), (iv) terminating

Essay type questions

16. 279

17. $2^4 \times 3^3 \times 5^2 \times 7$

18. 9:15 a.m.

20. $\frac{23091}{990}$

key points for selected questions

Short answer type questions

11. (i) By Euclid's Division Lemma, for any two integers say x and y there exists unique integers say q and r such that $x = yq + r$ where $0 \le r < y$
(ii) Now $264 = 72 \times 3 + 48$ where 48 is remainder.
(iii) Again consider 72, 48 and apply Euclid's division lemma.
(vi) Apply this concept until we get the remainder zero.
(v) In this process, HCF is the divisor obtained when the remainder is zero.

12. (i) First of all take out 7 as it is the common factor.
(ii) Now, the given number is a product of two numbers.
(iii) As neither of these numbers is 1, it is composite number.

13. (i) Prime factorize 108 and 360.
(ii) HCF is the product of common prime factors with least exponents.
(iii) LCM is the product of all prime factors with possible highest exponents.

14. (i) Assume that $\sqrt{6}$ is not irrational i.e., rational.
(ii) Let $\sqrt{6} = \frac{p}{q}$ (p and q have no common factor other than 1).
(iii) Square the above equation.
(vi) Then check whether 2 is common factor of p and q.

15. In the simplified form of a fraction, if the denominator has any prime factors other than 2 or 5 then the fraction is a non-terminating decimal otherwise it is a terminating decimal.

Essay type questions

16. (i) First of all find the HCF of 732 and 942.

(ii) The required number of students = $\frac{732+942}{\text{HCF}}$

17. (i) First of all find the least prime which divides 75600 and divide 75600 by it.

(ii) Now consider the quotient, then find the least prime which divides the quotient and divide it.

(iii) Continue this process, until you get a prime number as quotient.

(vi) 75600 is the product of all these prime numbers.

18. (i) Find the LCM of 150 and 180.

(ii) LCM is the time taken to meet for the first time.

19. (i) Assume that $5 - \sqrt{5}$ is not rational.

(ii) So, $5 - \sqrt{5} = \frac{p}{q}$ (p and q are integers in their lowest terms).

(iii) Then write $\sqrt{5}$ in terms of p and q.

(vi) Then check whether it contradicts our assumption.

20. $ab.c\overline{de} = \frac{abcde - abc}{990}$

KEY

Concept Application Level—1,2,3

1. 3	**2.** 2
3. 3	**4.** 2
5. 1	**6.** 1
7. 4	**8.** 2
9. 3	**10.** 1
11. 2	**12.** 4
13. 4	**14.** 4
15. 2	**16.** 1
17. 2	**18.** 2
19. 3	**20.** 3

Concept Application Level—1,2,3

Key points for select questions

1. (i) Given expression is in the form of $a^2 - b^2$.

(ii) $a^{2n} - b^{2n}$ is divisible by both $(a - b)$ and $(a + b)$.

2. Apply trial and error method.

3. Apply trial and error method.

4. (i) Express 432 and 504 as the product of prime factors and proceed.

(ii) Write 432 and 504 as the product of prime factors.

(iii) HCF is the product of common prime factors with least exponents.

5. For any natural number n, 6^n and 5^n end with 6 and 5 respectively.

6. (i) Apply the method of prime factorization for finding the HCF of the given numbers.

(ii) Then find the possibilities for q.

7. (i) Recall the concepts of rational and irrational numbers.

(ii) Recall the concept of RF.

8. (i) Multiply and divide $3 + \sqrt{5}$ by 2.

(ii) Convert the expression $\sqrt{x+2\sqrt{y}}$ in the form of $\sqrt{(\sqrt{m})^2+(\sqrt{n})^2+2\sqrt{mn}}$.

9. (i) First of all find the HCF of 720 and 405.
   (ii) The required number is $\frac{720}{\text{HCF}} + \frac{405}{\text{HCF}}$.

10. (i) $3^{2n} + 2^{2n} = 9^n + 4^n$
    (ii) $a^n + b^n$ is divisible by $(a + b)$ when n is odd.

11. (i) Write cubes of 1 to 9.
    (ii) Then check their unit digits.

12. Recall the concept of rational numbers and irrational numbers.

13. (i) Suppose the numbers to be ka and kb where k is their HCF and a and b are co-primes.
    (ii) Check from the options.

14. (i) Recall the concepts of surds and irrational numbers.
    (ii) Recall the concepts of RF.

15. Form the equations in LCM and HCF and solve for LCM and HCF.

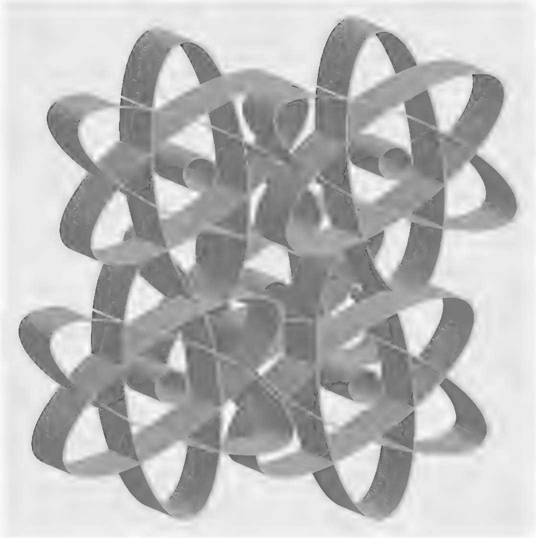
16. Any prime number greater than 3 is in the form $6k \pm 1$, where k is a natural number.

17. HCF is the factor of LCM.

18. (i) Write $156 = 1 \times 120 + 36$ and proceed further.
    (ii) $156 = 120 \times 1 + 36$
    $120 = 36 \times 3 + 12$
    (iii) Use division algorithm until we get zero as remainder.

19. Factorize the given expressions.

20. Apply trial and error method.



Chapter 2

Polynomials and Rational Expressions

INTRODUCTION

In this chapter we shall learn about HCF and LCM of polynomials and the method of finding them. Further we shall learn about rational expressions and also the method of expressing them in their lowest terms.

Polynomial of nth degree

The expression of the form $a_0x^n + a_1x^{n-1} + \dots\dots\dots + a_{n-1}x + a_n$ is a polynomial of nth degree ($a_0 \neq 0$) in one variable. Here $a_0, a_1, \dots\dots\dots, a_n$ are real numbers.

HCF of given polynomials

For the given two polynomials, f(x) and g(x), r(x) can be taken as the highest common factor, if

(i) r(x) is a common factor of f(x) and g(x) and
(ii) every common factor of f(x) and g(x) is also a factor of r(x).

Highest common factor is generally referred to as HCF.

Method for finding HCF of the given polynomials

Step 1
Express each polynomial as a product of powers of irreducible factors which also requires the numerical factors to be expressed as the product of the powers of primes.

Step 2
If there is no common factor then HCF is 1 and if there are common irreducible factors, we find the least exponent of these irreducible factors in the factorised form of the given polynomials.

Step 3

Raise the common irreducible factors to the smallest or the least exponents found in step 2 and take their product to get the HCF.

Examples

(i) Find the HCF of $42a^2b^2$ and $48ab^3$.

Let $f(x) = 42a^2b^2$ and $g(x) = 48ab^3$

Writing f(x) and g(x) as a product of powers of irreducible factors,

$f(x) = 2 \times 3 \times 7 \times a^2 \times b^2$

$g(x) = 2 \times 2 \times 2 \times 2 \times 3 \times a \times b^3 = 2^4 \times 3 \times a \times b^3$

The common factors with the least exponents are 2, 3 and ab^2.

$\therefore$ The HCF of the given polynomials $= 2 \times 3 \times ab^2 = 6ab^2$.

(ii) Find the HCF of $96(x - 1)(x + 1)^2(x + 3)^3$ and $64(x^2 - 1)(x + 3)(x + 2)^2$

Let $f(x) = 96(x - 1)(x + 1)^2(x + 3)^3$ and

$g(x) = 64(x^2 - 1)(x + 3)(x + 2)^2$

Writing f(x) and g(x) as the product of the powers of irreducible factors,

$f(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times (x - 1) \times (x + 1)^2 \times (x + 3)^3$

$g(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times (x - 1)(x + 1)(x + 3)(x + 2)^2$

The common factors with the least exponents are 2^5, $(x - 1)$, $(x + 1)$, $(x + 3)$.

$\therefore$ The HCF of the given polynomials $= 32 \times (x - 1)(x + 1)(x + 3)$

$= 32(x^2 - 1)(x + 3)$

LCM of the given polynomials

Least Common Multiple or the Lowest Common Multiple is the product of all the factors (taken once) of the polynomials given with their highest exponents respectively.

Method to calculate LCM of the given polynomials

Step 1

First express each polynomial as a product of powers of irreducible factors.

Step 2

Consider all the irreducible factors (only once) occurring in the given polynomials. For each of these factors, consider the greatest exponent in the factorised form of the given polynomials.

Step 3

Now raise each irreducible factor to the greatest exponent and multiply them to get the LCM.

Examples

(i) Find the LCM of $42a^2b^2$ and $48ab^3$.

Let $p(x) = 42a^2b^2$ and $q(x) = 48ab^3$

Writing p(x) and q(x) as the product of the powers of irreducible factors,

$p(x) = 2 \times 3 \times 7 \times a^2 \times b^2$

$q(x) = 2 \times 2 \times 2 \times 2 \times 3 \times a \times b^3$

Now all the factors (taking only once) with the highest exponents are $2^4, 3, 7, a^2, b^3$

$\Rightarrow$ The LCM of the given polynomials $= 2^4 \times 3 \times 7 \times a^2 \times b^3 = 336a^2b^3$

(ii) Find the LCM of $96(x-1)(x+1)^2(x+3)^3$ and $64(x^2-1)(x+3)(x+2)^2$

Let $f(x) = 96(x-1)(x+1)^2(x+3)^3$ and $g(x) = 64(x^2-1)(x+3)(x+2)^2$

$f(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 3\ (x-1)(x+1)^2\,(x+3)^3$

$g(x) = 2 \times 2 \times 2 \times 2 \times 2 \times 2\ (x-1)(x+1)\ (x+3)(x+2)^2$

Now all the factors (taking only once) with the highest exponents are 2^6, 3, $(x-1)$, $(x+1)^2$, $(x+3)^3$, $(x+2)^2$.

$\Rightarrow$ The LCM of the given polynomials $= 192(x-1)(x+1)^2(x+2)^2(x+3)^3$

Relation among the HCF, the LCM and the product of the polynomials

If f(x) and g(x) are two polynomials then we have the following relation,
(HCF of f(x) and g(x)) × (LCM of f(x) and g(x)) = ± (f(x) × g(x))

Example

Let $f(x) = (x+3)^2(x-1)(x+2)^3$ and $g(x) = (x+3)\ (x-1)^2(x+2)^2$ be two polynomials.

The common factors with the least exponents are $(x+3)$, $(x-1)$, $(x+2)^2$.

$\Rightarrow$ HCF $= (x-1)(x+3)(x+2)^2$

All the factors (taken only once) with the highest exponents are $(x+3)^2$, $(x-1)^2$, $(x+2)^3$.

$\Rightarrow$ LCM $= (x-1)^2(x+2)^3(x+3)^2$

Now $f(x) \times g(x) = \{(x+3)^2(x-1)(x+2)^3\} \times \{(x+3)(x-1)^2(x+2)^2\} = (x-1)^3(x+2)^5(x+3)^3$

LCM × HCF $= \{(x-1)(x+3)(x+2)^2\} \times \{(x-1)^2(x+2)^3(x+3)^2\} = (x-1)^3(x+3)^3\,(x+2)^5$

Thus, we say, LCM × HCF = Product of two polynomials

Rational expressions

We know that any number of the form $\frac{p}{q}$ where $p, q \in z$ and $q \neq 0$ is called a rational number. As integers and polynomials behave in the same manner, we observe most of the properties satisfied by rational numbers are also satisfied by the algebraic expressions of the form $\frac{f(x)}{g(x)}$ which are called rational expressions. Rational expression is 'an algebraic expression which is of the form $\frac{f(x)}{g(x)}$ where f(x) and g (x) are polynomials and g(x) is not a zero polynomial.

For any rational number of the form $\frac{p}{q}$ $p, q \in z$ and $q \neq 0$. p and q are called numerator and denominator respectively. Eventhough p and q are integers $\frac{p}{q}$ need not be an integer. Similarly for any rational expression $\frac{f(x)}{g(x)}$, f(x) is called numerator and g(x) is called denominator.

Eventhough f(x) and g(x) are polynomials $\frac{f(x)}{g(x)}$ need not be a polynomial.

Example

1. $\dfrac{2x-1}{x^2-3x+1}$ is a rational expression

2. $\dfrac{x^3+5x^2-\sqrt{3}x+\sqrt{5}}{2x^2-\sqrt{5}x+8}$ is a rational expression

3. $\dfrac{x^2-5\sqrt{x}-1}{3x-5}$ is not a rational expression as the numerator is not a polynomial.

NOTE:

1. Every polynomial is a rational expression as f(x) can be written as $\dfrac{f(x)}{1}$.
2. $\dfrac{f(x)}{g(x)}$ is not a rational expression if either numerator f(x) or denominator g(x) or both f(x) and g(x) are not polynomials.

Rational expressions in lowest terms

Let f(x) and g(x) have integer coefficients and HCF of f(x) and g(x) is 1, then the rational expression $\dfrac{f(x)}{g(x)}$ is said to be in its lowest terms.

If HCF of f(x) and g(x) is not equal to 1, then by cancelling HCF of f(x) and g(x) from both numerator and denominator, we can reduce the given rational expression to its lowest term.

Example

Verify whether the rational expression $\dfrac{x^2-1}{(2x+1)(x+2)}$ is in its lowest terms.

Solution

The given rational expression can be written as $\dfrac{(x+1)(x-1)}{(2x+1)(x+2)}$ and clearly HCF of numerator and denominator of the given expression is 1.

∴ It is in its lowest terms.

Example

Express the rational expression $\dfrac{x^2-2x-3}{2x^2-3x-5}$ in its lowest terms.

Solution

Factorizing both the numerator and the denominator of the given expression we have

$$\frac{x^2-2x-3}{2x^2-3x-5} = \frac{(x+1)(x-3)}{(2x-5)(x+1)}$$

HCF of numerator and denominator is x + 1

Dividing both numerator and denominator of the given expression by (x + 1), we have $\dfrac{x-3}{2x-5}$ which is in its lowest terms.

Addition/subtraction of rational expressions

The sum of any two rational expressions $\frac{f(x)}{g(x)}$ and $\frac{h(x)}{p(x)}$ is written as $\frac{f(x)}{g(x)} + \frac{h(x)}{p(x)} = \frac{f(x)\,p(x)+h(x)\,g(x)}{g(x)\,p(x)}$

If the denominators g(x) and p(x) are equal then $\frac{f(x)}{g(x)} + \frac{h(x)}{p(x)} = \frac{f(x)+h(x)}{g(x)}$

The difference of the above rational expressions can be written as $\frac{f(x)}{g(x)} - \frac{h(x)}{p(x)} = \frac{f(x)\,p(x)-h(x)\,g(x)}{g(x)\,p(x)}$

NOTE:

1. Sum or difference of two rational expressions is also a rational expression.
2. For any rational expression $\frac{f(x)}{g(x)}$, $\frac{-f(x)}{g(x)}$ is called the additive inverse of $\frac{f(x)}{g(x)}$.

   i.e., $\frac{f(x)}{g(x)}+\left(\frac{-f(x)}{g(x)}\right) = 0$

Example

Simplify $\frac{x+1}{2x-1}+\frac{3x-2}{x-1}$

Solution

$$\frac{x+1}{2x-1}+\frac{3x-2}{x-1} = \frac{(x+1)(x-1)+(3x-2)(2x-1)}{(2x-1)(x-1)} = \frac{x^2-1+6x^2-7x+2}{2x^2-3x+1} = \frac{7x^2-7x+1}{2x^2-3x+1}$$

Example

Find the sum of the rational expressions $\frac{2x-3}{x^2+x-2}$ and $\frac{3x-1}{2x^2+5x+2}$

Solution

$$\frac{2x-3}{x^2+x-2}+\frac{3x-1}{2x^2+5x+2} = \frac{2x-3}{(x+2)(x-1)}+\frac{3x-1}{(2x+1)(x+2)}$$

$$= \frac{(2x-3)(2x+1)+(3x-1)(x-1)}{(x-1)(x+2)(2x+1)}$$

$$= \frac{4x^2-4x-3+3x^2-4x-1}{(x-1)(x+2)(2x+1)} = \frac{7x^2-8x-4}{2x^3-3x^2-3x-2}$$

Example

If $A = \frac{x+1}{2x-1}$, $B = \frac{2x-1}{3x+2}$ and $C = \frac{4x-5}{2x^2+5x-3}$, then find $4A - 3B + C$.

Solution

$$4A - 3B + C = \frac{4(x+1)}{2x-1} - \frac{3(2x-1)}{3x+2} + \frac{4x-5}{(2x-1)(x+3)} \quad (\because 2x^2 + 5x - 3 = (2x-1)\ (x+3))$$

$$= \frac{4(x+1)(3x+2)(x+3) - 3(2x-1)^2(x+3) + (4x-5)(3x+2)}{(2x-1)(3x+2)(x+3)}$$

$$= \frac{4(3x^3 + 14x^2 + 17x + 6) - 3(4x^3 + 8x^2 - 11x + 3) + 12x^2 - 7x - 10}{(2x-1)(3x+2)(x+3)}$$

$$= \frac{44x^2 + 94x + 5}{6x^3 + 19x^2 + x - 6}$$

Multiplication of rational expressions

The product of two rational expressions $\frac{f(x)}{g(x)}$ and $\frac{h(x)}{p(x)}$ is given by $\frac{f(x)}{g(x)} \times \frac{h(x)}{p(x)} = \frac{f(x).h(x)}{g(x).\,p(x)}$

NOTE:

1. The process of finding the
   (i) product of two rational expressions is similar to the process of finding the product of two rational numbers.
   (ii) product of two rational expressions is also a rational expression.
2. After finding the product of two rational expressions the resultant rational expression must be put in its lowest terms.

☛ *Example*

Find the product of the rational expressions $\frac{3x^2 + 8x - 3}{2x^2 - x - 6}$ and $\frac{x^2 - 4}{x + 3}$

Solution

Product of the given expressions is $\frac{3x^2 + 8x - 3}{2x^2 - x - 6} \times \frac{x^2 - 4}{x + 3}$

$$= \frac{(x+3)(3x-1)}{(2x+3)(x-2)} \times \frac{(x-2)(x+2)}{(x+3)}$$

$$= \frac{(3x-1)(x+2)}{2x+3} = \frac{3x^2 + 5x - 2}{2x + 3}$$

☛ *Example*

Simplify $\left[\frac{2x-1}{x+3} - \frac{x^2 - 4}{2x+1}\right] \times \frac{2x^2 + 7x + 3}{x + 2}$

Solution

$$\left[\frac{2x-1}{x+3}-\frac{x^2-4}{2x+1}\right]\times\frac{2x^2+7x+3}{x+2}$$

$$\frac{(2x-1)(2x+1)-(x^2-4)(x+3)}{(x+3)(2x+1)}\times\frac{(x+3)(2x+1)}{x+2}$$

$$=\frac{4x^2-1-x^3-3x^2+4x+12}{x+2}=\frac{-(x^3-x^2-4x-11)}{x+2}$$

Note: For every rational expression of the form $\frac{f(x)}{g(x)}$, $(g(x) \neq 0)$ there exists a rational expression of the form $\frac{f(x)}{g(x)}$ such that $\frac{f(x)}{g(x)} \times \frac{g(x)}{f(x)} = 1$, then $\frac{g(x)}{f(x)}$ is called the multiplicative inverse of $\frac{f(x)}{g(x)}$ and vice versa.

Division of rational expressions

Let $\frac{f(x)}{g(x)}$ and $\frac{h(x)}{p(x)}$ be two non-zero rational expressions, then $\frac{f(x)}{g(x)} \div \frac{h(x)}{p(x)} = \frac{f(x)}{g(x)} \times \frac{h(x)}{p(x)}$ i.e., $\frac{f(x)p(x)}{g(x)h(x)}$ which is also a rational expression.

Note:

The process of dividing two rational expressions is similar to the process of dividing two rational numbers

☛ *Example*

Express $\frac{2x^2+6x}{3x^2+7x+2} \div \frac{x^2+x-6}{3x^2+7x+2}$ as a rational expression in its lowest terms.

Solution

Given $\frac{2x^2+6x}{3x^2+7x+2} \div \frac{x^2+x-6}{3x+7x+2}$

$$\frac{2x(x+3)}{(x+2)(3x+1)} \div \frac{(x+3)(x-2)}{(3x+1)(x+2)}$$

$$\frac{2x(x+3)}{(x+2)(3x+1)} \times \frac{(3x+1)(x+2)}{(x+3)(x-2)} = \frac{2x}{x-2}$$

test your concepts

Very short answer type questions

1. The LCM of $18x^2y^3$ and $8x^3y^2$ is ________.
2. The HCF of $24a^2b$ and $32ab^2$ is ________.
3. The HCF of $9a^2 - 16b^2$ and $12a^2 - 16\ ab$ is ________.
4. The LCM of $2\ (x - 3)^2$ and $3(x - 2)^2$ is ________.
5. The HCF of $24x^5$ and $36x^6y^k$ is $12x^5$, then the value of k is ________.
6. The HCF of $a^m + b^m$ and $a^n - b^n$, where m is odd positive integer and n is even positive integer is ________.
7. The LCM of $12x^3y^2$ and $18x^py^3$ is $36x^4y^3$. Then the number of integer values of p is ________.
8. The HCF of $(x + 2)\ (x^2 - 7x + k)$ and $(x - 3)\ (x^2 + 3x + \ell)$ is $(x + 2)\ (x - 3)$. Then the values of k and ℓ are ________ respectively.
9. The LCM of $x^2 + 2x - 8$ and $x^2 + 3x - 4$ is ________.
10. The rational expression $\dfrac{x^2+1}{x^2+4x+3}$ is not in its lowest terms. (True/False)
11. The additive inverse of $\dfrac{x^2+1}{x^2-1}$ is ________.
12. $\dfrac{x^2}{x^2-1} \div \dfrac{x^3}{x+1} =$ _______.
13. The product of two rational expressions is not always a rational expression. (True/False)
14. The rational expression $\dfrac{(x+1)^2}{x^2-1}$ in its lowest terms is ________.
15. The reciprocal of $x - \dfrac{1}{x}$ is $\dfrac{1}{x} - x$. (True/False)
16. If the rational expression $\dfrac{x-a}{x^2-b}$ is in its lowest terms, then ________. $[a \neq b/a^2 \neq b]$
17. Every polynomial is a rational expression. (True/False)
18. $\dfrac{x^3+\sqrt{6}x^2-7}{2x^2+\sqrt{x}+1}$ is a rational expression. (True/False)
19. The HCF of the polynomials $8a^3b^2c$, $16a^2bc^3$ and $20ab^3c^2$ is ________.
20. The LCM of the polynomials $8(x^3 + 8)$ and $12(x^2 - 4)$ is ________.
21. The HCF of the polynomials $12xy^2z^3$, $18x^2y^2z$ and $28x^3yz^2$ is ________.

22. The LCM of the polynomials $15a^2b(a^2 - b^2)$ and $40\ ab^2(a - b)$ is ________.

23. The HCF of the polynomials $15(a + 1)^2 (a - 2)$, $65(a - 1)^2 (a + 1)$ and $90\ (a - 2)^2\ (a - 1)$ is ________.

24. The HCF of the polynomials $(x + 3)^2\ (x - 2)\ (x + 1)^2$ and $(x + 1)^3\ (x + 3)\ (x + 4)$ is ________.

25. The LCM of the polynomials $x^2 - 1$, $x^2 + 1$ and $x^4 - 1$ is ________.

26. The multiplicative inverse of $x - \dfrac{x-1}{1-x}$ is ________.

27. Which of the following algebraic expressions are polynomials?

(a) $x^3 - \sqrt{7}\ x + 13$

(b) $3x^4 + 2x^3 - \sqrt{5x} + 8$

(c) $11 - x^2 + x\sqrt{x} + 7x$

(d) $-\sqrt{3x^3} + 8x^2 - 9$

28. The additive inverse of $\dfrac{x+1}{x^2-1}$ is ________.

29. If $A = \dfrac{p(x)}{q(x)}$ and $B = \dfrac{f(x)}{g(x)}$ are two rational expressions where $q(x), g(x) \neq 0$, then $A + B$ and $A - B$ are also rational expressions. (True/False)

30. The rational expression whose numerator is a linear polynomial with -3 as zero and whose denominator is a quadratic polynomial with zeroes $\dfrac{1}{2}$ and -1 is ________.

Short answer type questions

31. Find the HCF and the LCM of the following monomials.

(i) x^3y^6 and x^2y^8

(ii) $3a^2b^3c^4$ and $9a^4b^3c^2$

(iii) $p^4q^2r^3$, $q^3p^6r^5$ and

32. Find the LCM and HCF of the polynomials $(x^2 - 4)\ (x^2 - x - 2)$ and $(x^2 + 4x + 4)\ (x^2 - 3x + 2)$. Verify that the product of the LCM and HCF is equal to the product of the polynomials.

33. If HCF of the two polynomials $(x - 1)\ (x - 6)\ (x - 2)$ and $(x - 6)^3(x - 4)\ (x - 2)^2$ is $(x - 6)\ (x - 2)$, then find their LCM.

34. If LCM of the two polynomials $(x^2 + 3x)\ (x^2 + 3x + 2)$ and $(x^2 + 6x + 8)\ (x^2 + kx + 6)$ is $x\ (x + 1)\ (x + 2)^2\ (x + 3)\ (x + 4)$, then find k.

35. If $(x + 2)\ (x + 5)$ is the HCF of the polynomials $(x + 2)\ (x^2 + 6x + a)$ and $(x + 5)\ (x^2 + 8x + b)$, then find the values of a and b.

36. If the LCM and HCF of the two polynomials are $(x - 1)^2(x - 2)^2(x - 3)^3$ and $(x - 1)\ (x - 2)\ (x - 3)$ respectively and one of the polynomials is $(x - 1)\ (x - 2)^2\ (x - 3)$, then find the other polynomial.

37. Find the LCM and HCF of the polynomials $(8 - x^3)$ and $(x^2 - 4)(x + 3)$. Verify that the product of the LCM and HCF is equal to the product of the polynomials.

38. The HCF and LCM of the two polynomials are $(x - 3)(x + 1)$ and $(x^2 - 9)(x^2 - 1)$ respectively. If one of the polynomial is $(x - 3)(x^2 - 1)$, then find the second polynomial.

39. Find the HCF and LCM of $(x - 1)(x + 2)(x - 3)$, $(x + 2)^2(x + 1)$ and $(x + 3)(x^2 - 4)$.

40. Find the sum of the following rational expressions.

$\frac{x+1}{(x-1)^2}$ and $\frac{x-2}{x^2-1}$

41. If $P = \frac{3x+1}{3x-1}$ and $Q = \frac{3x-1}{3x+1}$, then find $P - Q$ and $P + Q$.

42. Reduce the following rational expressions to their lowest terms.

(i) $\frac{16x^2-(x^2-9)^2}{4x+9-x^2}$ (ii) $\frac{8x^5-8x}{(3x^2+3)(4x+4)}$

43. If $X = \frac{1+a}{1-a}$ and $Y = \frac{1-a}{1+a}$, then find $X^2 + Y^2 + XY$.

44. What should be added to $\frac{a}{a-b} + \frac{b}{a+b}$ to get 1?

45. Simplify:

$$\frac{x+1}{x-1} - \frac{x-1}{x+1} + \frac{8x}{1+x^2} - \frac{12x^3}{x^4-1}$$

Essay type questions

46. Simplify: $\left[\frac{x+2}{x-2} - \frac{x-2}{x+2} - \frac{8x}{x^2+4}\right] \div \frac{8x}{x^4-16}$

47. Simplify the rational expression $\frac{2x}{1+x^2+x^4} + \frac{1}{1+x+x^2} - \frac{1}{1-x+x^2}$.

48. Find the product of the following rational expressions and express the result in lowest terms:

$\frac{8x^2+10x-3}{12x^2+x-1}$ and $\frac{6x^2-7x-3}{8x^2-10x-3}$

49. Simplify the rational expression $\frac{1}{x+p} + \frac{1}{x+q} + \frac{1}{x+r} + \frac{px}{x^3+px^2} + \frac{qx}{x^3+qx^2} + \frac{rx}{x^3+rx^2}$.

50. Simplify: $\frac{8x^3 - y^3 + z^3 + 6xyz}{a^3 - 8b^3 + 27c^3 + 18abc} \div \frac{4x^2 + y^2 + z^2 + 2xy + yz - 2xz}{a^2 + 4b^2 + 9c^2 + 2ab + 6bc - 3ac}$

CONCEPT APPLICATION

Concept Application Level—1

1. The LCM of the polynomials $18(x^4 - x^3 + x^2)$ and $24(x^6 + x^3)$ is ________.

(1) $72x^2 (x + 1) (x^2 - x + 1)^2$ (2) $72x^3 (x^2 - 1) (x^3 - 1)$
(3) $72x^3 (x^3 - 1)$ (4) $72x^3 (x^3 + 1)$

2. The LCM of the polynomials $f(x) = 9(x^3 + x^2 + x)$ and $g(x) = 3(x^3 + 1)$ is ________.

(1) $27x (x + 1) (x^2 + x + 1)$ (2) $9x(x + 1) (x^2 + x + 1) (x^2 - x + 1)$
(3) $9(x + 1) (x^2 - x + 1) (x^2 + x + 1)$ (4) $9x(x + 1) (x^2 + x + 1)$

3. The LCM of polynomials $14(x^2 - 1) (x^2 + 1)$ and $18(x^4 - 1) (x + 1)$ is ________.

(1) $126(x + 1) (x^2 + 1) (x - 1)$ (2) $126(x + 1) (x^2 + 1) (x^2 - 1)$
(3) $126(x + 1)^2 (x^2 + 1) (x - 1)^2$ (4) $126(x + 1) (x^2 + 1) (x - 1)^2$

4. If HCF and LCM of two polynomials P(x) and Q(x) are $x(x + p)$ and $12x^2 (x - p)(x^2 - p^2)$ respectively. If $P(x) = 4x^2 (x +p)$, then $Q(x)$ = ________.

(1) $3x(x - p)^2 (x + p)$ (2) $3x(x - p) (x + p)$
(3) $3x(x + p) (x^2 - p^2)$ (4) $3x(x + p) (x^2 + p^2)$

5. The product of HCF and LCM of two polynomials is $(x^2 - 1) (x^4 - 1)$, then the product of the polynomials is ________.

(1) $(x^2 - 1) (x^2 + 1)$ (2) $(x^2 - 1)(x^2 + 1)^2$
(3) $(x^2 - 1)^2 (x^2 + 1)$ (4) None of these

6. If $(x + 4) (x - 2) (x + 1)$ is the HCF of the polynomials $f(x) = (x^2 + 2x - 8) (x^2 + 4x + a)$ and $g(x) = (x^2 - x - 2) (x^2 + 3x - b)$, then (a, b) = ________.

(1) (3, –4) (2) (–3, –4) (3) (–3, 4) (4) (3, 4)

7. Find the LCM of $x^3 - x^2 + x - 1$ and $x^3 - 2x^2 + x - 2$.

(1) $(x + 1) (x - 1)$ (2) $x - 1$
(3) $(x^2 + 1) (x - 1) (x - 2)$ (4) None of these

8. The HCF of the polynomials p(x) and q(x) is $6x - 9$, then p(x) and q(x) could be ________.

(1) $3, 2x - 3$ (2) $12x - 18, 2$
(3) $3(2x -3)^2, 6(2x - 3)$ (4) $3(2x - 3), 6(2x + 3)$

9. If the HCF of the polynomials f(x) and g(x) is $4x - 6$, then f(x) and g(x) could be ________.

(1) $2, 2x - 3$ (2) $8x - 12, 2$
(3) $2(2x - 3)^2, 4(2x - 3)$ (4) $2(2x + 3), 4 (2x + 3)$

10. If the HCF of the polynomials $f(x) = (x + 3) (3x^2 - 7x - a)$ and $g(x) = (x - 3)(2x^2 + 3x + b)$ is $(x + 3) (x - 3)$, then $a + b$ = ________.

(1) 3 (2) –15 (3) –3 (4) 15

11. Find the HCF of the polynomials $f(x) = x^3 + x^2 + x + 1$ and $g(x) = x^3 - x^2 + x - 1$.

(1) $x(x + 1)$ (2) $x - 1$ (3) $x^2 + 1$ (4) $x + 1$

12. If the HCF of $x^3 + 2x^2 - ax$ and $2x^3 + 5x^2 - 3x$ is $x(x + 3)$, then a = ________.

(1) 3 (2) −3 (3) 6 (4) − 4

13. The HCF of the polynomials $70(x^3 - 1)$ and $105(x^2 - 1)$ is ________.

(1) $15(x - 1)$ (2) $35(x - 1)$

(3) $35(x^2 - 1)(x^2 + x + 1)$ (4) $15(x^2 - 1)$

14. What should be subtracted from $\dfrac{7x}{(x^2 - x - 12)}$ to get $\dfrac{3}{x+3}$?

(1) $\dfrac{5}{x+4}$ (2) $\dfrac{4}{x-4}$ (3) $\dfrac{2}{x-4}$ (4) $\dfrac{1}{x-4}$

15. Which of the following algebraic expressions is/are not polynomials?

(a) $x^3 + 2x^3 + \sqrt{7}\,x + 4$ (b) $5x^2 + 4\sqrt{x} - 11$

(c) $\dfrac{x^3 + 3x^2 - 8x + 11}{4x\sqrt{x} - 3x + 3}$ (d) $\dfrac{x^3 + 3x^2 - 6x + 13}{x^2 + 1}$

(1) a,b and c (2) a and c (3) b and c (4) a and d

16. Which of the following is/are true?

(a) The sum of two rational expressions is always a rational expression.

(b) The difference of two rational expressions is always a rational expression.

(c) $\dfrac{p(x)}{q(x)}$ is in its lowest terms if LCM [p(x), q(x)] = 1.

(d) Reciprocal of $\dfrac{-2x}{x^2 - 1}$ is $\dfrac{x^2 - 1}{2x}$.

(1) a, b (2) a, b, d (3) a, c (4) a, b, c

17. The product of additive inverses of $\dfrac{x^2 - 1}{2x}$ and $\dfrac{x^2 - 4}{3 - x}$ is ________.

(1) $x^2 + 5x + 6$ (2) $x^2 + x - 6$ (3) $x^2 - x - 6$ (4) $x^2 - 5x + 6$

18. What should be added to $\dfrac{1}{x^2 - 7x + 12}$ to get $\dfrac{2}{x^2 - 6x + 8}$?

(1) $\dfrac{2}{(x+3)(x-2)}$ (2) $\dfrac{4}{(x+3)(x+2)}$ (3) $\dfrac{1}{x^2 - 5x + 6}$ (4) $\dfrac{-1}{x^2 + 5x - 6}$

19. The rational expression $\dfrac{x^3 - 3x^2 + 2x}{x^2y - 2xy}$ in lowest terms is ________.

(1) $\dfrac{x-2}{y}$ (2) $\dfrac{x+1}{xy}$ (3) $\dfrac{x-2}{y}$ (4) $\dfrac{x+1}{y}$

20. If (x – 4) is the HCF of p (x) = $x^2 - nx - 12$ and q(x) = $x^2 - mx - 8$, then the simplest form of $\frac{p(x)}{q(x)}$ is ________.

(1) $\frac{x-3}{x+2}$ (2) $\frac{x+3}{x-2}$ (3) $\frac{x+2}{x+3}$ (4) $\frac{x+3}{x+2}$

21. What should be added to $\frac{2}{(x^2+x-6)}$ to get $\frac{-4x}{(x^2-4)(x^2-9)}$?

(1) $\frac{4}{x^2-x-6}$ (2) $\frac{-2}{x^2-x-6}$ (3) $\frac{4x}{x^2+x+6}$ (4) $\frac{-3}{x^2+x+6}$

22. The HCF of the polynomials $4(x+8)^2(x^2-5x+6)$ and $6(x^2+12x+32)(x^2-7x+12)$ is ________.

(1) 2(x + 8) (x – 3) (2) (x – 3) (x + 8)
(3) (x – 8) (x + 3) (4) 2(x – 8)(x + 3)

23. The LCM of the polynomials $8(x^2-2x)(x-6)^2$ and $2x(x^2-4)(x-6)^2$ is _______.

(1) $8(x^2-4)(x^2-6x)^2$ (2) $(x^3-4x)(x-6)^2$ (3) $8(x^3-4x)(x-6)^2$ (4) $8x(x^2+4)(x-6)^2$

24. The HCF of polynomials $(x^2-2x+1)(x+4)$ and $(x^2+3x-4)(x+1)$ is _______.

(1) (x + 4)(x – 1) (2) (x + 1)(x + 4) (3) (x + 1)(x – 4) (4) $(x^2-1)(x+4)$

25. If the zeroes of the rational expression (3x + 2a) (2x + 1) are $\frac{-1}{2}$ and $\frac{b}{3}$, then the value of a is ________.

(1) – 2b (2) $\frac{-b}{2}$ (3) $\frac{-b}{3}$ (4) None of these

26. The LCM of the polynomials $(x^2-8x+16)(x^2-25)$ and $(x^2-10x+25)(x^2-2x-24)$ is ________.

(1) $(x^4-41x+400)(x-6)$ (2) $(x^4+41x+400)(x^2-9x+20)$
(3) $(x^4-41x+400)(x^2-9x+20)(x-6)$ (4) $(x^4-41x+400)(x^2-9x+20)(x+6)$

27. If the HCF of the polynomials $(x^2+8x+16)(x^2-9)$ and $(x^2+7x+12)$ is $x^2+7x+12$, then their LCM is _______.

(1) $(x+4)(x^2-9)$ (2) $(x+4)^2(x^2-9)$
(3) $(x^2-4)(x^2+9)$ (4) $(x+4)^2(x+3)$

28. If h(x) = $x^2 + x$ and g(y) = $y^3 - y$, then the HCF of h(b) – h(a) and g(b) – g(a) is _______.

(1) a + b (2) b – a (3) $b^2 + a^2$ (4) $b^2 + ab + a^2$

29. If h(y) = y^3 and g(z) = z^4, then HCF of h(b) – h(a) and g(b) – g(a) is _________.

(1) b – a (2) $b^2 - a^2$ (3) $b^3 - a^3$ (4) $b^2 + ab + a^2$

30. If the HCF of the polynomials $(x+4)(2x^2+5x+a)$ and $(x+3)(x^2+7x+b)$ is $(x^2+7x+12)$ then 6a + b is _______.

(1) –6 (2) 5 (3) 6 (4) – 5

Concept Application Level—2

31. The HCF and LCM of the polynomials p(x) and q(x) are $5(x-2)(x+9)$ and $10(x^2+16x+63)(x-2)^2$. If p(x) is $10(x+9)(x^2+5x-14)$, then q(x) is ________.

(1) $5(x+9)(x-2)$ (2) $10(x-2)^2(x+7)$
(3) $10(x+9)(x-2)$ (4) $5(x-2)^2(x+9)$

32. If the zeroes of the rational expression $(ax+b)(3x+2)$ are $\frac{-2}{3}$ and $\frac{1}{2}$, then a + b = ______.

(1) -1 (2) 0 (3) $-b$ (4) None of these

33. Simplify: $\frac{x^2-(y-2z)^2}{x-y+2z}+\frac{y^2-(2x-z)^2}{y+2x-z}+\frac{z^2-(x-2y)^2}{z-x+2y}$.

(1) 0 (2) 1 (3) $x+y+z$ (4) None of these

34. If the HCF of the polynomials x^2+px+q and x^2+ax+b is $x+\ell$, then their LCM is ________.

(1) $(x+a-\ell)(x+\ell-p)$ (2) $(x-(\ell+a))(x+\ell-p)(x+\ell)$
(3) $(x+a-\ell)(x+p-\ell)(x+\ell)$ (4) $(x-\ell+a)(x-p+\ell)(x+\ell)$

35. The expression $\frac{1}{1-x}-\frac{1}{1+x}-\frac{x^3}{1-x}+\frac{x^2}{1+x}$ in lowest terms is _______.

(1) $2x^3+1$ (2) x^2+2 (3) x^2+2x (4) x^2-2x

36. Simplify: $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

(1) 0 (2) 1 (3) $a+b+c$ (4) $\frac{1}{a+b+c}$

37. Simplify: $\frac{x+1}{x-1}+\frac{x-1}{x+1}-\frac{2x^2-2}{x^2+1}$

(1) $\frac{4x^4+2}{x^4-1}$ (2) $\frac{4x^2}{x^4-1}$ (3) $\frac{8x^2}{x^4-1}$ (4) 1

38. If the LCM of the polynomials $x^9+x^6+x^3+1$ and x^6-1 is $x^{12}-1$, then their HCF is ________.

(1) x^3+1 (2) x^6+1 (3) x^3-1 (4) x^6-1

39. If x^2+x-1 is a factor of $x^4+px^3+qx^2-1$, then the values of p and q can be

(1) 2, 1 (2) 1, -2 (3) -1, -2 (4) -2, -1

40. The HCF of two polynomials p(x) and q(x) using long division method was found to be x + 5, If their first three quotients obtained are x, 2x + 5, and x + 3 respectively. Find p(x) and q(x). (The degree of p(x) > the degree of q(x))

(1) $p(x)=2x^4+21x^3+72x^2+88x+15$
$q(x)=2x^3+21x^2+71x+80$

(2) $p(x)=2x^4-21x^3-72x^2-88x+15$
$q(x)=2x^3+21x^2-71x+80$

(3) $p(x)=2x^4+21x^3+88x+15$
$q(x)=2x^3+71x+80$

(4) $p(x)=2x^4-21x^2-72x^2+80x+15$
$q(x)=2x^3-21x^2+71x+80$

41. If the HCF of the polynomials $x^3 + px + q$ and $x^3 + rx^2 + lx + x$ is $x^2 + ax + b$, then their LCM is _______. $(r \neq 0)$

(1) $(x^2 + ax + b)(x + a)(x + a - r)$
(2) $(x^2 + ax + b)(x - a)(x - a + r)$
(3) $(x^2 + ax + b)(x - a)(x - a - r)$
(4) $(x^2 - ax + b)(x - a)(x - a + r)$

42. If the HCF of the polynomials $(x - 3)(3x^2 + 10x + b)$ and $(3x - 2)(x^2 - 2x + a)$ is $(x - 3)(3x - 2)$, then the relation between a and b is ______.

(1) $3a + 8b = 0$
(2) $8a - 3b = 0$
(3) $8a + 3b = 0$
(4) $a - 2b = 0$

43. The HCF of the polynomials $12(x + 2)^3 (x^2 - 7x + 10)$ and $18(x^2 - 4)(x^2 - 6x + 5)$ is ______.

(1) $(x^2 + 3x + 10)(x - 2)$
(2) $6(x^2 + 3x + 10)(x + 2)$
(3) $(x^2 - 3x - 10)(x - 2)$
(4) $6(x^2 - 3x - 10)(x - 2)$

44. The HCF of the polynomials $(x^2 - 4x + 4)(x + 3)$ and $(x^2 + 2x - 3)(x - 2)$ is

(1) $x + 3$
(2) $x - 2$
(3) $(x + 3)(x - 2)$
(4) $(x + 3)(x - 2)^2$

45. The HCF of the polynomials $5(x^2 - 16)(x + 8)$ and $10(x^2 - 64)(x + 4)$ is _______.

(1) $x^2 + 12x + 32$
(2) $5(x^2 + 12x + 32)$
(3) $x^2 - 12x + 32$
(4) $5(x^2 - 12x + 32)$

Concept Application Level—3

46. The HCF of two polynomials p(x) and q(x) using long division method was found in two steps to be $3x - 2$, and the first two quotients obtained are $x + 2$ and $2x + 1$. Find p(x) and q(x). (The degree of p(x) > the degree of q(x)).

(1) $p(x) = 6x^3 + 11x^2 + x + 6$, $q(x) = 6x^2 + x + 2$
(2) $p(x) = 6x^3 + 11x^2 - x + 6$, $q(x) = 6x^2 - x + 2$
(3) $p(x) = 6x^3 - 11x^2 + x - 6$, $q(x) = 6x^2 - x - 2$
(4) $p(x) = 6x^3 + 11x^2 - x - 6$, $q(x) = 6x^2 - x - 2$

47. Simplify: $\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2} - \dfrac{3x^2-3}{x^2+4}$

(1) $\dfrac{-x^4 + 31x^2 + 20}{x^4 - 16}$
(2) $\dfrac{x^4 + 31x^2 + 20}{x^4 - 16}$
(3) $-\dfrac{x^4 + 31x^2 - 20}{x^4 - 16}$
(4) $-\dfrac{x^4 - 21x^2 + 20}{x^4 - 16}$

48. If $P = \dfrac{x+1}{x-1}$ and $Q = \dfrac{x-1}{x+1}$, then $P^2 + Q^2 - 2PQ =$ ______.

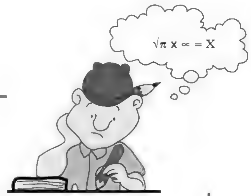
(1) $\dfrac{16x^2}{x^4 - 2x^2 + 1}$
(2) $\dfrac{4x^4 + 8x^2 + 4}{x^4 - 2x + 1}$
(3) $\dfrac{4x^2}{x^4 + 2x^2 + 1}$
(4) $\dfrac{8x^2}{x^4 - 2x^2 + 1}$

49. Simplify: $\dfrac{81x^4 - 16x^2 + 32x - 16}{9x^2 - 4x + 4}$

(1) $9x^2 + 4x - 4$
(2) $9x^2 - 4x - 4$
(3) $9x^2 - 2x - 8$
(4) $9x^2 + 2x - 8$

50. The rational expression $A = \left(\frac{x+1}{x-1} - \frac{x-1}{x+1} - \frac{4x}{x^2+1}\right)$ is multiplied with the additive inverse of $B = \frac{1-x^4}{4x}$ to get C. Then, C = ______.

(1) $\frac{32x^2}{x^4-1}$ (2) $\frac{2x}{x^4-1}$ (3) 2 (4) 1



KEY

Very short answer type questions

1. $72x^3y^3$

2. 8 ab

3. $3a - 4b$

4. $6(x-2)^2(x-3)^2$

5. Any whole number

6. $a + b$

7. 4

8. 12, 2

9. $(x-1)(x-2)(x+4)$

10. False

11. $\frac{x^2+1}{1-x^2}$

12. $\frac{1}{x(x-1)}$

13. False

14. $\frac{x+1}{x-1}$

15. False

16. $a^2 \neq b$

17. True

18. False

19. 4abc

20. $24(x+2)^2(x-2)(x^2-2x+4)$

21. 2xyz

22. $120a^2b^2(a^2-b^2)$

23. 5

24. $(x+3)(x+1)^2$

25. $(x^2+1)(x^2-1)$

26. $\frac{1}{1+x}$

27. a, d

28. $\frac{1}{1-x}$

29. True

30. $\frac{x+3}{\left(x-\frac{1}{2}\right)(x+1)}$

Short answer type questions

31. (i) HCF $= x^2y^6$
LCM $= x^3y^8$

(ii) HCF $= 3a^2b^3c^2$, LCM $= 9a^4b^3c^4$

(iii) HCF $= p^4q^2r^3$ and LCM $= p^6q^3r^5$

33. LCM $= (x-1)(x-2)^2(x-4)(x-6)^3$

34. $k = 5$

35. $a = 5, b = 12$

36. $(x-1)^2 (x-2) (x-3)^3$

38. $(x+1)(x^2-9)$

39. HCF = $(x+2)$
LCM = $(x^2-9)(x^2-4)(x^2-1)(x+2)$

40. $\dfrac{2x^2-x+3}{(x-1)^2(x+1)}$

41. $P - Q = \dfrac{12x}{(3x-1)(3x+1)}$

$P + Q = \dfrac{2(9x^2+1)}{(9x^2-1)}$

42. (i) x^2+4x-9

(ii) $\dfrac{2x(x-1)}{3}$

43. $\dfrac{(a^2+3)(3a^2+1)}{(1-a^2)^2}$

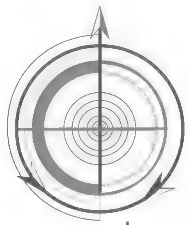
44. $\dfrac{2ab}{b^2-a^2}$ **45.** $\dfrac{4x}{1-x^4}$

Essay type questions

46. 8

47. 0 **48.** $\dfrac{2x+3}{4x+1}$

49. $\dfrac{3}{x}$ **50.** $\dfrac{(2x-y+z)}{a-2b+3c}$



key points for selected questions

Very short answer type questions

19. The product of the common factors with least exponents is HCF.

20. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

21. The product of the common factors with least exponents is HCF.

22. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

23. The product of the common factors with least exponents is HCF.

24. The product of the common factors with least exponents is HCF.

25. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

26. Multiplicative inverse of P(x) is $\dfrac{1}{P(x)}$

27. Recall the definition of polynomials.

28. Additive inverse of P(x) is –P(x).

Short answer type questions

31. (i) HCF is the product of all the common factors with least exponents.
(ii) LCM is the product of all the factors with highest exponents.

32. (i) Factorize the given expressions.
(ii) HCF is the product of all the common factors with least exponents.
(iii) LCM is the product of all the factors with highest exponents.

33. LCM = $\dfrac{\text{Product of two polynomials}}{\text{HCF}}$

34. (i) Factorize each of the given two polynomials.

(ii) By observation $(x^2 + kx + 6)$ must have $x + 2$ as its factor.

35. (i) Remainder theorem says that if $(x - a)$ is a factor of f(x), then f(a) = 0.

(ii) As $(x + 5)$ is a factor of $(x^2 + 6x + a)$ we can find a by applying remainder theorem.

(iii) As $(x + 2)$ is a factor of $(x^2 + 8x + b)$, we can find b by applying remainder theorem.

36. Use, Product of two polynomials = ± (LCM) (HCF) and proceed.

37. (i) Factorize the given expressions.

(ii) HCF is the product of all the common factors with least exponents.

(iii) LCM is the product of all the factors with highest exponents.

38. Product of the two polynomials = ± (LCM) (HCF)

39. (i) HCF is the product of all the common factors with least exponents.

(ii) LCM is the product of all the factors with highest exponents.

40. $\dfrac{x+1}{(x-1)^2} + \dfrac{x-2}{x^2-1} = \dfrac{x+1}{(x-1)^2} + \dfrac{x-2}{(x+1)(x-1)}$

LCM of $(x - 1)^2$ and $(x + 1)(x - 1)$ is $(x - 1)^2 (x + 1)$

41. LCM of $(3x - 1)$ and $(3x + 1)$ is $(3x - 1)(3x + 1)$

42. (i), (ii):

(a) Factorize the expression if possible.

(b) Then cancel the common factors if any.

(c) $16x^2 - (x^2 - 9)$
$(x^2 + 4x - 9)(4x + 9 - x^2)$

(d) $x^5 - x = x(x^2 + 1)(x + 1)(x - 1)$
$(3x^2 + 3)(4x + 4) = 12(x^2 + 1)(x + 1)$

43. Use, $X^2 + Y^2 + XY = (X + Y)^2 - XY$

44. The required expression is

$$1 - \frac{a}{a-b} - \frac{b}{a+b}$$

45. (i) LCM of the denominators is $(x^2 + 1)(x + 1)(x - 1)$ i.e., $(x^4 - 1)$

Essay type questions

46. (i) LCM of $(x - 2)$, $(x + 2)$ and $(x^2 + 4)$ is $x^4 - 16$.

(ii) Simplify the first expression and proceed.

47. (i) Write $x^4 + x^2 + 1$ as $(x^2 + x + 1)(x^2 - x + 1)$

(ii) Take LCM and simplify

48. (i) Factorize the numerator and denominator of each rational expressional.

(ii) Cancel the common terms.

49. (i) Write $\dfrac{px}{x^3 + px^2} = \dfrac{p}{x(x+p)}$,

$\dfrac{qx}{x^3 + qx^2} = \dfrac{q}{x(x+q)}$ and

$\dfrac{rx}{x^3 + rx^2} = \dfrac{r}{x(x+r)}$

(ii) Regroup the terms with same denominator then take LCM and simplify.

50. (i) Factorize the first rational expression by using, $a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

(ii) Cancel the common terms and simplify.

KEY

Concept Application Level—1,2,3

1. 4	**26.** 3
2. 2	**27.** 2
3. 2	**28.** 2
4. 1	**29.** 1
5. 3	**30.** 1
6. 4	**31.** 4
7. 3	**32.** 3
8. 3	**33.** 1
9. 3	**34.** 3
10. 3	**35.** 3
11. 3	**36.** 2
12. 1	**37.** 3
13. 2	**38.** 1
14. 2	**39.** 1
15. 3	**40.** 1
16. 1	**41.** 2
17. 2	**42.** 2
18. 3	**43.** 4
19. 3	**44.** 3
20. 4	**45.** 2
21. 2	**46.** 4
22. 1	**47.** 1
23. 3	**48.** 1
24. 1	**49.** 1
25. 2	**50.** 3

Concept Application Level—1,2,3

Key points for select questions

1. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

2. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

3. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

4. We gave, P(x) Q(x) = LCM × HCF.

5. We have, P(x) Q(x) = LCM × HCF.

6. If $(x - k)$ is a factor if $f(x)$, then $f(k) = 0$

7. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

10. If $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$.

11. Factorize the given polynomials.

12. If $(x - k)$ is a factor of $f(x)$, then $f(k) = 0$.

14. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

15. Recall definition of polynomials.

16. Sum or difference of two rational expressions is always a rational expression.

18. Subtract the first expression from the second expression.

19. Factorise numerator and denominator and eliminate the common factors.

21. Factorize the given polynomials. The product of all the factors with highest exponents is LCM.

22. (i) Factorize the given polynomials.
(ii) $x^2 - 5x + 6 = (x - 2)(x - 3)$
$x^2 + 12x + 32 = (x + 8)(x + 4)$
$x^2 - 7x + 12 = (x - 4)(x - 3)$

23. (i) Factorize the given polynomials.
(ii) $x^2 - 2x = x(x - 2)$.
$x^2 - 4 = (x + 2)(x - 2)$.

24. (i) Factorize the given polynomials.
(ii) $x^2 - 2x + 1 = (x - 1)^2$
$x^2 + 3x - 4 = (x + 4)(x - 1)$

25. (i) Equate the given expression to zero and compare with the given values.
(ii) As zero of $2x + 1$ is $-\frac{1}{2}$, zero of the expression $3x + 2a$ is $\frac{b}{3}$.

26. (i) Factorize the given polynomials.
(ii) $(x^2 - 8x + 16)(x^2 - 25) = (x - 4)^2 (x + 5)(x - 5)$.

$(x^2 - 10x + 25)(x^2 - 2x - 24)$
$= (x - 5)^2 (x - 6)(x + 4)$.

27. (i) LCM × HCF = $\pm f(x) \times g(x)$
(ii) $x^2 + 8x + 16 = (x + 4)^2$

28. (i) $h(b) - h(a) = (b^2 + b) - (a^2 + a)$.
(ii) Now factorize the above expression.
(iii) Similarly factorize $g(b) - g(a)$.

29. (i) Divide the given polynomials with $(x + \ell)$.
(ii) $h(b) - h(a) = b^3 - a^3$, $g(b) - g(a) = b^4 - a^4$.
(iii) Now, factorize the above expressions.

30. (i) Find the values of a and b using the concept of HCF then obtain the required relation.
(ii) $x^2 + 7x + 12 = (x + 4)(x + 3)$
(iii) $x + 3$ is a factor of $2x^2 + 5x + a$ and $x + 4$ is a factor of $x^2 + 7x + b$.

31. (i) $p(x)\, q(x)$ = LCM × HCF.
(ii) LCM × HCF = $\pm f(x) \times g(x)$
(iii) $x^2 + 16x + 63 = (x + 7)(x + 9)$
$x^2 + 5x - 14 = (x + 7)(x - 2)$.

32. (i) Equate the given expression to zero to get the values of a and b then find $a + b$.
(ii) As the zero of $3x + 2$ is $\frac{-2}{3}$, zero of $ax + b$ is $\frac{1}{2}$.

33. (i) Factorize and then simplify.
(ii) $x^2 - (y - 2z)^2 = (x + y - 2z)(x - y + 2z)$ and so on.

34. (i) If $(x - k)$ is a of factor $f(x)$, then $f(k) = 0$
(ii) Find the quotients by dividing each of the expressions by $x + \ell$.

35. (i) Add the terms which have the same denominators.
(ii) LCM of denominators is $(1 - x)(1 + x)$.
(iii) $a^2 - 1 = (a + 1)(a - 1)$; $a^3 - 1 = (a - 1)(a^2 + a + 1)$

36. (i) Factorize numerator and denominator then simplify.
(ii) $a^2 - (b - c)^2 = (a + b - c)(a - b + c)$; $(a + c)^2 - b^2 = (a + c + b)(a + c - b)$

37. (i) Simplify the first two terms.
(ii) LCM of denominators is $(x^2 - 1)(x^2 + 1)$.

38. (i) We have, $P(x)\, Q(x)$ = LCM × HCF.
(i) $P^2 + Q^2 - 2PQ = (P - Q)^2$.
(ii) Find $P - Q$ and then find $(P - Q)^2$.

40. (i) Use division rule.
(ii) Apply the concept of finding HCF by long division method.

41. Find the quotients by dividing each of the given expressions by $x^2 + ax + b$.

42. (i) Find the values of a, b using the concept of HCF then obtain the relation between a and b.
(ii) $3x - 2$ is a factor of $3x^2 + 10x + b$.
(iii) $x - 3$ is a factor of $x^2 - 2x + a$.
(iv) Apply remainder theorem to find the values of a and b.

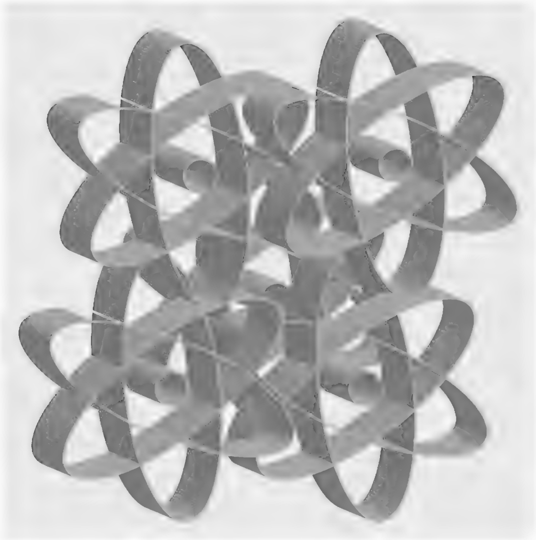
43. (i) Factorize the given polynomials.
(ii) $x^2 - 7x + 10 = (x - 5)(x - 2)$
$(x^2 - 6x + 5) = (x - 5)(x - 1)$

45. (i) Factorize the given polynomials.
(ii) $x^2 - 16 = (x + 4)(x - 4)$
$x^2 - 64 = (x + 8)(x - 8)$

46. (i) Use division rule.
(ii) Apply the concept of finding HCF by division method.

48. (i) $P^2 + Q^2 - 2PQ = (P - Q)^2$.
(ii) Find $P - Q$ and then find $(P - Q)^2$.

50. (i) Simplify the expression A
(ii) Find additive inverses of the expression B and then multiply with A.



CHAPTER 3

Linear Equations in Two Variables

INTRODUCTION

While solving the problems, in most cases, first we need to frame an equation. In this chapter, we learn how to frame and solve equations. Framing an equation is more difficult than solving an equation. Now, let us review the basic concepts related to this chapter.

Algebraic expressions

Expressions of the form $2x$, $(3x + 5)$, $(4x - 2y)$, $2x^2 + 3\sqrt{y}$, $3x^3/2\sqrt{y}$ are algebraic expressions. $3x$ and 5 are the terms of $(3x + 5)$, and $4x$ and $2y$ are the terms of $4x - 2y$. Algebraic expressions are made of numbers, symbols and the basic arithmetical operations. In the term $2x$, 2 is the numerical coefficient of x and x is the variable coefficient of 2.

Equation

An equation is a sentence in which there is an equality sign between two algebraic expressions.

For example, $2x + 5 = x + 3$, $3y - 4 = 20$ and $5x + 6 = x + 1$ are some examples of equations. Here x and y are unknown quantities and 5, 3, 20, etc are known quantities.

Linear equation

An equation, in which the highest index of the unknowns present is one, is a linear equation.

$2(x + 5) = 18$, $3x - 2 = 5$

$x + y = 20$ and $3x - 2y = 5$ are some linear equations.

Simple equation

A linear equation which has only one unknown is a simple equation.

$3x + 4 = 16$ and $2x - 5 = x + 3$ are examples of simple linear equations.

The part of an equation which is to the left side of the equality sign is known as the left hand side, abbreviated as L.H.S. The part of an equation which is to the right side of the equality sign is known as the right hand side, abbreviated as R.H.S. The process of finding the value of an unknown in an equation is called solving the equation. The value/values of the unknown found after solving an equation is/are called the solutions or the roots of the equation.

Before we learn how to solve an equation, let us review the basic properties of equality.

1. When a term is added to both sides of the equality, the equality does not change.

☛ *Example*

If $a + b = c + d$ then $a + b + x = c + d + x$.
This property holds good for difference also.

2. When the expressions on the L.H.S. and R.H.S. of the equation are multiplied by a non-zero term, the equation does not change.

☛ *Example*

If $a + b = c + d$, then $x(a + b) = x(c + d)$

This property holds good for division also.

Solving an equation in one variable

The following steps are involved in solving an equation.

Step 1: Always ensure that the unknown quantities are on the L.H.S. and the known quantities or constants on the R.H.S.
Step 2: Add all the terms containing the unknowns on the L.H.S. and all the knowns on the R.H.S. so that each side of the equation contains only one term.
Step 3: Divide both sides of the equation by the coefficient of the unknown.

☛ *Example*

If $4x + 15 = 35$, find the value of x.

Solution

Step 1: Group the known quantities as the R.H.S. of the equation i.e., $4x = 35 - 15$

Step 2: Simplify the numbers on the R.H.S. $\Rightarrow 4x = 20$

Step 3: Since 4 is the coefficient of x, divide both the sides of the equation by 4.

$$\frac{4x}{4} = \frac{20}{4} \Rightarrow x = 5$$

☛ *Example*

Solve: $3x + 11 = 6x - 13$

Solution

Step 1: $6x - 3x = 11 + 13$

Step 2: $3x = 24$

Step 3: $\frac{3x}{3} = \frac{24}{3}$

$\Rightarrow x = 8$

Example

A swarm of 62 bees flies in a garden. If 3 bees land on each flower, 8 bees are left with no flowers. Find the number of flowers in the garden.

Solution

Let the number of flowers be x.

Number of bees on the flowers = 3x

Total number of bees = 62

$\therefore$ 3x + 8 = 62

$\Rightarrow$ 3x = 62 – 8 = 54 $\Rightarrow$ x = 18

$\therefore$ There are 18 flowers in the garden.

Transposition

In the above problem, 3x + 8 = 62 can be written as 3x = 62 – 8. When a term is moved (transposed) from one side of the equation to the other side, the sign is changed. The positive sign is changed to the negative sign and multiplication is changed to division. Moving a term from one side of the equation to the other side is called transposition. Thus solving a linear equation, in general, comprises two kinds of transposition.

Simultaneous linear equations

We have learnt to solve an equation with one unknown. Very often we come across equations involving more than one unknown. In such cases we require more than one condition or equation. Generally, when there are two unknowns, we require two equations to solve the problem. When there are three unknowns, we require three equations and so on.

We need to find the values of the unknowns that satisfy all the given equations. Since the values satisfy all the given equations we call them simultaneous equations. In this chapter, we deal with simultaneous (linear) equations in two unknowns.

Let us consider the equation, 2x + 5y = 19, which contains two unknown quantities x and y.

Here, 5y = 19 – 2x

$\Rightarrow y = \frac{19-2x}{5}$ ------------- (1)

In the above equation for every value of x, there exists a corresponding value for y.

When x = 1, $y = \frac{17}{5}$

When x = 2, y = 3 and so on.

If there is another equation, of the same kind, say, 5x – 2y = 4

From this, we get, $y = \frac{5x-4}{2}$ ------------- (2)

If we need the values of x and y such that both the equations are satisfied, then $\frac{19-2x}{5} = \frac{5x-4}{2}$

$\Rightarrow$ 38 – 4x = 25x – 20

$\Rightarrow$ 29x = 58

$\Rightarrow$ x = 2

On substituting the value of x = 2 in the first equation, we get,

$$y = \frac{19-2(2)}{5} = \frac{15}{5} = 3$$

$\Rightarrow y = 3$

Both the equations are satisfied by the same values of x and y. Thus we can say that when two or more equations are satisfied by the same values of unknown quantities then those equations are called simultaneous equations.

When two equations, each in two variables, are given, they can be solved in five ways.

(a) Elimination by cancellation.
(b) Elimination by substitution.
(c) Adding the two equations once and subtracting one equation from the other.
(d) Cross-multiplication method.
(e) Graphical method.

(a) Elimination by cancellation

☛ *Example*

If $2x + 3y = 19$ and $5x + 4y = 37$, then find the values of x and y.

Solution

In this method, the two equations are reduced to a single variable equation by eliminating one of the variables.

Step 1: Here, let us eliminate the y term, and in order to eliminate the y term, we have to multiply the first equation with the coefficient of y in the second equation and the second equation with the coefficient of y in the first equation so that the coefficients of y terms in both the equations become equal.

$(2x + 3y = 19)4 \Rightarrow 8x + 12y = 76$ ----------- (3)

$(5x + 4y = 37)3 \Rightarrow 15x + 12y = 111$ ----------- (4)

Step 2: Subtract equation (3) from (4),

$(15x + 12y) - (8x + 12y) = 111 - 76$

$\Rightarrow 7x = 35$

$\Rightarrow x = 5$

Step 3: Substitute the value of x in equation (1) or (2) to find the value of y. Substituting the value of x in the first equation, we get,

$2(5) + 3y = 19$

$\Rightarrow 3y = 19 - 10 \Rightarrow 3y = 9$

$\Rightarrow y = 3$

$\therefore$ The solution of the given pair of equation is $x = 5$; $y = 3$.

(b) Elimination by substitution

Example

If $4x - 3y = 32$ and $x + y = 1$, then find the values of x and y.

Solution

In this method, the two equations are reduced to a single variable equation by substituting the value of one variable, obtained from one equation, in the other equation.

Step 1: Using the second equation, find x in terms of y.
i.e., $x + y = 1$
$\Rightarrow y = 1 - x$

Step 2: Substitute the value of y in the first equation to find the value of x.
$\therefore\ 4x - 3(1 - x) = 32$

Step 3: Simplify the equation in terms of x and find the value of x.
$4x - 3 + 3x = 32$
$\Rightarrow 7x = 32 + 3 = 35$
$\Rightarrow x = 5$

Step 4: Substituting the value of x in equation (1) or (2), we have,
$y = 1 - x = 1 - 5$
$\Rightarrow y = -4$

$\therefore$ The solution for the given pair of equations is $x = 5$; $y = -4$.

(c) Adding two equations and subtracting one equation from the other

Example

Solve $3x + 7y = 32$ and $7x + 3y = 48$

Solution

Given,
$3x + 7y = 32$ ------------ (1)
$7x + 3y = 48$ ------------ (2)

Step 1: Adding both the equations, we get
$10x + 10y = 80$
$\Rightarrow 10(x + y) = 10 \times 8$
$\Rightarrow x + y = 8$ ------------ (3)

Step 2: Subtracting equation (1) from equation (2),
$(7x + 3y) - (3x + 7y) = 48 - 32$
$\Rightarrow 4x - 4y = 16$
$\Rightarrow 4(x - y) = 4 \times 4$
$\Rightarrow x - y = 4$ ------------ (4)

Step 3: Adding the equations (3) and (4),
$x + y + x - y = 12$

$\Rightarrow 2x = 12$

$\Rightarrow x = 6$

Substituting $x = 6$ in any of the equations (1), (2), (3) or (4), we get, $y = 2$

$\therefore$ The solution of the pair of equations is $x = 6; y = 2$.

(d) Cross-multiplication method

Example

Solve $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Solution

Given,

$a_1x + b_1y + c_1 = 0$ ------------- (1)

$a_2x + b_2y + c_2 = 0$ ------------- (2)

Solving the above equations using elimination by cancellation method, we get,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Applying alternendo on the above ratios, we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \text{ and } \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

i.e., $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

The above result can be better remembered using the following diagram.

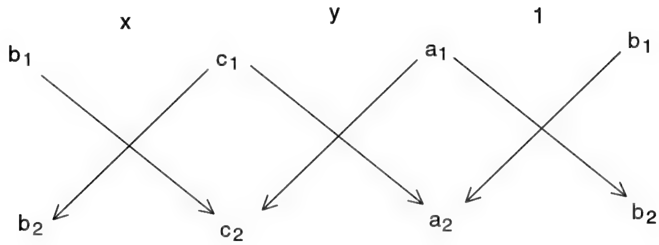


Figure 3.1

The arrows between the two numbers indicate that they are to be multiplied and second product is to be subtracted from the first.

While using this method, the following steps are to be followed.

Step 1: Write the given equations in the form $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

Step 2: Write the coefficients of the equations as mentioned above.

Step 3: Find the values of x and y.

☛ *Example*

Solve $4x + 5y = 71$ and $5x + 3y = 66$.

Solution

Step 1: Rewriting the equations, we get,

$4x + 5y - 71 = 0$ ---------- (1)

$5x + 3y - 66 = 0$ ----------- (2)

Step 2: Write the coefficients of x and y in the specified manner.

x	y	1	
5	−71	4	5
3	−66	5	3

Step 3: Find the values of x and y.

$$\frac{x}{-330+213} = \frac{y}{-355+264} = \frac{1}{12-25}$$

$$\Rightarrow \frac{x}{-117} = \frac{y}{-91} = \frac{1}{-13}$$

$$\Rightarrow x = \frac{-117}{-13};\ y = \frac{-91}{-13}$$

$$\Rightarrow x = 9 \text{ and } y = 7.$$

NOTE: Choosing a particular method to solve a pair of equations makes the simplification easier. One can learn as to which method is the easiest to solve a pair of equations by becoming familiar with the different methods of solving the equation.

(e) Graphical method

Plotting the points

If we consider any point in a plane, then we can determine the location of the given point i.e., we can determine the distance of the given point from x-axis and y-axis. Therefore, each point in the plane represents the distance from both the axes. So, each point is represented by an ordered pair and it consists of x-coordinate and y-coordinate. The first element of an ordered pair is called x-coordinate and the second element of an ordered pair is called y-coordinate. In the first quadrant Q_1, both the x-coordinate and y-coordinate are positive real numbers. In the second quadrant Q_2, x-coordinates are negative real numbers and y-coordinates are positive real numbers. In the third quadrant Q_3 both the x-coordinate and y-coordinate are negative real numbers. In the fourth quadrant Q_4, x-coordinates are positive real numbers and y-coordinates are negative real numbers. And the origin is represented by (0, 0).

Consider the point (2, 3). Here 2 is the x-coordinate and 3 is the y-coordinate. The point (2, 3) is 2 units away from the y-axis and 3 units away from the x-axis. The point (2, 3) belongs to the first quadrant. If we consider the point (−3, −5), −3 is x-coordinate and −5 is y-coordinate. The point (−3, −5) belongs to Q_3 and is 3 units away from the y-axis and 5 units away from the x-axis.

To plot a point say P(–3, 4), we start from the origin and proceed 3 units towards the left hand side along the X-axis (i.e., negative direction as x-coordinate is negative), and from there we move 4 units upwards along the y-axis (i.e., positive direction as y-coordinate is positive). The method of plotting a point in a coordinate plane was explained by a Rene DesCartes, a French mathematician.

Example

Plot the following points on the coordinate plane.

P(2, 3), Q(–4, –5), R(2, –3), S(–4, 4), T(3, 0), U(0, 5).

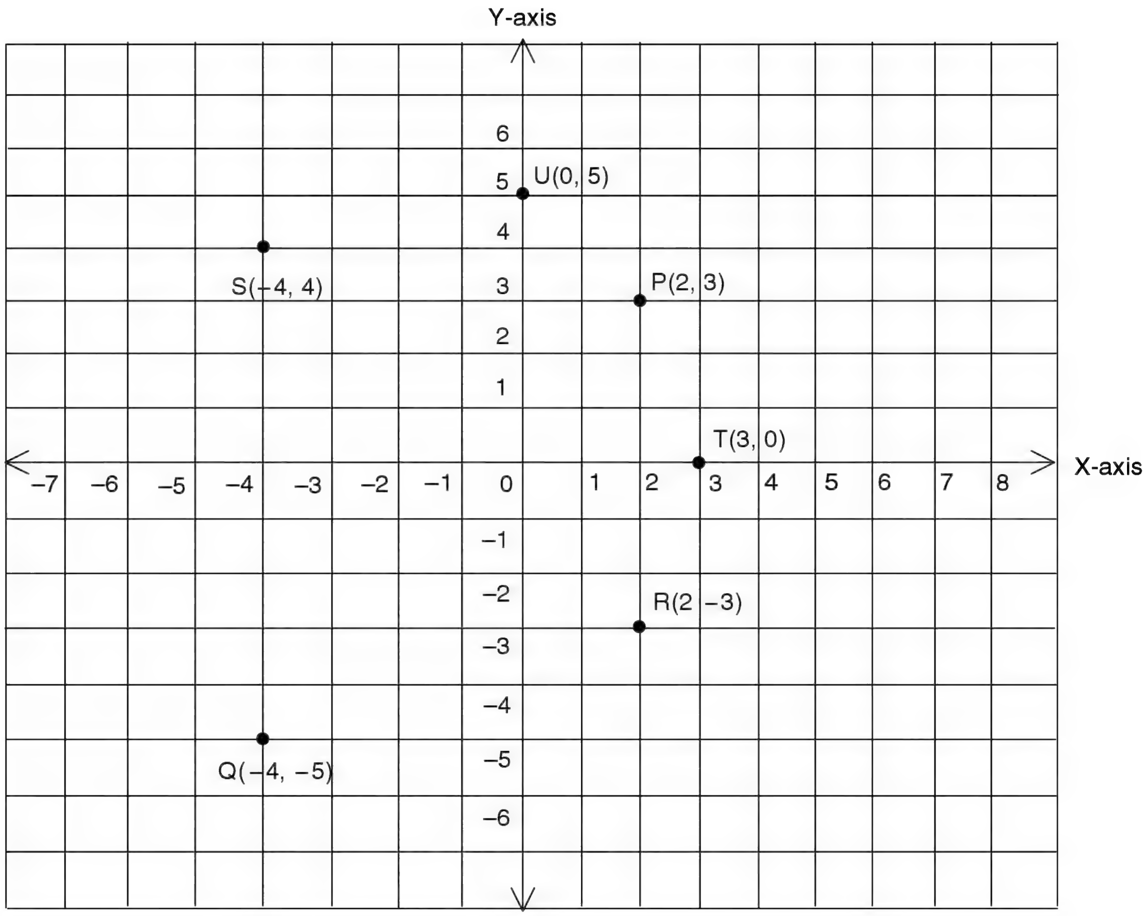


Figure 3.2

Example

Plot the following points on the coordinate plane. What do you observe?

(i) (–2, 3), (–1, 3), (0, 3), (1, 3), (2, 3)

(ii) (4, 2), (4, 1), (4, 0), (4, –1), (4, –2)

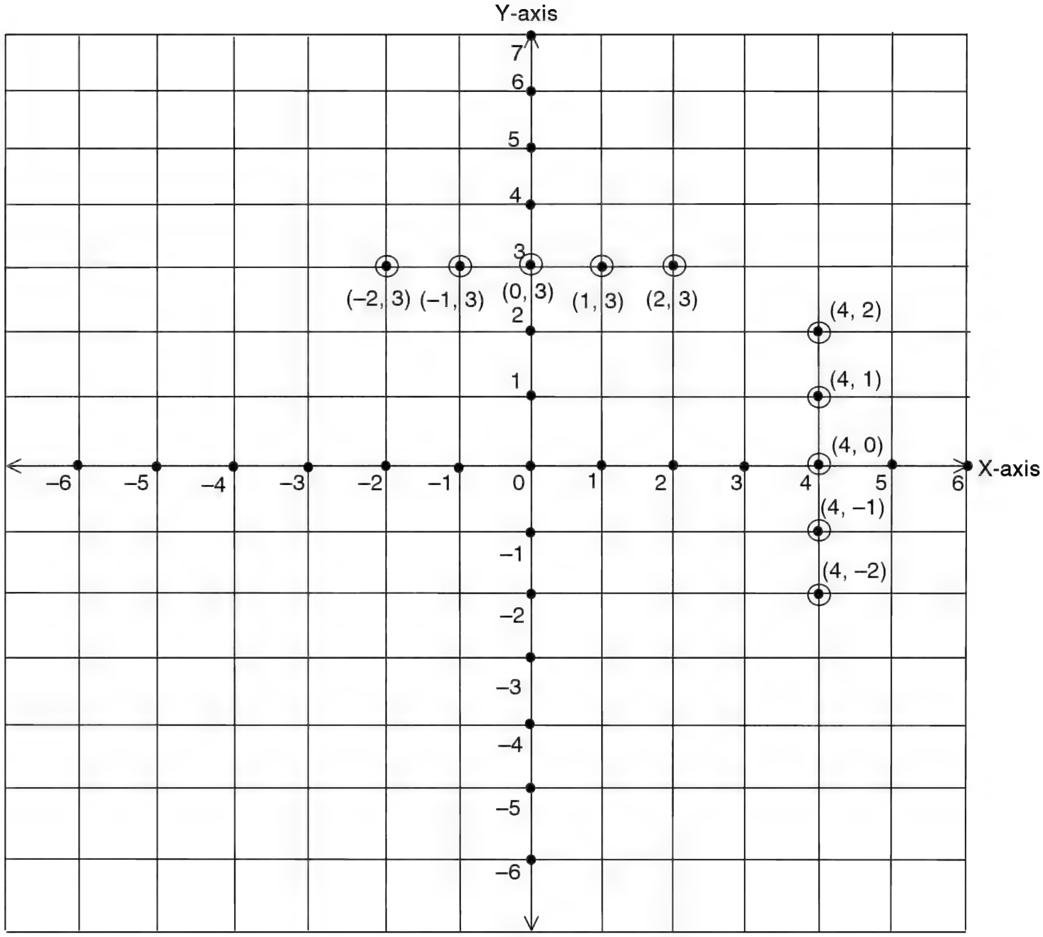


Figure 3.3

Solution

(i) (−2, 3), (−1, 3), (0, 3), (1, 3), (2, 3)

1. The above points lie on the same straight line which is perpendicular to the Y-axis.
2. The y-coordinates of all the given points are the same i.e., y = 3.
3. Hence, the straight line passing through the given points is represented by y = 3.
4. Therefore, the line y = 3 is parallel to X-axis which intersects Y-axis at (0, 3).

(ii) (4, 2), (4, 1), (4, 0), (4, −1), (4, −2)

1. The above points lie on the same straight line which is perpendicular to the X-axis.
2. The x-coordinates of all the given points is the same i.e., x = 4.
3. Hence, the straight line passing through the given points is represented by x = 4.
4. Therefore, the line x = 4 is parallel to the Y-axis which intersects the X-axis at (4, 0).

NOTE:

1. The y-coordinate of every point on the x-axis is zero i.e., y = 0. Therefore the x-axis is denoted by y = 0.

2. The x-coordinate of every point on the y-axis is zero i.e., $x = 0$. Therefore the y-axis is denoted by $x = 0$.

Example

Plot the following points on the coordinate plane and what do you observe?

(–3, 3), (–2, 2), (–1, 1), (0, 0), (1, –1), (2, –2), (3, –3)

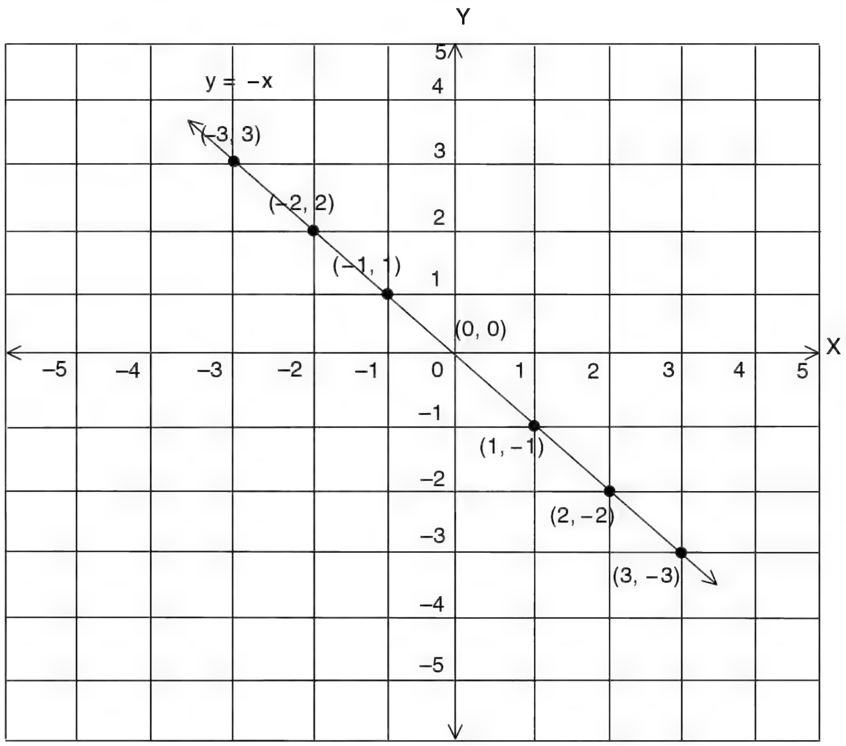


Figure 3.4

1. All the given points lie on the same straight line.
2. Every point on the straight line represents $y = -x$.
3. The above line with the given ordered pairs is represented by the equation $y = -x$.

Example

Draw the graph of the equation $y = 3x$ where R is the replacement set for both x and y.

Solution

x	–2	–1	0	1	2
y = 3x	–6	–3	0	3	6

Some of the ordered pairs which satisfy the equation $y = 3x$ are (–1, –3), (–2, –6), (0, 0), (1, 3), (2, 6).

By plotting the above points on the graph sheet, we get the following.

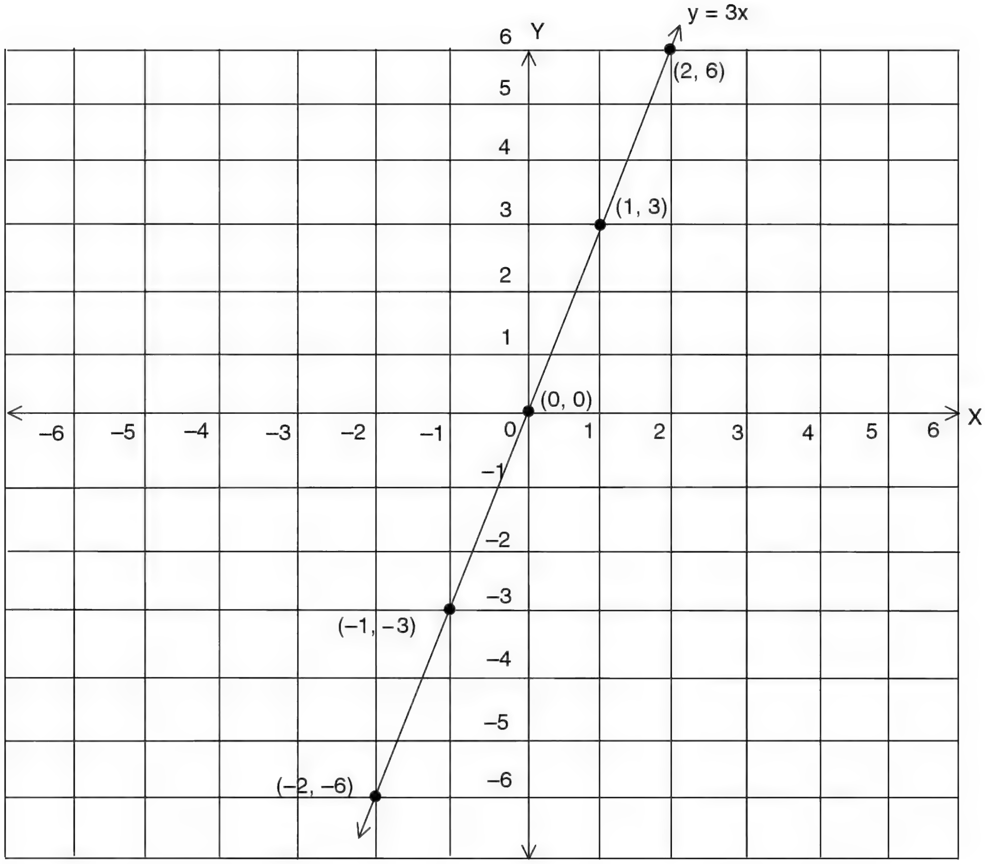


Figure 3.5

Example

Draw the graph of the equations x + y = –1 and x – y = 5.

Solution

(i) x + y = –1

x	–3	–2	–1	0	1	2	3
y = –1 – x	2	1	0	–1	–2	–3	–4

Some of the ordered pairs which satisfy the equation x + y = –1 are (–3, 2), (–2, 1), (–1, 0), (0, –1), (1, –2), (2, –3), (3, –4).

(ii) x – y = 5

x	–2	–1	0	1	2	3	4
y = x – 5	–7	–6	–5	–4	–3	–2	–1

∴ Some of the ordered pairs which satisfy the equation $x - y = 5$ are
$(-2, -7), (-1, -6), (0, -5), (1, -4), (2, -3), (3, -2), (4, -1)$

The ordered pairs which satisfy the equations $x + y = -1$ and $x - y = 5$ are plotted on a graph paper. We find that each equation represents a line.

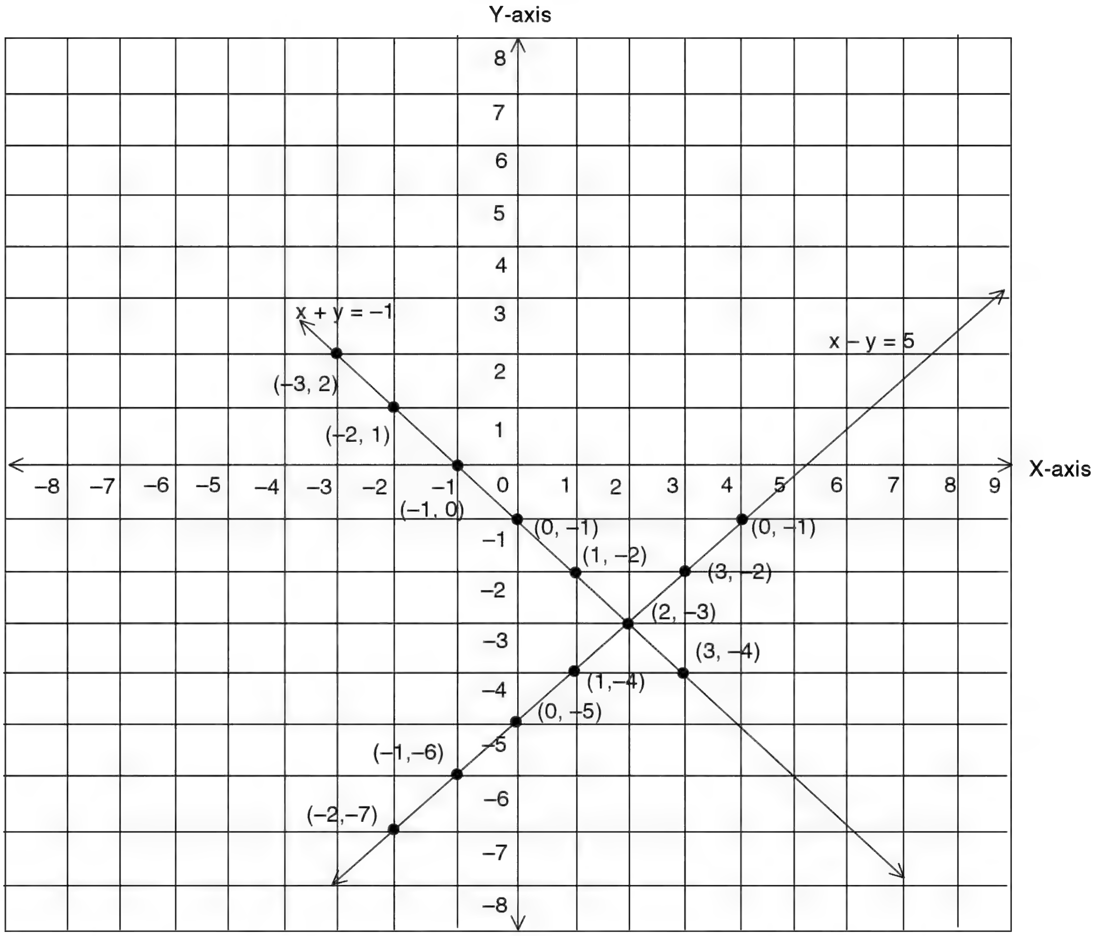


Figure 3.6

From the graph, we notice that the two given lines intersect at the point $(2, -3)$.
That is, lines $x + y = -1$ and $x - y = 5$ have a common point $(2, -3)$. Therefore, $(2, -3)$ is the solution of the equations $x + y = -1$ and $x - y = 5$.

Verification

$x + y = -1$ ------------ (1)
$x - y = 5$ ------------ (2)

Solving (1) and (2), we get,

$x = 2$ and $y = -3$

∴ $(2, -3)$ is the solution of $x + y = -1$ and $x - y = 5$.

NOTE: From the above example, we notice that we can find the solution for simultaneous equations by representing them in graphs i.e., by using the graphical method.

Nature of solutions

When we try to solve a pair of equations we could arrive at three possible results. They are, having

(a) a unique solution.
(b) an infinite number of solutions.
(c) no solution.

Let the pair of equations be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where a_1 and b_1 are the coefficients of x; b_1 and b_2 are the coefficients of y; while c_1 and c_2 are the known constant quantities.

(a) A pair of equations having a unique solution

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the pair of equations will have a unique solution.

We have solved such equations in the previous examples of this chapter.

(b) A pair of equations having infinite solutions

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will have infinite number of solutions.

NOTE:

In fact this means that there are no two equations as such and one of the two equations is simply obtained by multiplying the other with a constant. These equations are known as dependent equations.

Example

$3x + 4y = 8$

$9x + 12y = 24$

For these two equations $a_1 = 3, a_2 = 9, b_1 = 4, b_2 = 12, c_1 = -8, c_2 = -24$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Since, $\frac{3}{4} = \frac{4}{12} = \frac{-8}{-24}$

The above pair of equations will have infinite solutions.

(c) A pair of equations having no solution at all

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will have no solution.

NOTE:

1. In other words, the two equations will contradict each other or be inconsistent with each other.
2. A pair of equations is said to be consistent if it has a solution (finite or infinite).

☛ *Example*

$5x + 6y = 30$
$10x + 12y = 40$
For these two equations, $a_1 = 5, a_2 = 10, b_1 = 6, b_2 = 12, c_1 = -30, c_2 = -40$

Here, $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

$\frac{5}{10} = \frac{6}{12} \neq \frac{-30}{-40}$

Hence, the pair of equations has no solution at all.

Word problems and application of simultaneous equations

In this chapter, we have discussed earlier that it is essential to have as many equations as there are unknown quantities to be determined. In word problems also, it is necessary to have as many independent conditions as there are unknown quantities to be determined.

Let us understand with the help of the following examples as to how word problems can be solved using simultaneous equations.

☛ *Example*

The sum of the successors of two numbers is 42 and the difference of their predecessors is 12. Find the numbers.

Solution

Let the two numbers be x and y.
Given that,
$(x + 1) + (y + 1) = 42$
$\Rightarrow x + y = 40$ ------------ (1)
Also,
$(x - 1) - (y - 1) = 12$
$\Rightarrow x - y = 12$ ------------ (2)
Adding (1) and (2), we get,
$2x = 52$
$\Rightarrow x = 26$
Substituting $x = 26$ in any of the equations (1) and (2) we get $y = 14$.
$\therefore$ The two numbers are 26 and 14.

☛ *Example*

In a fraction, if numerator is increased by 2 and denominator is decreased by 3, then the fraction becomes 1. Instead, if numerator is decreased by 2 and denominator is increased by 3, then the fraction becomes $\frac{3}{8}$. Find the fraction.

Solution

Let the fraction be $\frac{a}{b}$.

Applying the first condition, we get $\frac{a+2}{b-3} = 1$

$\Rightarrow a + 2 = b - 3$

$\Rightarrow a - b = -5$ ------------ (1)

Applying the second condition, we get $\frac{a-2}{b+3} = \frac{3}{8}$

$\Rightarrow 8a - 16 = 3b + 9$

$\Rightarrow 8a - 3b = 25$ ------------ (2)

Solving the equations (1) and (2) using any of the methods discussed earlier, we get a = 8 and b = 13.

$\therefore$ The fraction is $\frac{8}{13}$.

Example

In a box, the total number of Rs 2 coins and Rs 5 coins is 20. If the total coins amount to Rs 76, find the number of coins of each denomination.

Solution

Let the number of Rs 2 coins and Rs 5 coins be x and y respectively.

Given, $x + y = 20$ ------------ (1)

$2x + 5y = 76$ ------------ (2)

Solving equations (1) and (2), we get

$x = 8$ and $y = 12$

i.e., The number of Rs 2 coins = 8 and

$\therefore$ Number of Rs 5 coins = 12

Example

Four years ago, the age of a person was thrice that of his son. Eight years later, the age of the person will be twice that of his son. Find the present ages of the person and his son.

Solution

Let the present ages of the person and his son be x years and y years respectively.

Given,

$x - 4 = 3(y - 4)$

$\Rightarrow x - 4 = 3y - 12$

$\Rightarrow x - 3y = -8$ ------------ (1)

Also,

$x + 8 = 2(y + 8)$

$\Rightarrow x + 8 = 2y + 16$

$\Rightarrow x - 2y = 8$ ------------ (2)

Solving the equations (1) and (2), we get

$x = 40$ and $y = 16$.

☛ *Example*

For what value of k do the set of equations $4x - (3k + 2)y = 20$ and $(11k - 3)x - 10y = 40$ have infinite solutions?

Solution

Given equations are

$4x - (3k + 2)y = 20$ ------------ (1)

$(11k - 3)x - 10y = 40$ ------------ (2)

System of equations (1) and (2) have infinite solutions, if $\frac{4}{11k-3} = \frac{-(3k+2)}{-10} = \frac{-20}{-40}$

$\Rightarrow \frac{4}{11k-3} = \frac{1}{2}$

$\Rightarrow 8 = 11k - 3$

$\Rightarrow 11k = 11$

$\Rightarrow k = 1$

test your concepts

Very short answer type questions

1. If $99x + 101y = 400$ and $101x + 99y = 600$, then $x + y$ is ________.
2. The number of common solutions for the system of linear equations $5x + 4y + 6 = 0$ and $10x + 8y = 12$ is ________.
3. Sum of the heights of A and B is 320 cm and the difference of heights of A and B is 20 cm. The height of B can be ________.(140 cm / 145 cm / 150 cm).
4. If $a : b = 7 : 3$ and $a + b = 20$, then $b =$ ________.
5. If $\frac{1}{x} + \frac{1}{y} = k$ and $\frac{1}{x} - \frac{1}{y} = k$, then the value of y ________. (is 0/does not exist).
6. If $p + q = k$, $p - q = n$ and $k > n$, then q is ________ (positive/negative).
7. If $a + b = x$, $a - b = y$ and $x < y$, then b is ________ (positive/negative).
8. If $3a + 2b + 4c = 26$ and $6b + 4a + 2c = 48$, then $a + b + c =$ ________.
9. Sum of the ages of X and Y, 12 years ago, was 48 years and sum of the ages of X and Y, 12 years hence will be 96 years. Present age of X is ________.
10. If the total cost of 3 chairs and 2 tables is Rs 1200 and the total cost of 12 chairs and 8 tables is Rs 4800, then the cost of each chair must be Rs 200 and each table must be Rs 300. (True/False)
11. If the total cost of 2 apples and 3 mangoes is Rs 22, then the cost of each apple and each mango must be Rs 5 and Rs 4 respectively, (where cost of each apple and mango is an integer). (True/False)

12. Number of non-negative integral solutions for the equation $2x + 3y = 12$ is ________.

13. Two distinct natural numbers are such that the sum of one number and twice the other number is 6. The two numbers are ________.

14. If $2x + 3y = 5$ and $3x + 2y = 10$, then $x - y$ = ________.

15. If $a + b = p$ and $ab = p$, then find the value of p. (where a and b are positive integers).

Short answer type questions

16. Solve: $331a + 247b = 746$ and $247a + 331b = 410$.

17. Six gallery seats and three balcony seats for a play were sold for Rs 162. Four gallery seats and five balcony seats were sold for Rs 180. Find the price of a gallery seat and the price of a balcony seat.

18. If the numerator of a fraction is increased by 2 and the denominator is decreased by 4, then it becomes 2. If the numerator is decreased by 1 and the denominator is increased by 2, then it becomes $\frac{1}{3}$. Find the fraction.

19. Solve: $\frac{1}{x} + \frac{1}{y} = 6$, $\frac{1}{y} + \frac{1}{z} = 7$ and $\frac{1}{z} + \frac{1}{x} = 5$.

20. Solve: $\frac{2}{x+y} - \frac{1}{x-y} = 11$ and $\frac{5}{x+y} + \frac{4}{x-y} = 8$.

21. Jaydeep starts his job with a certain monthly salary and earns a fixed increment in his monthly salary at the middle of every year, starting from the first year. If his monthly salary was Rs 78000 at the end of 6 years of service and Rs 84000 at the end of 12 years of service, find his initial salary and annual increment.

22. Alok was asked to find, $\frac{6}{7}$ of a number but instead he multiplied it by $\frac{7}{6}$. As a result he got an answer, which was more than the correct answer by 299. What was the number?

23. Shriya has certain number of 25 paise and 50 paise coins in her purse. If the total number of coins is 35 and their total value is Rs 15.50, find the number of coins of each denomination.

24. For what value of k, will the following pair of linear equations have no solution?
$2x + 3y = 1$ and $(3k - 1)x + (1 - 2k)y = 2k + 3$.

25. Solve: $\frac{x}{a} + \frac{y}{b} = a^2 + b^2$ and $\frac{x}{a^2} + \frac{y}{b^2} = a + b$.

Essay type questions

26. Solve: $x - 2y + z = 0$, $9x - 8y + 3z = 0$ and $2x + 3y + 5z = 36$.

27. Four friends P, Q, R and S have some money. The amount with P equals the total amount with the others. The amount with Q equals one-third of the total amount with the others. The amount with R equals one-fifth of the total amount with the others. The amount with S equals one-eleventh of the

total amount with the others. The sum of the smallest and the largest amounts with them is Rs 210. Find the sum of the amounts with the other two (in Rs).

28. The population of a town is 25000. If in the next year the number of males were to increase by 5% and that of females by 3%, the population would grow to 26010. Find the number of males and females in the town at present.

29. What is the solution set of $\frac{12}{2x + 3y} + \frac{5}{3x - 2y} = -7$ and $\frac{8}{2x + 3y} + \frac{6}{3x - 2y} = -10$?

30. Tito purchased two varieties of ice cream cups, vanilla and strawberry – spending a total amount of Rs 330. If each vanilla cup costs Rs 25 and each strawberry cup costs Rs 40, then in how many different combinations could he have purchased the ice cream cups?

CONCEPT APPLICATION

Concept Application Level—1

1. For what value of k do the equations $3(k - 1)x + 4y = 24$ and $15x + 20y = 8(k + 13)$ have infinite solutions?

(1) 1 (2) 4 (3) 3 (4) 2

2. If the system of equations $4x + py = 21$ and $px - 2y = 15$ has unique solution, then which of the following could be the value of p?

(a) 103 (b) 105 (c) 192 (d) 197

(1) Both (a) and (b) (2) Both (c) and (d)

(3) (a), (b) and (d) (4) All of (a), (b), (c) and (d)

3. If the system of equations $2x - 3y = 3$ and $-4x + qy = \frac{p}{2}$ is inconsistent, which of the following cannot be the value of p?

(1) –24 (2) –18 (3) –12 (4) –36

4. The semi perimeter of a triangle exceeds each of its side by 5, 3 and 2 respectively. What is the perimeter of the triangle?

(1) 12 (2) 10 (3) 15 (4) 20

5. If (p, p) is the solution of system of equations $ax + by + (t - s) = 0$ and $bx + ay + (s - r) = 0$, $(a \neq b)$, then which of the following must be true?

(1) $2r = s + t$ (2) $2t = r + s$ (3) $2s = r + t$ (4) $r + s + t = 0$

6. If $173x + 197y = 149$ and $197x + 173y = 221$, then find (x, y).

(1) (3, –2) (2) (2, 1) (3) (1, –2) (4) (2, –1)

7. Mallesh has some cows and some hens in his shed. The total number of legs is 92 and the total number of heads is 29. Find the number of cows in his shed.

(1) 12 (2) 14 (3) 17 (4) 19

8. Total cost of 14 pens and 21 books is Rs 130 and the total cost of 6 pens and p books is Rs 90. Which of the following cannot be the value of p?

(1) 8 (2) 9 (3) 10 (4) 11

9. If an ordered pair satisfying the equations $2x - 3y = 18$ and $4x - y = 16$ also satisfies the equation $5x - py - 23 = 0$, then find the value of p.

(1) 1 (2) 2 (3) –1 (4) –2

10. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are such that a_1, b_1, c_1, a_2, b_2 and c_2 are consecutive integers in the same order, then find the values of x and y.

(1) 1, –2 (2) 2, –3 (3) 1, 3 (4) data insufficient

11. If we increase the length by 2 units and the breadth by 2 units, then the area of rectangle is increased by 54 square units. Find the perimeter of the rectangle (in units).

(1) 44 (2) 50
(3) 56 (4) Cannot be determined

12. A mother said to her son, "the sum of our present ages is twice my age 12 years ago and nine years hence, the sum of our ages will be thrice my age 14 years ago". What is her son's present age? (in years)

(1) 8 (2) 12 (3) 15 (4) 10

13. A told B, "when I was as old as you are now, then your age was four years less than half of my present age". If the sum of the present ages of A and B is 61 years, what is B's present age? (in years)

(1) 9 (2) 25 (3) 43 (4) 36

14. If the system of equations $4x - 5y = 6$ and $-12x + ay = b$ is inconsistent, which of the following cannot be the value of b?

(1) –16 (2) –18 (3) –20 (4) –22

15. A sum of Rs 400 was distributed among the students of a class. Each boy received Rs 8 and each girl received Rs 4. If each girl had received Rs 10, then each boy would have received Rs 5. Find the total number of students of the class.

(1) 40 (2) 50 (3) 60 (4) 70

Concept Application Level—2

16. Dheeraj has twice as many sisters as he has brothers. If Deepa, Dheeraj's sister has the same number of brothers as she has sisters, then Deepa has how many brothers?

(1) 2 (2) 3
(3) 4 (4) Cannot be determined

17. The total cost of six books, five pencils and seven sharpeners is Rs 115 and that of eight books, ten pencils and fourteen sharpeners is Rs 190, then which of the following article's cost can be found uniquely?

(1) Book (2) Pencil
(3) Sharpener (4) None of these

18. The sum of the speeds of a boat in still water and the speed the current is 10 kmph. If the boat takes 40% of the time to travel downstream when compared to that upstream, then find the difference of the speeds of the boat when travelling upstream and down stream.

(1) 3 kmph (2) 6 kmph
(3) 4 kmph (4) 5 kmph

19. In a fraction, if the numerator is decreased by 1 and the denominator is increased by 1, then the fraction becomes $\frac{1}{2}$. Instead, if the numerator is increased by 1 and the denominator is decreased by 1, then the fraction becomes $\frac{4}{5}$. Find the numerator of the fraction.

(1) 5 (2) 6 (3) 7 (4) 8

20. Ram has 18 coins in the denominations of Re1, Rs 2 and Rs 5. If their total value is Rs 54 and the number of Rs 2 coins are greater than that of Rs 5 coins, then find the number of Re 1 coins with him.

(1) 2 (2) 3
(3) 4 (4) Cannot be determined

21. If the ordered pair ($\sin\theta$, $\cos\theta$) satisfies the system of equations $mx + ny + a + b = a - b$ and $nx + my + 2b = 0$, then find the value of θ where $0 \leq \theta \leq 90°$. ($m \neq n$)

(1) 30° (2) 45°
(3) 50° (4) Cannot be determined

22. Swaroop can row 16 km downstream and 8 km upstream in 6 hours. He can row 6 km upstream and 24 km downstream in 6 hours. Find the speed of Swaroop in still water.

(1) 6 kmph (2) 8 kmph (3) 3 kmph (4) 5 kmph

23. A two-digit number is formed by either subtracting 17 from nine times the sum of the digits or by adding 21 to 13 times the difference of the digits. Find the number.

(1) 37 (2) 73
(3) 71 (4) Cannot be determined

24. Swathi starts her job with certain monthly salary and earns a fixed increment every year. If her salary was Rs 22500 per month after 6 years of service and Rs 30000 per month after 11 years of service. Find her salary after 8 years of service (in Rs).

(1) 24000 (2) 25500 (3) 26000 (4) 24500

25. A three digit number abc is 459 more than the sum of its digits. What is the sum of the 2 digit number ab and the 1-digit number a?

(1) 71 (2) 61
(3) 51 (4) Cannot be determined

Concept Application Level—3

26. An examination consists of 100 questions. Two marks are awarded for every correct option. If one mark is deducted for every wrong option and half mark is deducted for every question left, then a person scores 135. Instead, if half mark is deducted for every wrong option and one mark is deducted for every question left, then the person scores 133. Find the number of questions left unattempted by the person.

(1) 14 (2) 16 (3) 10 (4) 12

27. Ram, Shyam, Tarun and Varun together had a total amount of Rs 240 with them. Ram had half of the total amount with the others. Shyam had one-third of the total amount with the others. Tarun had one-fourth of the total amount with the others. Find the amount with Varun (in Rs).

(1) 64 (2) 70 (3) 52 (4) 58

28. Ramu had 13 notes in the denominations of Rs 10, Rs 50 and Rs 100. The total value of the notes with him was Rs 830. He had more of Rs 100 notes than that of Rs 50 notes with him. Find the number of Rs 10 notes with him.

(1) 4 (2) 3 (3) 2 (4) 5

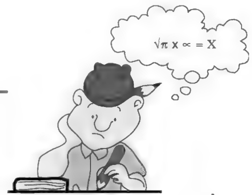
29. A, B, C and D share a certain amount amongst themselves. B sees that the other three get 3 times what he himself gets. C sees that the other three get 4 times what he gets, while D sees that the other three get 5 times what he gets. If the sum of the largest and smallest shares is 99, what is the sum of the other two shares?

(1) 99 (2) 81 (3) 64 (4) 54

30. If $3|x| + 5|y| = 8$ and $7|x| - 3|y| = 48$, then find the value of $x + y$.

(1) 5 (2) –4

(3) 4 (4) The value does not exist



KEY

Very short answer type questions

1. 5
2. zero
3. 150 cm
4. 6
5. Does not exist.
6. Positive
7. Negative
8. 10
9. Cannot be determined
10. False
11. False
12. 3
13. 4 and 1
14. 5
15. 4

Short answer type questions

16. $a = 3$ and $b = -1$
17. Rs 15, Rs 24
18. $\frac{4}{7}$
19. $x = \frac{1}{2}, y = \frac{1}{4}$
20. $x = \frac{-1}{24}, y = \frac{7}{24}$
21. Rs 72000, Rs 1000
22. 966
23. Number of 25 paise coins is 8 and number of 50 paise coins is 27.
24. $\frac{5}{13}$
25. $x = a^3, y = b^3$

Essay type questions

26. $x = 1, y = 3, z = 5$
27. $x = 4$ and $y = 5$
28. 13000, 12000
29. $(\frac{1}{2}, 1)$
30. 2

key points for selected questions

Short answer type questions

16. (i) Add the two equations and get a + b.
(ii) Subtract one from the other and get a – b.
(iii) Solve the above equations for a and b.

17. (i) Let the fare of each gallery seat be Rs x. and each balcony seat be Rs y.
(ii) Given $6x + 3y = 162$ i.e., $2x + y = 34$
(iii) Solve the above equations.

18. (i) Let the numerator be x and denominator be y.
(ii) Given $\frac{x+2}{y-4} = 2$ and $\frac{x-1}{y+2} = \frac{1}{3}$.
(iii) Solve the above equations by using elimination method.

19. (i) Let $\frac{1}{x} = a, \frac{1}{y} = b$ and $\frac{1}{z} = c$.
(ii) Given, $a + b = 6, b + c = 7$ and $c + a = 5$
(iii) Add the above obtained three equations and find $a + b + c$.
(iv) Find a on subtracting of $b + c = 7$ from $a + b + c$.
(v) Similarly find b and c.
(vi) Then $x = \frac{1}{a}, y = \frac{1}{b}$ and $z = \frac{1}{c}$ and evaluate the values of x, y and z.

20. (i) Let $\frac{1}{x+y} = a$ and $\frac{1}{x-y} = b$.
(ii) Solve the equations for a and b.
(iii) Replace a and b by $x + y$ and $x - y$ and solve for x and y.

21. (i) Let the initial salary and annual increment be Rs x and Rs y respectively.
(ii) Frame the equations from the data given.
(iii) Solve the equations obtained in step (ii).

22. (i) Let the number b x.
(ii) Given $\frac{7x}{6} - \frac{6x}{7} = 299$
(iii) Find the value of x by using the above equation.

23. (i) Let the number of 25 paise and 50 paise coins be x and y respectively.
(ii) Given $x + y = 35$ and $\frac{x}{4} + \frac{y}{2} = 15.5$
(iii) solve the above equation we get x and y values.

24. (i) If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, have no solution then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
(ii) Substitute the values in the above relation and find k.

25. Solve the equations by elimination method.

Essay type questions

26. (i) First of all eliminate z and obtain two equations in x and y.
(ii) Then solve the two equations obtained above.
(iii) Substitute x and y in one of the given equations to find z.

27. (i) Let $\frac{1}{x}$ and $\frac{1}{y} = b$
(ii) Then write the given equations in terms of a and b.
(iii) Then solve the equations by using cancellation method.

28. (i) Let the number of males and females presently be x and y respectively.
(ii) $x + y = 25000$
(iii) $\frac{105x}{100} + \frac{103y}{100} = 26010$
(iv) Solve the above equations.

30. (i) Frame the linear equation and check possibilities.
(ii) Let the number of ice cream cups of vanilla be x and that of strawberry be y.
(ii) Frame the equation and identify the possible values of x and y.

Concept Application Level—1,2,3

1. 4	**2.** 4
3. 3	**4.** 4
5. 3	**6.** 4
7. 3	**8.** 2
9. 2	**10.** 1
11. 2	**12.** 2
13. 2	**14.** 2
15. 3	**16.** 2
17. 1	**18.** 2
19. 3	**20.** 2
21. 2	**22.** 4
23. 2	**24.** 2
25. 3	**26.** 1
27. 3	**28.** 2
29. 2	**30.** 4

Concept Application Level—1,2,3

Key points for select questions

1. Condition for infinite solutions is,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

2. For unique solution, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

3. If two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2x + c_2 = 0$ are inconsistent, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

4. (i) Semi perimeter (s) $= \frac{a+b+c}{2}$

(ii) $a + 5 = s; b + 3 = s; c + 2 = s$

5. If $x = y$, then $a_1 = a_2, b_1 = b_2$ and $c_1 = c_2$.

6. Add two equations.

7. Each cow has 4 legs and each hen has 2 legs.

8. Frame the linear equation and write $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

9. Solve the first two equations and substitute (x, y) in the third equation.

10. Take any consecutive integers for a_1, b_1, c_1, a_2, b_2 and c_2 and solve.
or
Substitute any convenient consecutive integers for a_1, b_1, c_1, a_2, b_2 and c_2 and solve.

11. Frame linear equation by taking length and breadth as ℓ and b respectively.

12. Frame linear equations and solve.

13. (i) Let the present ages of A and B be x years and y years respectively.

(ii) $y - (x - y) = \frac{x}{2} - 4.$

(iii) $x + y = 61.$

14. If the system of equations $a_1x + b_1y + c_1, a_2x + b_2y + c_2 = 0$ is inconsistent, $\frac{a_1}{a_2} = \frac{b_1}{b_2} + \frac{c_1}{c_2}$

15. Let the cost of each book and each pencil and each sharpener be b, p and s respectively. Then frame the equations.

16. (i) Let the number of sons and daughters that Dheeraj's parents have be b and d respectively.

(ii) Form the equations and solve them.

17. Let the cost of each book and each pencil and each sharpener be b, p and s respectively. Then frame the equations.

19. Let the fraction be $\frac{n}{a}$ and frame the linear equations.

20. (i) Let the number of Re 1, Rs 2 and Rs 5 coins be x, y and z respectively.

(ii) From the information given, obtain an equation in y and z and proceed.

21. If (p, q) is a solution of $ax + by + c = 0$ and $bx + ay + c = 0$, then $p = q$.

23. (i) $10x + y = 9(x + y) - 17$

(ii) $10x + y = 13(x - y) + 21.$

(iii) Solve the above equations.

24. Frame linear equations and solve.

25. (i) Let the three digit number be $100x + 10y + z$.

(ii) Frame the equations and solve them.

26. (i) $x + y + z = 100$, $2x - y - \frac{1}{2}z = 135$; $2x - \frac{y}{2} - z = 133$

where x, y, and z are the number of correct answers, wrong answers and unattempted questions respectively.

(ii) Solve the above equations.

27. (i) Let r, s, t, v, be the amounts, with Ram, Syam, Tarun, Varun respectively.

(ii) $r = \frac{1}{2}(s + t + v)$, $s = \frac{1}{3}(r + t + v)$,

$t = \frac{1}{4}(r + s + v)$, $v + s + t + v = 240$.

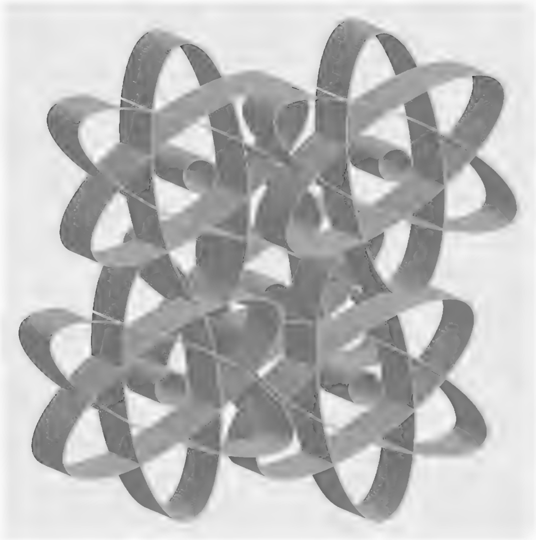
(iii) Solve the above equation set to get v.

29. Let the total amounts be 4B or 5C or 6D and $B + D = 99$.

30. (i) Let $|x| = a$ and $|y| = b$ and solve linear equations which are in a and b.

(ii) Let $|x| = k_1$ and $|y| = k_2$ and solve them.

(iii) $|x| = \pm x$, $|y| = \pm y$.



CHAPTER 4

Quadratic Equations and Inequalities

INTRODUCTION

Very often we come across many equations involving several powers of one variable. If the indices of all these powers are integers then the equation is called a polynomial equation. If the highest index of a polynomial equation in one variable is two, then it is a quadratic equation.

A quadratic equation is a second degree polynomial in x usually equated to zero. In other words, for an equation to be a quadratic, the coefficient of x^2 should not be zero and the coefficients of any higher power of x should be 0.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real). The following are some quadratic equations.

1. $x^2 - 5x + 6 = 0$
2. $x^2 - x - 6 = 0$
3. $2x^2 + 3x - 2 = 0$
4. $2x^2 + x - 3 = 0$

Roots of the equation

Just as a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x has two values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or complex.

The roots of the four quadratic equations given above are:

Equation (1) $x = 2$ and $x = 3$

Equation (2) $x = -2$ and $x = 3$

Equation (3) $x = \frac{1}{2}$ and $x = -2$

Equation (4) $x = 1$ and $x = \frac{-3}{2}$

In general, the roots of a quadratic equation can be found in two ways.

(i) by factorising the expression on the left hand side.

(ii) by using the standard formula.

All the expressions may not be easy to factorize, whereas applying the formula is simple and straightforward.

Finding the roots by factorization

If the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is written in the form, $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first express it in the form of $(x - \alpha)(x - \beta) = 0$, i.e., the left hand side, $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorized.

For this purpose, we should go through the following steps. We will understand these steps with the help of the equation $x^2 - 5x + 6 = 0$.

Here $a = 1$, $b = -5$ and $c = 6$

First write down b (the coefficient of x) as the sum of two quantities whose product is equal to ac. In this case, -5 has to be written as the sum of two quantities whose product is 6. We write -5 as $(-3) + (-2)$, because the product of (-3) and (-2) is equal to 6.

Now, rewrite the equation. In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$.

Consider the first two terms and rewrite them together after taking out their common factor. Similarly, the third and the fourth terms should be rewritten after taking out their common factor. In this process, we should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and fourth terms (after removing their common factor).

In this case, the equation can be rewritten as $x(x - 3) -2(x - 3) = 0$; now $(x - 3)$ is a common factor.

If we take out $(x - 3)$ as the common factor, we can rewrite the given equation as $(x - 3)(x - 2) = 0$

We know that if α and β are the roots of the given quadratic equation $(x - \alpha)(x - \beta) = 0$.

Hence, the roots of the given equation are 3 and 2.

Consider equation (2): $x^2 - x - 6 = 0$, the coefficient of x is -1 which can be rewritten as $(-3) + (+2)$, because the product of (-3) and 2 is -6, which is equal to 'ac' (1 multiplied by -6). Then, we can rewrite the equation as $(x - 3)(x + 2) = 0$ to get the roots as 3 and -2.

Consider equation (3): $2x^2 + 3x - 2 = 0$, the co-efficient of x is 3, which can be rewritten as $(+4) + (-1)$ so that their product is -4, which is the value of 'ac' (-2 multiplied by 2). Then, we can rewrite the equation as $(2x - 1)(x + 2) = 0$, obtaining the roots as $\frac{1}{2}$ and -2.

Consider equation (4): $2x^2 + x - 3 = 0$, the coefficient of x is 1, which can be rewritten as $(+3) + (-2)$ because their product is -6, which is equal to 'ac' (2 multiplied by -3). Then we can rewrite the given equation as $(x - 1)(2x + 3) = 0$ to get the roots as 1 and $\frac{-3}{2}$.

Finding the roots by using the formula

For the quadratic equation $ax^2 + bx + c = 0$, we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

For the quadratic equation $ax^2 + bx + c = 0$, Let α and β be the roots, then

the sum of the roots $(\alpha + \beta) = \frac{-b}{a}$

the product of the roots $(\alpha\beta) = \frac{c}{a}$

These two rules will be very helpful in solving problems on quadratic equation.

Nature of the roots

We have already learnt that the roots of a quadratic equation with real coefficients can be real or complex. When the roots are real, they can be rational or irrational and, also, they can be equal or unequal.

Consider the expression $b^2 - 4ac$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the "DISCRIMINANT" of the quadratic equation.

A quadratic equation has real roots only if $b^2 - 4ac \geq 0$.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation are complex conjugates.

The following table gives us a clear idea about the nature of the roots of a quadratic equation when a, b and c are all rational.

when $b^2 - 4ac < 0$	the roots are complex conjugates
when $b^2 - 4ac = 0$	the roots are rational and equal.
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal.
when $b^2 - 4ac > 0$ and not a perfect square	the roots are irrational and unequal.

NOTE:

1. Whenever the roots of the quadratic equation are irrational, (a, b, c being rational), are of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e., whenever $a + \sqrt{b}$ is one root of a quadratic equation, $a - \sqrt{b}$ is the other root of the quadratic equation and vice-versa. In other words, if the roots of a quadratic equation are irrational, then they are conjugate to each other.

2. If the sum of the coefficients of a quadratic equation, say $ax^2 + bx + c = 0$, is zero, then its roots are 1 and $\frac{c}{a}$.

   That is if $a + b + c = 0$, then the roots of $ax^2 + bx + c = 0$ are 1 and $\frac{c}{a}$.

Signs of the roots

We can comment on the signs of the roots, i.e., whether the roots are positive or negative, based on the sign of the sum of the roots and the product of the roots of the quadratic equation. The following table indicates the signs of the roots when the signs of the sum and the product of the roots are given.

Sign of product of the roots	Sign of sum of the roots	Sign of the roots
+ ve	+ ve	Both the roots are positive
+ ve	– ve	Both the roots are negative
– ve	+ ve	One root is positive and the other negative. The numerically greater root is positive
– ve	– ve	One root is positive and the other negative. The numerically greater root is negative

Constructing a quadratic equation

We can build a quadratic equation in the following cases:

(i) when the roots of the quadratic equation are given.

(ii) when the sum of the roots and the product of the roots of the quadratic equation are given.

Case (i): If the roots of the quadratic equation are α and β, then its equation can be written as $(x - \alpha)(x - \beta) = 0$ i.e., $x^2 - x(\alpha + \beta) + \alpha\beta = 0$

Case (ii): If p is the sum of the roots of the quadratic equation and q is their product, then the equation can be written as $x^2 - px + q = 0$.

Constructing a new quadratic equation by changing the roots of a given quadratic equation

If we are given a quadratic equation, we can build a new quadratic equation by changing the roots of this equation in the manner specified to us.

For example, consider the quadratic equation $ax^2 + bx + c = 0$ and let its roots be α and β respectively. Then, we can build new quadratic equations as per the following points:

(i) A quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, i.e., the roots are reciprocal to the roots of the given quadratic equation can be obtained by substituting $(1/x)$ for x in the given equation, which gives us $cx^2 + bx + a = 0$, i.e., we get the equation required by interchanging the coefficient of x^2 and the constant term.

(ii) A quadratic equation whose roots are $(\alpha + k)$ and $(\beta + k)$ can be obtained by substituting $(x - k)$ for x in the given equation.

(iii) A quadratic equation whose roots are $(\alpha - k)$ and $(\beta - k)$ can be obtained by substituting $(x + k)$ for x in the given equation.

(iv) A quadratic equation whose roots are $(k\alpha)$ and $(k\beta)$ can be obtained by substituting with $\left(\frac{x}{k}\right)$ for x in the given equation.

(v) A quadratic equation whose roots are $\left(\frac{\alpha}{k}\right)$ and $\left(\frac{\beta}{k}\right)$ can be obtained by substituting (kx) for x in the given equation.

(vi) A quadratic equation whose roots are $(-\alpha)$ and $(-\beta)$ can be obtained by replacing x by $(-x)$ in the given equation.

Finding the roots of a quadratic equation by graphical method

First, let us learn how to draw the graph of $y = x^2$

We assume certain real values for x, i.e., we substitute some values for x in $y = x^2$. We can find the corresponding values of y. We tabulate the values, as shown below.

x	5	4	3	2	1	0	–1	–2	–3	–4	–5
$y = x^2$	25	16	9	4	1	0	1	4	9	16	25

Plotting the points corresponding to the ordered pairs (5, 25), (4, 16), (3, 9), (2, 4), (1, 1), (0, 0) (–1, 1), (–2, 4), (–3, 9), (–4, 16) and (–5, 25) on the graph paper and joining them with a smooth curve we obtain the graph of $y = x^2$, as shown below.

We observe the following about the graph of $y = x^2$.

1. It is a U shaped graph and it is called a parabola. The arms of the 'U' spread outwards.
2. For every value of x ($\neq 0$) we notice that y is always positive. Hence, the graph lies entirely in the first and second quadrants.
3. When $x = 0, y = 0 \Rightarrow y = x^2$ passes through origin.
4. The graph is symmetric about the y-axis.
5. Using the graph of $y = x^2$, we can find the square of any real number as well as the square root of any non-negative real number
   (a) for any given x value, the corresponding y value on the graph is its square and
   (b) for any given y (≥ 0) value, the corresponding x value on the graph is its square root.
6. The graph of $y = kx^2$, when $k > 0$ lies entirely in Q_1 and Q_2 and when $k < 0$ the graph lies entirely in Q_3 and Q_4

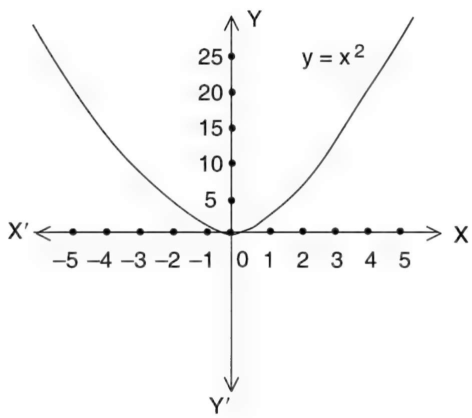


Figure 4.1

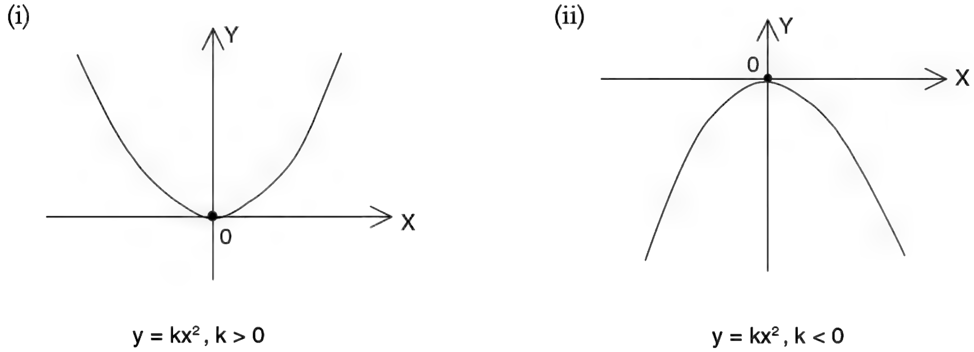


Figure 4.2

The method of solving the quadratic equation of the form $px^2 + qx + r = 0$ whose roots are real is shown in the following example.

Example

Solve $x^2 - 5x + 6 = 0$ using the graphical method.

Solution

Let $y = x^2 - 5x + 6$

Prepare the following table by assuming different values for x.

x	0	1	2	3	– 1	– 2	4	5
x^2	0	1	4	9	1	4	16	25
5x	0	5	10	15	– 5	10	20	25
$y = x^2 - 5x + 6$	6	2	0	0	12	20	2	6

Plot the points (0, 6), (1, 2), (2, 0), (3, 0), (–1, 12), (–2, 20), (4, 2) and (5, 6) on the graph and join the points with a smooth curve, as shown below.

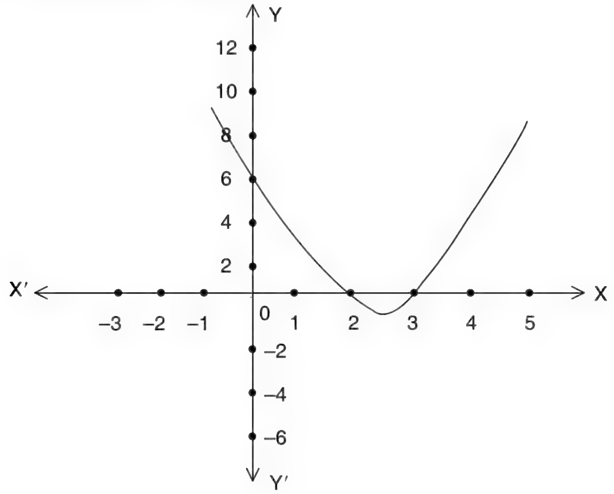


Figure 4.3

Here we notice that the given graph (parabola) intersects the x-axis at (2, 0) and (3, 0)
The roots of the given quadratic equation $x^2 - 5x + 6 = 0$ are x = 2 and x = 3.
∴ The roots of the given equation are the x-coordinates of the points of intersection of the curve with x-axis.

Note:

1. If the graph meets the x-axis at two distinct points, then the roots of the given equation are real and distinct

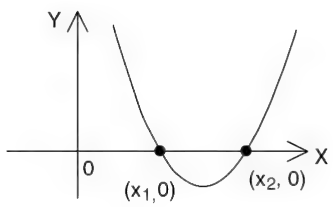


Figure 4.4

2. If the graph touches the x-axis at only one point, then the roots of the quadratic equation are real and equal

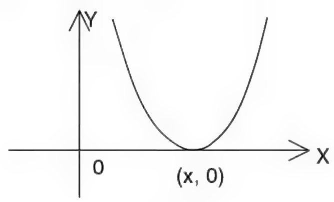


Figure 4.5

3. If the graph does not meet the x-axis, then the roots of the quadratic equation are not real, i.e., they are complex.

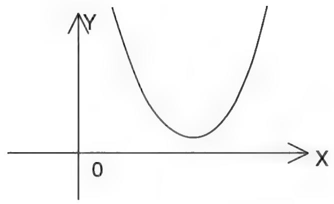


Figure 4.6

Second method

We can also solve the quadratic equation $px^2 + qx + r = 0$ by considering the following equations $y = px^2$ ---------- (1) and $y = -qx - r$ ---------- (2)
Clearly, $y = px^2$ is a parabola and $y = -qx - r$ is a straight line.

Step (i) Draw the graph of $y = px^2$ and $y = -qx - r$ on the same graph paper.
Step (ii) Draw perpendiculars from the points of intersection of parabola and the line onto the x-axis. Let the points of intersection on the x-axis be $(x_1, 0)$ and $(x_2, 0)$.
Step (iii) The x-coordinates of the points in step (ii), i.e., x_1 and x_2 are the two distinct roots of $px^2 + qx + r = 0$

Example

Solve $2x^2 - x - 3 = 0$

Solution

We know that the roots of $2x^2 - x - 3 = 0$ are the x coordinates of the points of intersection of the parabola $y = 2x^2$ and the line $y = x + 3$

(1) $y = 2x^2$

x	0	1	2	–1	–2
$y = 2x^2$	0	2	8	2	8

(2) $y = x + 3$

x	0	1	2	–1	–2	–3
$y = x + 3$	3	4	5	2	1	0

Draw the graph of $y = 2x^2$ and $y = x + 3$.
Clearly, the perpendiculars drawn from the points of intersection of parabola and the line meet the x-axis at $\left(\frac{3}{2}, 0\right)$ and $(-1, 0)$.

$\therefore$ The roots of the given quadratic equation $2x^2 - x - 3 = 0$ are $\frac{3}{2}$ and -1.

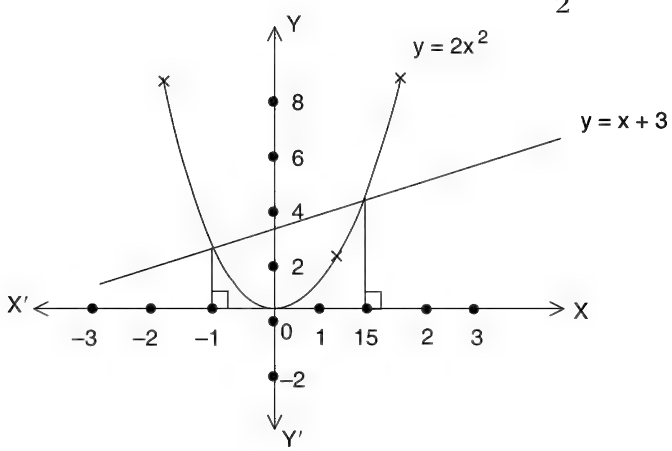


Figure 4.7

NOTE:

(i) If the line meets the parabola at two points, then the roots of the quadratic equation are real and distinct.

(ii) If the line touches the parabola at only one point, then the quadratic equation has real and equal roots.

(iii) If the line does not meet the parabola, i.e., when the line and the parabola have no points in common, then the quadratic equation has no real roots. In this case, the roots of the quadratic equation are imaginary

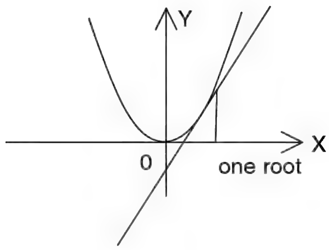


Figure 4.8

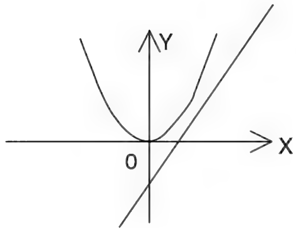


Figure 4.9

Equations of higher degree

The index of the highest power of x in the equation is called the degree of the equation. For example, if the highest power of x in the equation is x^3, then the degree of the equation is 3. An equation whose degree is 3 is called a cubic equation. A cubic equation will have three roots.

NOTE: An equation whose degree is n will have n roots.

Maximum or minimum value of a quadratic expression

The quadratic expression $ax^2 + bx + c$ takes different values as x takes different values.

For all the values of x, as x varies from $-\infty$ to $+\infty$, (i.e., when x is real), the quadratic expression $ax^2 + bx + c$

(i) has a minimum value if $a > 0$ (i.e., a is positive). The minimum value of the quadratic expression is $\frac{(4ac - b^2)}{4a}$ and it occurs at $x = \frac{-b}{2a}$.

(ii) has a maximum value if $a < 0$ (i.e., a is negative). The maximum value of the quadratic expression is $\frac{(4ac - b^2)}{4a}$ and it occurs at $x = \frac{-b}{2a}$.

Examples

1. Find the roots of the equation $x^2 + 3x - 4 = 0$.

Solution

$x^2 + 3x - 4 = 0 \Rightarrow x^2 - x + 4x - 4 = 0$

$\Rightarrow x(x - 1) + 4(x - 1) = 0$

$\Rightarrow (x + 4)(x - 1) = 0$

$\therefore x = -4$ or $x = 1$

2. Find the roots of the equation $4x^2 - 13x + 10 = 0$

Solution

$4x^2 - 13x + 10 = 0$

$\Rightarrow 4x^2 - 8x - 5x + 10 = 0$

$\Rightarrow 4x(x - 2) - 5(x - 2) = 0$

$\Rightarrow (4x - 5)(x - 2) = 0$

$\therefore x = \dfrac{5}{4}$ or $x = 2$.

3. Find the roots of the equation $26x^2 - 43x + 15 = 0$

Solution

We have to write 43 as the sum of two parts whose product should be equal to $(26) \times (15)$.

$26 \times 15 = 13 \times 30$ and $13 + 30 = 43$

$\therefore 26x^2 - 43x + 15 = 0$

$\Rightarrow 26x^2 - 13x - 30x + 15 = 0$

$\Rightarrow (13x - 15)(2x - 1) = 0$

$\Rightarrow x = \dfrac{15}{13}$ or $x = \dfrac{1}{2}$

We can also find the roots of the equation by using the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{43 \pm \sqrt{(43)^2 - (1560)}}{52}$$

$$= \frac{43 \pm \sqrt{(1849) - (1560)}}{52}$$

$$= \frac{43 \pm \sqrt{289}}{52} \Rightarrow x = \frac{43 \pm 17}{52}$$

$$x = \frac{43 + 17}{52} \text{ or } \frac{43 - 17}{52} = \frac{60}{52} \text{ or } \frac{26}{52}$$

$$\therefore x = \frac{15}{13} \text{ or } \frac{1}{2}$$

4. Discuss the nature of the roots of the equation $4x^2 - 2x + 1 = 0$.

Solution

Discriminant $= (-2)^2 - 4(4)(1) = 4 - 16 = -12 < 0$

Since the discriminant is negative, the roots are imaginary.

5. If the sum of the roots of the equation $kx^2 - 3x + 9 = 0$ is $\frac{3}{11}$, then find the product of the roots of that equation.

Solution

Sum of roots of the equation $= \frac{3}{k} = \frac{3}{11}$ (given)

$\therefore$ k = 11

In the given equation, product of roots $= \frac{9}{k}$

As k = 11, product of roots $= \frac{9}{11}$

6. Form the quadratic equation whose roots are 2 and 7.

Solution

Sum of the roots = 2 + 7 = 9
Product of roots = $2 \times 7 = 14$
We know that if p is the sum of the roots and q the product of the roots of a quadratic equation, then its equation is $x^2 - px + q = 0$
Hence the required equation is $x^2 - 9x + 14 = 0$

7. Form a quadratic equation with rational coefficients one of whose roots is $3 + \sqrt{5}$.

Solution

If $(3 + \sqrt{5})$ is one root, then the other root is $(3 - \sqrt{5})$

Sum of the roots = 6

Product of the roots = 4

Thus, the required equation is $x^2 - 6x + 4 = 0$

8. A person can buy 15 books less for Rs 900 when the price of each book goes up by Rs 3. Find the original price and the number of copies he could buy at the initial price.

Solution

Let the number of books bought initially for Rs 900 be 'x'. The original price of book was $\frac{900}{x}$. Now the price of the book is increased by Rs 3

i.e., the new cost is Rs $\left(\frac{900}{x}\right) + 3$

And the number of books bought is reduced by 15 i.e., (x – 15)

Since the total amount spent is still Rs 900, the product of the price and the number of books are still 900.

$$\left[\left(\frac{900}{x}\right)+3\right](x-15)=900$$

$\Rightarrow (900 + 3x)(x - 15) = 900x$

$\Rightarrow 3x^2 + 855x - 13500 = 900x$

$\Rightarrow 3x^2 - 45x - 13500 = 0$

$\Rightarrow x^2 - 15x - 4500 = 0$

$\Rightarrow x^2 - 75x + 60x - 4500 = 0$

$\Rightarrow x(x - 75) + 60(x - 75) = 0$

$\Rightarrow (x - 75)(x + 60) = 0 \Rightarrow x = 75$ or -60

Since x cannot be negative, $x = 75$

Thus, the original price of the book $= \dfrac{900}{75} =$ Rs 12

9. If α and β are the roots of the equation $x^2 - 6x + 8 = 0$, then find the values of

(i) $\alpha^2 + \beta^2$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(iii) $\alpha - \beta$ $(\alpha > \beta)$

Solution

From the given equation, we get $\alpha + \beta = 6$ and $\alpha\beta = 8$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (6)^2 - 2(8) = 20$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{6}{8} = \dfrac{3}{4}$

(iii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow (\alpha - \beta) = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \sqrt{6^2 - 4(8)}$

$\Rightarrow (\alpha - \beta) = \pm 2$

$\therefore \alpha - \beta = 2, (\because \alpha > \beta)$

10. Solve for x: $3^{x+1} + 3^{2x+1} = 270$.

Solution

$3^{x+1} + 3^{2x+1} = 270$

$\Rightarrow 3.3^x + 3^{2x}.3 = 270$

$\Rightarrow 3^x + 3^{2x} = 90$

Substituting $3^x = a$, we get,

$a + a^2 = 90$

$\Rightarrow a^2 + a - 90 = 0$

$\Rightarrow a^2 + 10a - 9a - 90 = 0$

$\Rightarrow (a + 10)(a - 9) = 0$
$\Rightarrow a = 9$ or $a = -10$
If $3^x = 9$, then $x = 2$.
If $3^x = -10$, which is not possible.
$\therefore x = 2$

11. Solve $|x|^2 - 7|x| + 12 = 0$.

Solution

Given equation is $|x|^2 - 7|x| + 12 = 0$
$\Rightarrow (|x| - 3)(|x| - 4) = 0$
$\Rightarrow |x| = 3$ or $|x| = 4$
$\Rightarrow x = \pm 3$ or $x = \pm 4$.

12. Solve $|x|^2 + 7|x| + 10 = 0$.

Solution

Given equation is $|x|^2 + 7|x| + 10 = 0$
$\Rightarrow (|x| + 2)(|x| + 5) = 0$
$\Rightarrow |x| = -2$ or $|x| = -5$
But, absolute value of any number can never be negative.
$\therefore$ No roots are possible for the given equation.

Quadratic Inequations

Consider the quadratic equation $ax^2 + bx + c = 0, (a \neq 0)$ where a, b and c are real numbers.

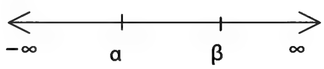
The quadratic inequations related to $ax^2 + bx + c = 0$ are $ax^2 + bx + c < 0$ and $ax^2 + bx + c > 0$.

Assume that $a > 0$.

The following cases arise:

Case I

If $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has real and unequal roots.
Let α and $\beta (\alpha < \beta)$ be the roots.
Then,
$\therefore ax^2 + bx + c = a(x - \alpha)(x - \beta)$



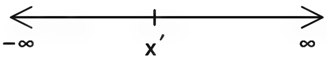
(a) If $x < \alpha$, then $(x - \alpha) < 0$ and $(x - \beta) < 0$
$\therefore ax^2 + bx + c > 0$

(b) If $\alpha < x < \beta$, then $(x - \alpha) > 0$ and $(x - \beta) < 0$
$\therefore ax^2 + bx + c < 0$

(c) If $x > \beta$, then $x - \alpha > 0$ and $x - \beta > 0$
$\therefore ax^2 + bx + c > 0$

Case II

If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has real and equal roots.
Let x' be the equal root.
$\Rightarrow ax^2 + bx + c = a(x - x')(x - x')$



(a) If $x < x'$. Then $x - x' < 0$
$\therefore\ ax^2 + bx + c > 0$
(b) If $x > x'$, then $x - x' > 0$
$\therefore\ ax^2 + bx + c > 0$

Case III

If $b^2 - 4ac < 0$, then $ax^2 + bx + c = 0$ has imaginary roots.
In this case, $ax^2 + bx + c > 0, \forall\ x \in R$.
The above concept can be summarised as

(i) If $\alpha < x < \beta$, then $(x - \alpha)(x - \beta) < 0$ and vice-versa.
(ii) If $x < \alpha$ and $x > \beta$ $(\alpha < \beta)$, then $(x - \alpha)(x - \beta) > 0$ and vice-versa.

NOTE: If $a < 0$ and $b^2 - 4ac < 0$ then the solution for $ax^2 + bx + c > 0$ does not exist.

Example

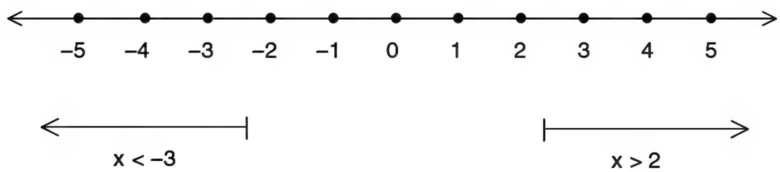
Solve the inequation $x^2 + x - 6 < 0$.

Solution

Given inequation is $x^2 + x - 6 < 0$
$\Rightarrow (x + 3)(x - 2) < 0$
$\Rightarrow (x + 3) < 0, (x - 2) > 0$ or $(x + 3) > 0, (x - 2) < 0$
$\Rightarrow x < -3, x > 2$ (case I) (or)
$x > -3, x < 2$ (case II)

Case I

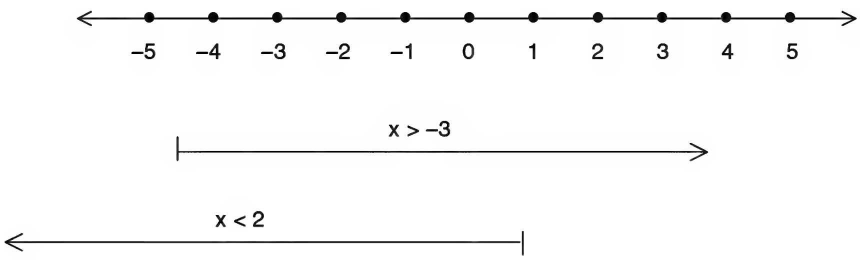
$x < -3$ and $x > 2$



There exists no value of x so that $x < -3$ and $x > 2$ (as there is no overlap of the regions). Hence in this case no value of x satisfies the given inequation.

Case II

$x > -3$ and $x < 2$



All the points in the overlapping region, i.e., $-3 < x < 2$, satisfy the inequation. Hence, the solution set of the inequation $x^2 + x - 6 < 0$ is $\{x/-3 < x < 2\}$ or $(-3, 2)$.

Example

Solve for x: $x^2 - 4x + 3 \geq 0$.

Solution

Given inequation is $x^2 - 4x + 3 \geq 0$

$\Rightarrow (x - 1)\ (x - 3) \geq 0$

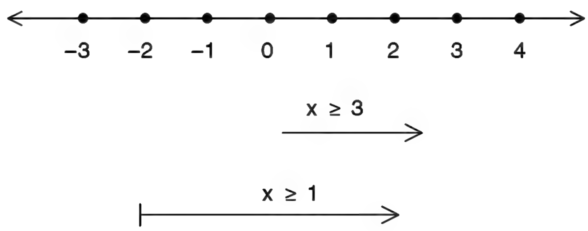
$\Rightarrow x - 1 \geq 0; x - 3 \geq 0$ or $x - 1 \leq 0; x - 3 \leq 0$

$\Rightarrow x \geq 1; x \geq 3$ (case I) (or)

$x \leq 1; x \leq 3$ (case II)

Case I

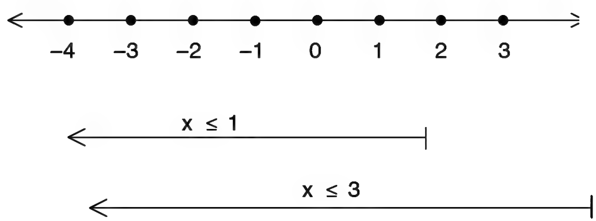
$x \geq 1$ and $x \geq 3$



All the points in the overlapping region, i.e., $x \geq 3$, satisfy the given inequation.

Case II

$x \leq 1$ and $x \leq 3$



All the points in the overlapping region, i.e., $x \leq 1$, satisfy the given inequation
Hence, the solution for the given inequation is
$x \in (-\infty, 1] \cup [3, \infty)$

Example

Solve $x^2 + 6x + 13 > 0$

Solution

Given inequation is $x^2 + 6x + 13 > 0$
Here, factorization is not possible.
Rewriting the given inequation we get,
$(x^2 + 6x + 9) + 4 > 0$
$\Rightarrow (x + 3)^2 + 4 > 0.$
We know that $(x + 3)^2 \geq 0 \ \forall \ x \in R$,
$(x + 3)^2 + 4 \geq 4 > 0 \ \forall \ x \in R$,
$\therefore$ The required solution is the set of all real numbers, i.e., $(-\infty, \infty)$.

Example

Solve $\dfrac{x^2 + 5x + 3}{x + 2} < x$.

Solution

$$\frac{x^2 + 5x + 3}{x + 2} < x$$

$$\Rightarrow \frac{x^2 + 5x + 3}{x + 2} - x < 0$$

$$\Rightarrow \frac{x^2 + 5x + 3 - x^2 - 2x}{x + 2} < 0$$

$$\Rightarrow \frac{3x + 3}{x + 2} < 0$$

$$\Rightarrow \frac{x + 1}{x + 2} < 0 \text{ ----- } (1)$$

To solve (1), it is sufficient to solve $(x + 1)(x + 2) < 0$.
We know that $(x - \alpha)(x - \beta) < 0$
$\Rightarrow \alpha < x < \beta$ where $(\alpha < \beta)$
$\therefore -2 < x < -1$
Thus, the required solution is $-2 < x < -1$

Example

Solve $\dfrac{1}{x-1} < \dfrac{-2}{1-2x}$

Solution

$$\frac{1}{x-1} < \frac{-2}{1-2x}$$

$$\Rightarrow \frac{1}{x-1} + \frac{2}{1-2x} < 0$$

$$\Rightarrow \frac{1-2x+2x-2}{(x-1)(1-2x)} < 0$$

$$\Rightarrow \frac{-1}{(x-1)(1-2x)} < 0$$

$$\Rightarrow \frac{1}{(x-1)(1-2x)} > 0 \qquad \text{-----} \qquad (1)$$

(1) holds good if $(x-1)(1-2x) > 0 \Rightarrow (x-1)(2x-1) < 0$

We know that $(x-\alpha)(x-\beta) < 0 \Rightarrow \alpha < x < \beta$ where $(\alpha < \beta)$

$\therefore$ The solution of the given inequation is $\dfrac{1}{2} < x < 1$ i.e., $x \in \left(\dfrac{1}{2}, 1\right)$

test your concepts

Very short answer type questions

1. For a quadratic equation $ax^2 + bx + c = 0$, if $a + b + c = 0$, then $x = 1$. (True/False)
2. If $a = c$, then the roots of $ax^2 + bx + c = 0$ are ________ to each other.
3. If $a^2 - b^2 > 0$, then $a > b$ or $a < -b$. (True/False)
4. If $b = 0$, then $ax^2 + bx + c = 0$ is called ________ quadratic equation.
5. If the sum of the roots of a quadratic equation is positive and the product of the roots is negative, then the numerically greater root is ________.
6. If we divide 8 into two parts such that their product is 15, then the parts are ________ and ________.
7. One side of a rectangle exceeds its other side by 2 cm. If its area is 24 cm^2, then the length and breadth are ________ and ________ respectively.
8. If the sum of a number and its reciprocal is 2, then the number is ________.

9. If we divide 4 into two parts such that the sum of their squares is 8, then the parts are _________ and _________.

10. If the product of two consecutive positive numbers is 20, then the numbers are _________ and _________.

11. The area of a right angled triangle is 32 sq.units. If its base is 4 times the altitude, then the altitude is _________.

12. The diagonal of a rectangle is 5 units. If one side is twice the other side, then the length is _________.

13. If α and β are the roots of the quadratic equation such that $\alpha > \beta$, then for $(x - \alpha)(x - \beta) > 0$, the solution set of x is _________.

14. The positive number which is less than its square by 12 is _________.

15. If the product of two successive even numbers is 48, then the numbers are _________ and _________.

16. The point $(1, -6)$ satisfies the equation $y = x^2 - x - 6$. Is the statement true?

17. The line $y = 5x - 6$ meets the parabola $y = x^2$ in two points where x-coordinates are _________ and _________.

18. The solution set of $x^2 - 4x + 4 < 0$ does not exist. (True/False)

19. If the product of Shiva's age 3 years ago and his age three years later is 16, then Shiva's present age is _________.

20. The solution set of $x^2 - 10x + 25 \geq 0$ exists. (True/False)

21. The sides of two squares are x cm and $(x + 2)$ cm. If the sum of their areas is 20 sq.cm, then the sides of the squares are _________ and _________.

22. The solution set of $x^2 - 3x - 4 \leq 0$ is _________.

23. The solution set of $x^2 - 8x + 15 \geq 0$ is _________.

24. The solution set of $x^2 - 5x + 6 > 0$ is _________.

25. Find the positive number which is less than its square by 42.

26. If $mx^2 < nx$, where m and n are positive, then find the range of x.

27. The solution set of $x^2 - x - 6 < 0$ is _________.

28. Find the dimensions of a rectangular hall, if its length exceeds its breadth by 7 m and the area of the hall being 228 m^2.

29. If $(x + 1)^2 > mx$, and $x \in Z^+$, then the possible values of m are

30. If $y = \sqrt{k\sqrt{k\sqrt{k\sqrt{k\ldots\ldots\ldots\infty}}}}$ where $k > 0$; find the value of y.

Short answer type questions

31. The equation $(m + 1)x^2 + 2mx + 5x + m + 3 = 0$ has equal roots. Find the value of m.

32. The denominator of a fraction exceeds the numerator by 3. The sum of the fraction and its multiplicative inverse is $2\frac{9}{28}$. Find the fraction.

33. Find the value of q^2 in terms of p and r so that the roots of the equation $px^2 + qx + r = 0$ are in the ratio of 3 : 4.

34. A number consists of two digits whose product is 18. If 27 is added to the number, the number formed will have the digits in reverse order, when compared to the original number. Find the number.

35. Find the minimum value of $x^2 + 12x$.

36. If the sides of a right angled triangle are x, 3x + 3, and 3x + 4, then find the value of x.

37. Find the roots of the equation $\frac{1}{x} - \frac{1}{x-a} = \frac{1}{b} - \frac{1}{b-a}$, where $a \neq 0$.

38. Find the values of x which satisfy the inequation, $x - 5 < x^2 - 3x - 50$.

39. In a boys' hostel, there are as many boys in each room as the number of rooms there are. If the number of rooms is doubled and the number of boys in each room is reduced by 10, then the number of boys in the hostel becomes 1200. Find number of rooms in the hostel.

40. If the equations $x^2 + 3x + 2 = 0$ and $x^2 + kx + 6 = 0$ have a common root, then find the values of k.

41. If $x^4 - 17x^2 + 16 = 0$, then find the sum of the squares of the roots.

42. If the quadratic equation $x^2 - mx - 4x + 1 = 0$ has real and distinct roots, then find the values of m.

43. $\sqrt{y+1} - \sqrt{y-1} = \sqrt{4y-1}$. Find the value of y.

44. Umesh and Varun are solving an equation of form $x^2 + bx + c = 0$. In doing so, Umesh commits a mistake in noting down the constant term and finds the roots as –3 and –12. And Varun commits a mistake in noting down the coefficient of x and finds the roots as –27 and –2. If so, find the roots of original equation.

45. Solve $\frac{1}{4x+4} > \frac{2}{4x-2}$.

Essay type questions

46. If $-(4x + 27) < (x + 6)^2 < -4(6 + x)$, then find all the integral values of x.

47. Determine the values of x which satisfy the simultaneous inequations $x^2 + 5x + 4 > 0$ and $-x^2 - x + 42 > 0$.

48. Solve $x^2 - 2x + 1 = 0$ graphically.

49. Draw the graph of $y = x^2 - x - 12$.

50. Solve $x^2 - x - 42 = 0$ graphically.

CONCEPT APPLICATION

Concept Application Level—1

1. The roots of the equation $3x^2 - 2x + 3 = 0$ are

(1) real and distinct.
(2) real and equal.
(3) imaginary.
(4) irrational and distinct.

2. Find the sum and the product of the roots of the equation $\sqrt{3}\,x^2 + 27x + 5\sqrt{3} = 0$.

(1) $-9\sqrt{3}$, 5
(2) $9\sqrt{3}$, 5
(3) $6\sqrt{3}$, -5
(4) $6\sqrt{3}$, 5

3. If α and β are the roots of the equation $x^2 - 12x + 32 = 0$, then find the value of $\dfrac{\alpha^2 + \beta^2}{\alpha + \beta}$.

(1) $\dfrac{-8}{3}$
(2) $\dfrac{8}{3}$
(3) $\dfrac{-20}{3}$
(4) $\dfrac{20}{3}$

4. Find the maximum or minimum value of the quadratic expression, $x^2 - 3x + 5$ whichever exists.

(1) The minimum value is $\dfrac{9}{10}$.
(2) The minimum value is $\dfrac{11}{4}$.
(3) The maximum value is $\dfrac{9}{10}$.
(4) The maximum value is $\dfrac{11}{4}$.

5. Find the values of x which satisfy the equation $\sqrt{3x + 7} - \sqrt{2x + 3} = 1$.

(1) 2, -2
(2) 4, 3
(3) 5, -1
(4) 3, -1

6. If one of the roots of a quadratic equation having rational coefficients is $\sqrt{7} - 4$, then the quadratic equation is

(1) $x^2 - 2\sqrt{7}\,x - 9 = 0$.
(2) $x^2 - 8x + 9 = 0$.
(3) $x^2 + 8x + 9 = 0$.
(4) $x^2 - 2\sqrt{7}\,x + 9 = 0$.

7. If the quadratic equation $px^2 + qx - r = 0$ $(p \neq 0)$ is to be solved by the graphical method, then which of the following graphs have to be drawn?

(1) $y = x^2$, $y = r - qx$
(2) $y = px^2$, $y = qx - r$
(3) $y = x^2$, $qx + py - r = 0$
(4) $y = x^2$, $qx - py = r$

8. If the roots of the equation $ax^2 + bx + c = 0$ are α and β, then the quadratic equation whose roots are $-\alpha$ and $-\beta$ is

(1) $ax^2 - bx - c = 0$.
(2) $ax^2 - bx + c = 0$.
(3) $ax^2 + bx - c = 0$.
(4) $ax^2 - bx + 2c = 0$.

9. Find the sum and the product of the roots of the quadratic equation $-x^2 - \dfrac{25}{3}x + 25 = 0$

(1) $\dfrac{25}{3}$, 25
(2) $\dfrac{-25}{3}$, 25
(3) $\dfrac{25}{3}$, -25
(4) $\dfrac{-25}{3}$, -25

10. For what value of k is one root of the quadratic equation $9x^2 - 18x + k = 0$ double the other?

(1) 36 (2) 9 (3) 12 (4) 8

11. The sum of a number and its square is greater than 6, then the number belongs to _______.

(1) $(-\infty, 2) \cup (3, \infty)$ (2) $(-\infty, -3) \cup (2, \infty)$ (3) $(2, 3)$ (4) $[2, 3]$

12. For which of the following intervals of x is $x^2 > \frac{1}{x^2}$?

(1) $(-\infty, -1) \cup (1, \infty)$ (2) $(-\infty, -1) \cup (1, \infty)$ (3) $(-1, 1)$ (4) $[-1, 1]$

13. If x and y are two successive multiples of 2 and their product is less than 35, then find the range of x.

(1) $\{2, 4, 0\}$
(2) $\{-6, -4, -2, 2, 4, 6\}$
(3) $\{-6, -4, -2, 0, 2, 4\}$
(4) $\{-6, -4, -2, 0, 2, 4, 6\}$

14. If $x^2 < n$, and $n \in (-\infty, 0)$, then x

(1) is any real number.
(2) is only positive number.
(3) has no value.
(4) is any negative number.

15. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$ such that x does not lie between α and β, then

(1) $a > 0$ and $ax^2 + bx + c < 0$.
(2) $ax^2 + bx + c < 0$ and $a < 0$.
(3) $a > 0$ and $ax^2 + bx + c > 0$.
(4) both (2) and (3).

16. The condition for the sum and the product of the roots of the quadratic equation $ax^2 - bx + c = 0$ to be equal, is

(1) $b + c = 0$. (2) $b - c = 0$. (3) $a + c = 0$. (4) $a + b + c = 0$.

17. The quadratic equation having rational coefficients and one of the roots as $4 + \sqrt{15}$ is

(1) $x^2 - 8x + 1 = 0$.
(2) $x^2 + x - 8 = 0$.
(3) $x^2 - x + 8 = 0$.
(4) $x^2 + 8x + 8 = 0$.

18. If α and β are the zeros of the quadratic polynomial $ax^2 + bx + c$ and x lies between α and β, then which of the following is true?

(1) If $a < 0$ then $ax^2 + bx + c > 0$.
(2) If $a > 0$ then $ax^2 + bx + c < 0$.
(3) If $a > 0$ then $ax^2 + bx + c > 0$.
(4) Both (1) and (2).

19. Find the nature of the roots of the equation $4x^2 - 2x - 1 = 0$.

(1) real and equal
(2) rational and unequal
(3) irrational and unequal
(4) imaginary

20. The solution of the inequation, $15x^2 - 31x + 14 < 0$ is given by

(1) $x \in \left(\frac{7}{5}, \infty\right)$ (2) $\frac{2}{3} < x < \frac{7}{5}$ (3) $x \in \left(\frac{7}{5}, \infty\right)$ (4) $x \in R$

21. If $mx^2 < nx$ such that m and n have opposite signs, then which of the following can be true?

(1) $x \in \left(\frac{n}{m}, \infty\right)$ (2) $x \in \left(-\infty, \frac{n}{m}\right)$ (3) $x \in \left(\frac{n}{m}, 0\right)$ (4) None of these

22. Find the range of values of x which satisfy the inequation, $(x + 1)^2 + (x - 1)^2 < 6$.

(1) $(-\sqrt{2}, \sqrt{2})$ (2) $(-1, 1)$
(3) $(-\infty, -2) \cup (2, \infty)$ (4) $(-\infty, -1) \cup (1, \infty)$

23. If the sum of the squares of three consecutive odd natural numbers is 155, then their product will be equal to

(1) 99 (2) 105 (3) 693 (4) 315

24. If $x^2 > 0$, then find the range of the values that x can take.

(1) $x = 0$ (2) $x \in R$ (3) $x \in (0, \infty)$ (4) $x \in R - \{0\}$

25. Find the range of the values of x which satisfy the inequation, $x^2 - 7x + 3 < 2x + 25$.

(1) $(-2, 11)$ (2) $(2, 11)$
(3) $(-\infty, -1) \cup (2, 11)$ (4) $(-8, -2) \cup [11, \infty)$

26. If A and B are the roots of the quadratic equation $x^2 - 12x + 27 = 0$, then $A^3 + B^3$ is ________.

(1) 27 (2) 729 (3) 756 (4) 64

27. By drawing which of the following graphs can the quadratic equation $4x^2 + 6x - 5 = 0$ be solved by graphical method?

(1) $y = x^2$, $3x - 2y - 5 = 0$ (2) $y = 4x^2$, $6x - 2y - 5 = 0$
(3) $y = x^2$, $6x - y - 5 = 0$ (4) $y = 2x^2$, $6x + 2y - 5 = 0$

28. If the quadratic equation $(a^2 - b^2)x^2 + (b^2 - c^2)x + (c^2 - a^2) = 0$ has equal roots, then which of the following is true?

(1) $b^2 + c^2 = a^2$ (2) $b^2 + c^2 = 2a^2$ (3) $b^2 - c^2 = 2a^2$ (4) $a^2 = b^2 + 2c^2$

29. Which of the following are the roots of the equation $|x|^2 + |x| - 6 = 0$?

(a) 2 (b) −2 (c) 3 (d) −3

(1) Both (a) and (b). (2) Both (c) and (d).
(3) (a), (b), (c) and (d). (4) None of the above.

30. What are the values of x which satisfy the equation, $\sqrt{5x - 6} + \frac{1}{\sqrt{5x - 6}} = \frac{10}{3}$?

(1) 3 (2) $4, \frac{11}{9}$ (3) $\frac{11}{9}$ (4) $3, \frac{11}{9}$

Concept Application Level—2

31. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β, then the equation whose roots are α^2 and β^2 is

(1) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$ (2) $a^2x^2 + b^2x + c^2 = 0$
(3) $a^2x^2 + (b^2 + 2ac)x + c^2 = 0$ (4) $a^2x^2 - (b^2 + 2ac)x + c^2 = 0$

32. If $3.2^{2x+1} - 5.2^{x+2} + 16 = 0$ and x is an integer, find the value of x.

(1) 1 (2) 2 (3) 3 (4) 4

33. If $(x + 1)(x + 3)(x + 5)(x + 7) = 5760$, find the real values of x.

(1) 5, −13 (2) −5, 13 (3) −5, −13 (4) 5, 13

34. If the roots of the equation $3ax^2 + 2bx + c = 0$ are in the ratio $2 : 3$, then

(1) $8ac = 25b$.
(2) $8ac = 9b^2$.
(3) $8b^2 = 9ac$.
(4) $8b^2 = 25ac$.

35. Find the roots of the equation $\ell^2(m^2 - n^2)x^2 + m^2(n^2 - \ell^2)x + n^2(\ell^2 - m^2) = 0$.

(1) $1, \dfrac{n^2(\ell^2 - m^2)}{\ell^2(m^2 - n^2)}$
(2) $1, \dfrac{-m^2(\ell^2 - n^2)}{\ell^2(m^2 - n^2)}$
(3) $1, \dfrac{n^2(\ell^2 + m^2)}{\ell^2(m^2 - n^2)}$
(4) $1, \dfrac{-m^2(\ell^2 + n^2)}{\ell^2(m^2 - n^2)}$

36. In writing a quadratic equation of the form $x^2 + bx + c = 0$, a student writes the coefficient of x incorrectly and finds the roots as –6 and 7. Another student makes a mistake in writing the constant term and finds the roots as 4 and 11. Find the correct quadratic equation.

(1) $x^2 + 15x - 42 = 0$
(2) $x^2 + x + 44 = 0$
(3) $x^2 - 15x - 42 = 0$
(3) $x^2 - x + 44 = 0$

37. Comment on the sign of the quadratic expression $x^2 - 5x + 6$ for all $x \in R$.

(1) $x^2 - 5x + 6 \geq 0$ when $2 \leq x \leq 3$ and $x^2 - 5x + 6 < 0$ when $x < 2$ or $x > 3$
(2) $x^2 - 5x + 6 \leq 0$ when $2 \leq x \leq 3$ and $x^2 - 5x + 6 > 0$ when $x < 2$ or $x > 3$
(3) $x^2 - 5x + 6 \leq 0$ when $-1 \leq x \leq 6$ and $x^2 - 5x + 6 > 0$ when $x < -1$ or $x > 6$
(4) $x^2 - 5x + 6 \geq 0$ when $-1 \leq x \leq 6$ and $x^2 - 5x + 6 < 0$ when $x < -1$ or $x > 6$

38. If $a - b, b - c$ are the roots of $ax^2 + bx + c = 0$, then find the value of $\dfrac{(a-b)(b-c)}{c-a}$.

(1) $\dfrac{b}{c}$
(2) $\dfrac{c}{b}$
(3) $\dfrac{ab}{c}$
(4) $\dfrac{bc}{a}$

39. The values of x for which $\dfrac{x+3}{x^2 - 3x - 54} \geq 0$ are ________.

(1) $(-6, -3) \cup (9, \infty)$
(2) $[-6, -3] \cup [9, \infty]$
(3) $(-6, -3) \cup (9, \infty)$
(4) $(-6, \infty)$

40. In a right triangle, the base is 3 units more than the height. If the area of the triangle is less than 20 sq.units, then the possible values of the base lie in the region ________.

(1) (4, 6)
(2) (3, 8)
(3) (6, 8)
(4) (5, 8)

41. A man bought 50 dozen fruits consisting of apples and bananas. An apple is cheaper than a banana. The number of dozens of apples he bought is equal to the cost per dozen of bananas in rupees and vice versa. If he had spent a total amount of Rs 1050, find the number of dozens of apples and bananas he bought respectively.

(1) 12 and 38
(2) 14 and 36
(3) 15 and 35
(4) 18 and 32

42. The values of x for which $-2x - 4 \leq (x + 2)^2 \leq -2x - 1$ is satisfied are

(1) $[-5, -1]$.
(2) $[-5, 0]$
(3) $[-5, -4] \cup [-2, -1]$.
(4) $[-5, -4] \cup [-2, -1]$.

43. If $\frac{x^2 + x - 12}{x^2 - 3x + 2} < 0$, then x lies in ________.

(1) $(-4, 3)$
(2) $(-4, 2)$
(3) $[-4, 1] \cup [2, 3]$
(4) $(-4, 1) \cup (2, 3)$

44. For all real values of x, $\frac{x^2 - \left(\frac{x}{2}\right) + 1}{x^2 + 1} - \frac{5}{4}$ is ________.

(1) equal to 1
(2) non-negative
(3) greater than $\frac{1}{4}$
(4) non-positive

45. If $x^2 - 4x + 3 > 0$ and $x^2 - 6x + 8 < 0$, then ______.

(1) $x > 3$
(2) $x < 4$
(3) $3 < x < 4$
(4) $1 < x < 2$

Concept Application Level—3

46. If the roots of the equation $2x^2 + 7x + 4 = 0$ are in the ratio p : q, then find the value of $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}$.

(1) $\pm\frac{7}{\sqrt{2}}$
(2) $\pm 7\sqrt{2}$
(3) $\pm\frac{7\sqrt{2}}{16}$
(4) $\pm\frac{7\sqrt{2}}{4}$

47. In a forest, a certain number of apes equal to the square of one-eighth of the total number of their group are playing and having great fun. The rest of them are twelve in number and are on an adjoining hill. The echo of their shrieks from the hills frightens them. They come and join the apes in the forest and play with enthusiasm. What is the total number of apes in the forest?

(1) 16
(2) 48
(3) 16 or 48
(4) 64

48. If the roots of the quadratic equation $x^2 - 2kx + 2k^2 - 4 = 0$ are real, then the range of the values of k is ________.

(1) $[-2, 2]$
(2) $[-\infty, -2] \cup [2, \infty]$
(3) $[0, 2]$
(4) None of the above

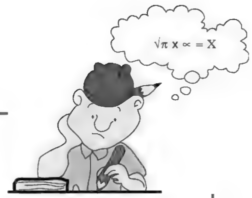
49. Find the values of x for which the expression $x^2 - (\log_5 2 + \log_2 5)\, x + 1$ is always positive.

(1) $x > \log_2 5$ or $x < \log_5 2$
(2) $\log_5 2 < x < \log_2 5$
(3) $-\log_5 2 < x < \log_2 5$
(4) $x < -\log_5 2$ or $x > \log_2 5$

50. Find the values of x which satisfy the quadratic inequation $|\,x\,|^2 - 2\,|\,x\,| - 8 \le 0$.

(1) $[-4, 4]$
(2) $[0, 4]$
(3) $[-4, 0]$
(4) $[-4, 2]$

KEY



Very short answer type questions

1. True **2.** Reciprocal
3. True **4.** Pure
5. Positive **6.** 3, 5
7. 6 cm and 4 cm **8.** 1
9. 2, 2 **10.** 4 and 5
11. 4 units **12.** $2\sqrt{5}$
13. $x < \beta$ or $x > \alpha$ **14.** 4
15. 6 and 8 **16.** Yes
17. 3, 2 **18.** True
19. 5 years **20.** True
21. 2, 4 **22.** $[-1, 4]$
23. $(-\infty, 3) \cup (5, \infty)$ **24.** $x \in (-\infty, 2) \cup (3, \infty)$
25. 7 **26.** $0 < x < \frac{n}{m}$
27. $-2 < x < 3$
28. Length = 19 m, breadth = 12 m
29. $(-\infty, 4)$
30. k

Short answer type questions

31. $\frac{-13}{4}$ **32.** $\frac{4}{7}$
33. $q^2 = \frac{49pr}{12}$ **34.** 36
35. -36 **36.** $x = 7$
37. b and $(a - b)$ **38.** $\{x / -7 < x < 9\}$
39. $\{x / x < -12 \text{ or } x > -8\}$
40. 5 or 7
41. 34
42. $m \in (-\infty, -6) \cup (-2, \infty)$
43. There is no solution for the given equation.
44. $-9, -6$
45. $x \in \left(-\infty, \frac{-5}{2}\right) \cup \left(-1, \frac{1}{2}\right)$

Essay type questions

46. $\varnothing$ **47.** $x \in (-7, -4) \cup (-1, 6)$
48. $(1, 1)$ **50.** $\{-6, 7\}$

key points for selected questions

Very short answer type questions

25. (i) Let the number be x.
(ii) frame the equation i.e., $x^2 - x - 42 = 0$
(iii) Solve the above equation for x by factorization.

26. Given, $\frac{n}{m}$ is always positive.

28. (i) Given $\ell - b = 7$ and $\ell b = 228$.
(ii) Substitute $\ell = b + 7$ in $\ell b = 228$.
(iii) Then solve the quadratic equation obtained.

29. $m < \frac{(x+1)^2}{x}$.

30. (i) Square on both the sides of given equation and put $\sqrt{c\sqrt{c\sqrt{c \ldots\ldots \infty}}} = y$
(ii) Solve the above equation for y.

Short answer type questions

31. (i) If $ax^2 + bx + c = 0$ has equal roots, then $b^2 = 4ac$.
(ii) Given, $a = m + 1$, $b = 2m + 5$ and $c = m + 3$.

32. (i) Let the fraction be $\frac{n}{d}$.
(ii) Given $d - n = 3$ $\frac{n}{d} + \frac{d}{n} = \frac{65}{28}$
(iii) Substitute $d = n + 3$ in second equation.
(iv) Then solve the quadratic equation in n.

33. (i) Let the roots of $px^2 + qx + r = 0$ be 3m and 4m.
(ii) $3m + 4m = \frac{-q}{p}$ and $(3m)(4m) = \frac{r}{p}$
(iii) Substitute, $m = \frac{-q}{7p}$ in second equation and find q^2.

34. (i) Let the two digit number be $10x + y$.
(ii) Given $xy = 18$ and $10x + y + 27 = 10y + x$ $\Rightarrow y - x = 3$
(iii) Substitute $y = x + 3$ in first equation and solve for x and y.

35. (i) Minimum value of $ax^2 + bx + c$ $(a > 0)$ is $\frac{4ac - b^2}{4a}$
(ii) Given $a = 1$, $b = 12$

36. (i) Apply Pythagoras theorem i.e., $x^2 + (3x + 3)^2 = (3x + 4)^2$
(ii) Simplify and solve the quadratic equation obtained above.

37. (i) Simplify LHS and RHS and frame the quadratic equation.
(ii) Solve the above equation for x.

38. (i) Factorize $x^2 - 2x - 63$
(ii) Use, $(a - x)(x - b) < 0 \Rightarrow a < x < b$ (where $a < b$) and find x.

39. (i) Factorize $x^2 + 20x + 96$
(ii) Use, $(x - a)(x - b) > 0 \Rightarrow x < a$ or $x > b$ $(a < b)$ and find the solution set of x.

40. (i) Solve $x^2 + 3x + 2 = 0$ for x.
(ii) Substitute the values of x in the other equation and find k.

41. (i) Put $x^2 = y$ and reframe the equation in y.
(ii) Then solve for y and then replace x^2 by y and solve for x.

42. (i) If $ax^2 + bx + c = 0$ has real and distinct roots, then $b^2 - 4ac > 0$
(ii) Given $a = 1$, $b = (m + 4)$ and $c = 1$.

43. (i) Squaring the given equation until we get quadratic equation in y.
(ii) Solve it.

44. (i) In case of Umesh sum of the roots is correct.
(ii) In case of Varun product of the roots is correct.
(iii) Use, x^2 – (sum of the roots)x + product of the roots = 0 and write the equation.

45. (i) Reframe the inequation as $\frac{2}{4x - 2} - \frac{1}{4x + 4} < 0$
(ii) The solution of $\frac{x - a}{x - b} < 0$ is same as the solution of $(x - a)(x - b) < 0$.

Essay type questions

46. (i) Solve $-(4x + 27) < (x + 6)^2$ and $(x + 6)^2 < -4(6 + x)$.
(ii) Then take common solution.

47. (i) Use, $(x - a)(x - b) > 0 \Rightarrow x < a$ or $x > b$ $(a < b)$ and
(ii) Find the solutions of given inequations individually and then take their common solution.

48. (i) Reframe the equation as $x^2 = 2x - 1$
(ii) Let $y = x^2$ and $y = 2x - 1$
(iii) Draw the parabola and line which represents $y = x^2$ and $y = 2x - 1$ respectively.
(iv) The x-co-ordinates of the common points of the parabola and the line is the solution.

49. (i) Find the values of y for the different values (+ve and –ve) of x.
(ii) Then plot the points on the graph sheet.

50. (i) As $y = x^2$, reframe the quadratic equation as $y = x + 42$
(ii) Draw the parabola and line which represents $y = x^2$ and $y = x + 42$ respectively.
(iii) The x-co-ordinates of the common points of the parabola and the line is the solution.

KEY

Concept Application Level—1,2,3

1. 3	**2.** 1
3. 4	**4.** 2
5. 4	**6.** 3
7. 3	**8.** 2
9. 4	**10.** 4
11. 2	**12.** 2
13. 3	**14.** 3
15. 4	**16.** 2
17. 1	**18.** 4
19. 3	**20.** 2
21. 3	**22.** 1
23. 4	**24.** 4
25. 1	**26.** 3
27. 4	**28.** 2
29. 1	**30.** 4
31. 1	**32.** 1
33. 1	**34.** 4
35. 1	**36.** 3
37. 2	**38.** 2
39. 1	**40.** 2
41. 3	**42.** 4
43. 4	**44.** 4
45. 3	**46.** 4
47. 3	**48.** 1
49. 1	**50.** 1

Concept Application Level—1,2,3

Key points for select questions

1. Find the discriminant.

2. Use $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

3. Find the roots α and β or write $\alpha^2 + \beta^2$ as $(\alpha + \beta)^2 - 2\alpha\beta$ and $\alpha + \beta$ = sum of roots and $\alpha\beta$ product of roots.

4. Use the formula and check the coefficient of x^2.

5. Square both sides.

6. If $a + \sqrt{b}$ is one root then $a - \sqrt{b}$ is other root.

7. Take the first degree expression and the constant on the other side.

8. Use $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

9. Use $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

10. Simplify and solve by finding $\alpha + \beta$ and $\alpha\beta$. Where $\beta = 2\alpha$.

11. Let the number be x and frame the inequations.

12. Simplify and solve the inequation.

13. Assume the two successive multiples as n and n + 2.

14. x^2 is always positive and n is always negative.

16. Find the sum and the product of the roots then equate them.

17. If one of the roots is $a + \sqrt{b}$, this the other roots is $a - \sqrt{b}$.

19. Use the formula of discriminant.

20. Factorise the LHS of the inequation.

21. Take nx on the left hand side and solve.

22. Simplify and solve the inequation.

23. Let the three consecutive odd natural numbers be $x - 2$, x and $x + 2$.

24. Observe the options.

25. Simplify and solve the inequation.

26. Find the roots or write $A^3 + B^3$ in terms of $A + B$ and AB.

27. Take $6x - 5$ on other side.

28. Observe the coefficient of each term and guess one root.

29. (i) Put $|x| = y$ and frame the equation.
 (ii) Solve for y.

30. Find the LCM and then square on both sides.

31. Use $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

32. (i) Let $2^x = P$.
 (ii) Frame the quadratic equation in terms of P and solve it.

33. (i) (a) (b) (c) (d) = e can be written as (a d) (b c) = e
 (ii) Write the equation in the form of a quadratic equation.

34. If the roots of $ax^2 + bx + c = 0$ are in the ratio $m : n$, then $(m + n)^2 a.c = mnb^2$.

35. In the equation $ax^2 + bx + c = 0$ when $a + b + c = 0$, then the roots are 1 and $\frac{c}{a}$.

36. (i) Use sum of roots $= \frac{-b}{a}$, product of the roots $= \frac{c}{a}$.
 (ii) Identify that the first student got the correct product and the second student got the correct sum.

37. $(x - a)(x - b) \le 0$ when $a \le x \le b$.
 $(x - a)(x - b) \ge 0$ when $x \le a$ or $x \ge b$.

38. (i) $(a - b)(b - c)$ = product of the roots $= \frac{c}{a}$.
 (ii) $c - a = -(a - b + b - c) = -$ (sum of the roots) $= \frac{b}{a}$.

39. (i) Express the quadratic equations given in the numerator and the denominator as the product of two linear factors.
 (ii) Form the various regions by using critical points
 (iii) Identify the regions in which the inequality holds good.

40. (i) Let the height of the triangle be x. Hence base = x + 3.
 (ii) Form the quadratic inequation.

41. Form a quadratic equation in terms of apples or bananas.

42. Form two different inequalities and find the common solution to them.

43. (i) Express the quadratic equations given in the numerator and the denominator as the product of two linear factors.
 (ii) Form the various regions by using critical points.
 (iii) Identify the regions in which the given inequality holds good.

44. Simplify the expression and identify whether it is non-positive or non-negative.

45. (i) If $(x - a)(x - b) > 0$, then $x \in (-\infty, a) \cup (b, \infty)$.
 If $(x - a)(x - b) < 0$, then $x \in (a, b)$.
 (ii) Take the common range of both the inequations.

46. (i) Let the roots be pk and qk.
 (ii) Use sum of roots $= \frac{-b}{a}$ and product of roots $= \frac{c}{a}$.

47. (i) Let the number of apes be equal to n.

(ii) The number of apes which are on the adjoining hills is $n - \frac{n^2}{64}$, which is equal to 12.

48. Discriminant ≥ 0.

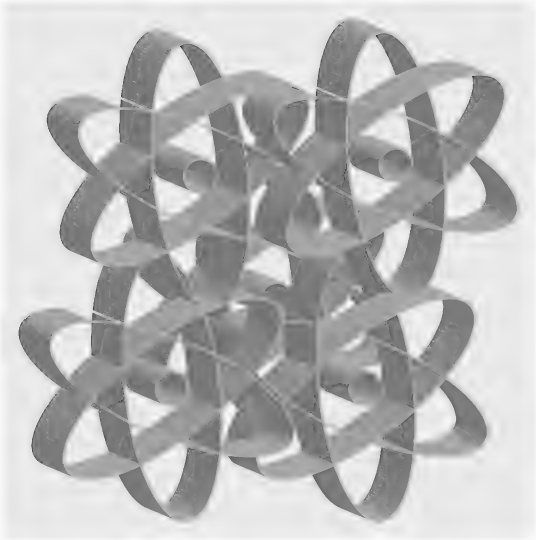
49. (i) $x^2 - (\log_5 2 + \log_2 5)\, x + 1 > 0$

(ii) $x^2 - (\log_5 2 + \log_2 5)\, x + 1$
$= (x - \log_5 2)\,(x - \log_2 5)$ and proceed.

50. (i) Put $|x| = y$

(ii) Form a quadratic inequation in terms of y.

(iii) If $(x - a)\,(x - b) \leq 0$ then $x \in (a, b)$.



CHAPTER 5

Statements

INTRODUCTION

In this chapter, we shall learn about statements, truth tables of different compound statements, laws of algebra of statements, application of truth tables to switching networks etc.

Statement

A sentence with can be judged either true or false but not both is called a **statement**. Statements are denoted by lower case letters like p, q, r etc.

☛ *Examples*

1. p : 2 is a prime number. This statement is true.
2. q : $2 + 3 = 6$. This statement is false.

Truth value

The truthness or falsity of a statement is called its truth value.
Truthness of a statement is denoted by T, while its falsity is denoted by F.

☛ *Examples*

1. The truth value of the statement p : The sun rises in the east, is True.
2. The truth value of the statement q : All odd numbers are prime, is False.

Negation of a statement

The denial of a statement is called its negation. Negation of a statement p is denoted by ~p and read as *not p* or *negation p*.

Truth table

p	~ p
T	F
F	T

Examples

1. p : 2 + 4 = 6
   ~ p : 2 + 4 ≠ 6
2. p : 3 is a factor of 10
   ~ p : 3 is not a factor of 10
3. p : Charminar is in Delhi
   ~ p : Charminar is not in Delhi

Compound statement

A statement obtained by combining two or more simple statements using connectives is called a compound statement.

Examples

Consider the two statements.

p : 2 is a prime number and

q : 2 is an even number.

Some compound statements that can be formed by using the statements p and q are:

(i) 2 is a prime number and 2 is an even number.
(ii) 2 is a prime number or 2 is an even number.
(iii) 2 is neither a prime number nor an even number.

Let us look at some basic compound statements.

1. Conjunction

If p and q are two simple statements, the compound statement *p and q* is called the conjunction of p and q. It is denoted by $p \wedge q$.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

We observe that $p \wedge q$ is true only when both p and q are true.

☛ *Examples*

(i) Let p : 4 is a perfect square and q : 2 is an odd number.
p ∧ q : 4 is a perfect square and 2 is an odd number.

As p is T and q is F, the truth value of p ∧ q is F.

(ii) Let p : 3 > 2 and q : $\sqrt{2}$ is an irrational number.

Then, p ∧ q : 3 > 2 and $\sqrt{2}$ is an irrational number.

The truth value of p ∧ q is true as both p and q are true.

2. Disjunction

If p and q are two simple statements, then the compound statement *p or q* is called the disjunction of p and q. It is denoted by p ∨ q.

Truth table

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

We observe that, p ∨ q is false only when both p and q are false.

☛ *Examples*

(i) Let p: The set of even primes is an empty set.
q: 1 is a factor of every natural number.
p ∨ q: The set of even primes is an empty set or 1 is a factor of every natural number.

As p is false and q is true, truth value of p ∨ q is T.

(ii) Let p: 5 is a factor of 18.
q: 12 divides 6.
Then, p ∨ q : 5 is factor of 18 or 12 divides 6.

The truth value of p ∨ q is false as both p and q are false.

3. Implication or conditional

If p and q are two statements, the compound statement *if p then q*, is called a conditional statement. It is denoted by $p \Rightarrow q$.

The statement p is called the hypothesis (or given) and the statement q is called the conclusion (or result).

Truth table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

We observe that, a true statement cannot imply a false statement.

Examples

(i) Let p: Every set is a subset of itself.
q: $3 + 5 = 8$
$p \Rightarrow q$: If every set is a subset of itself, then $3 + 5 = 8$.

As p is true and q is true, the truth value of $p \Rightarrow q$ is true.

(ii) Let p: ABC is a right triangle if $\angle A = 100°$.
q: $\angle A + \angle B + \angle C = 180°$
$p \Rightarrow q$: If ABC is a right triangle if $\angle A = 100°$, then $\angle A + \angle B + \angle C = 180°$.

As p is false and q is true, the truth value of $p \Rightarrow q$ is true.

4. Bi-conditional or bi-implication

If p and q are two statements, then the compound statement *p if and only if q* is called the bi-conditional of p and q. It is denoted by $p \Leftrightarrow q$.

Truth table

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We observe that, $p \Leftrightarrow q$ is true if both p and q have the same truth values.

Examples

1. Let p: $2 \times 3 = 6$
   q: $2 + 8 = 10$
   $p \Leftrightarrow q$: $2 \times 3 = 6$ if and only if $2 + 8 = 10$.

   Since both p and q are true, the truth value of $p \Leftrightarrow q$ is T.

2. Let p: Every triangle is equilateral

   q: Charminar is in Hyderabad
   $p \Leftrightarrow q$: Every triangle is equilateral if and only if Charminar is in Hyderabad.

   As p is false and q is true, the truth value of $p \Leftrightarrow q$ is F.

Converse, inverse and contrapositive of a conditional
Let $p \Rightarrow q$ or *if p then q* be a conditional,

(i) If q then p i.e., $q \Rightarrow p$, is called the converse of $p \Rightarrow q$.
(ii) If not p then not q i.e., $\sim p \Rightarrow \sim q$, is called the inverse of $p \Rightarrow q$
(iii) If not q then not p i.e., $\sim q \Rightarrow \sim p$ is called the contrapositive of $p \Rightarrow q$.

Truth table

[redacted]

T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Examples

1. Write the converse, inverse and contrapositive of the conditional "If x is odd then x^2 is odd".

Solution

Conditional: If x is odd, then x^2 is odd.

Converse : If x^2 is odd, then x is odd.

Inverse : If x is not odd, then x^2 is not odd.

Contrapositive : If x^2 is not odd, then x is not odd.

2. Write the converse, and the contrapositive of the conditional, "If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$".

Solution

Conditional : If ABC is a triangle, then $\angle A + \angle B + \angle C = 180°$.

Converse : If $\angle A + \angle B + \angle C = 180°$, then ABC is a triangle.

Contrapositive : If $\angle A + \angle B + \angle C \neq 180°$, then ABC is not a triangle.

Examples

Let us now look at the truth tables of some compound statements:

1. The truth table of $p \vee \sim q$:

p	q	~ q	p ∨ ~ q
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2. The truth table of $\sim p \vee (p \wedge q)$:

p	q	~p	p ∧ q	~p ∨ (p ∧ q)
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

3. Write the truth table of $\sim p \Rightarrow p \vee q$.

Solution

p	q	~ p	p ∨ q	~p ⇒ p ∨ q
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

Tautology

A compound statement which always takes **True** as its truth value is called a tautology.

☛ *Examples*

1. The truth table of $p \vee \sim p$ is

p	~ p	p ∨ ~ p
T	F	T
F	T	T

We observe that $p \vee \sim p$ takes T as its truth value always. So, $p \vee \sim p$ is a tautology.

2. Show that the compound statement $p \Rightarrow p \vee q$ is a tautology.

Solution

Truth table

P	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

We observe that $p \Rightarrow p \vee q$ is always True.
Hence, $p \Rightarrow p \vee q$ is a tautology.

Contradiction

A compound statement which always takes **False** as its truth value is called a contradiction.

Examples

1. The truth table of $p \wedge \sim p$ is

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

We observe that $p \wedge \sim p$ takes F as its truth value always. So $p \wedge \sim p$ is a contradiction.

2. Show that the compound statement $(p \vee \sim p) \Rightarrow (q \wedge \sim q)$ is a contradiction.

Solution

Truth table

p	q	$\sim p$	$\sim q$	$p \vee \sim p$	$q \wedge \sim q$	$p \wedge \sim p \Rightarrow q \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

We observe that $p \vee \sim p \Rightarrow q \vee \sim q$ is always False.
Hence, $p \vee \sim p \Rightarrow q \wedge \sim q$ is a contradiction.

Contingency

A compound statement which is neither a tautology nor a contradiction is called a contingency.

Example

$p \vee q \Rightarrow \sim p$

Truth table

p	q	~p	$p \vee q$	$p \vee q \Rightarrow \sim p$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Logically equivalent statements

Two statements *r* and *s* are said to be logically equivalent, if the last columns of their truth tables are identical. (OR)

Two statements *r* and *s* are said to be logically equivalent if $r \Leftrightarrow s$ is a tautology. Generally, r and s will be compound statements.

If the statements *r* and *s* are logically equivalent, then we denote this as $r \equiv s$.

Note: that $r \Leftrightarrow s$ is always true only if both *r* and *s* have same truth values.

Examples

1. Show that $p \wedge q \equiv q \wedge p$.

Solution

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

We observe that $p \wedge q$ and $q \wedge p$ have the same truth values. Hence, $p \wedge q \equiv q \wedge p$.

2. Show that $p \Rightarrow q \equiv \sim p \vee q$.

Solution

p	q	~p	$p \Rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that $p \Rightarrow q$ and $\sim p \vee q$ have the same truth values.
Hence, $p \Rightarrow q \equiv \sim p \vee q$

Laws of algebra of statements

Some logical equivalences are listed under the following laws:

1. **Commutative Laws**
   (a) $p \vee q \equiv q \vee p$
   (b) $p \wedge q \equiv q \wedge p$

2. **Associative laws**
   (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3. **Distributive Laws**
   (a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   (b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

4. **Idempotent Laws**
   (a) $p \vee p \equiv p$
   (b) $p \wedge p \equiv p$

5. **De Morgan's laws**
   (a) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
   (b) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

6. **Identity Laws**
   (a) $p \vee f \equiv p$, $p \vee t \equiv t$.
   (b) $p \wedge f \equiv f$, $p \wedge t \equiv p$.

7. **Complement Laws**
   (a) $p \vee (\sim p) \equiv t$ (b) $p \wedge (\sim p) \equiv f$ (c) $\sim(\sim p) \equiv p$
   (d) $\sim t \equiv f$ (e) $\sim f \equiv t$

List of equivalences based on implications

(i) $p \Rightarrow q \equiv \sim p \vee q$
(ii) $\sim(p \Rightarrow q) \equiv p \wedge \sim q$
(iii) $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$
(i.e., a conditional and its contrapositive are logically equivalent)
(iv) $q \Rightarrow p \equiv \sim q \Rightarrow \sim p$
(i.e., converse and inverse of a conditional are logically equivalent)
(v) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$
(vi) $\sim(p \Leftrightarrow q) \equiv (\sim p) \Leftrightarrow q$ or $p \Leftrightarrow (\sim q)$

Open sentence

A sentence involving one or more variables is called an open sentence, if it becomes TRUE or FALSE when the variables are replaced by some specific values from the given set. The set from which the values of a variable can be considered is called the replacement set or domain of the variable.

Examples

1. $x + 2 = 9$ is an open sentence.
   For $x = 7$, it becomes True and for other real values of x it becomes False.

2. $x^2 + 1 > 0$ is an open sentence.
   For all real values of x it is True.

Quantifiers

A quantifier is a word or phrase which quantifies a variable in the given open sentence.
There are two types of quantifiers.
(a) Universal quantifier.
(b) Existential quantifier.

Universal quantifier

The quantifiers like *for all*, *for every*, *for each* are called universal quantifiers. A universal quantifier is denoted by '$\forall$'.

☛ *Examples*

1. Consider an open sentence, $|x| \geq 0$.
   This is true for all $x \in R$. So, we write $|x| \geq 0, \forall x \in R$.
2. Consider the sentence,
   $$1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}, \forall n \in N.$$

Existential quantifier

The quantifiers like *for some*, *not all*, *there is/exists at least one* are called existential quantifiers.
An existential quantifier is denoted by '$\exists$'.

☛ *Examples*

1. Not all prime numbers are odd.
2. $\exists x \in R$ such that $x + 4 = 11$

Negation of statements involving quantifiers

1. p: All odd numbers are prime.
   ~p: Not all odd numbers are prime
   (or)
   Some odd numbers are not prime.
   (or)
   There is an odd number which is not prime.
2. p: All questions are difficult.
   ~ p: Not all questions are difficult.
   (or)
   Some questions are not difficult.
   (or)
   There is at least one question which is not difficult.
3. p: All birds can fly
   ~ p: Not all birds can fly
   (or)

There are some birds which cannot fly.
(or)
There is at least one bird which cannot fly.

Methods of proof

Statements in mathematics are usually examined for their validity. The various steps involved in the process is referred as proof.

There are two important types of proofs

1. **Direct Proof**
   In this method, we begin with the hypothesis and end up with the desired result through a logical sequence of steps.

☛ *Example*

If x is odd, then x^2 is odd.
Given, x is an odd number
Conclusion: x^2 is an odd number

Proof

$\Rightarrow x = 2k + 1$ for some $k \in Z$
$\Rightarrow x^2 = (2k + 1)^2$
$= 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
$= 2m + 1$, where $m = 2k^2 + 2k \in Z$ ($\because k \in Z \Rightarrow 2k^2 + 2k \in Z$)
$\therefore x^2 = 2m + 1, m \in Z$
$\Rightarrow x^2$ is an odd number.
Hence, if x is an odd number, then x^2 is an odd number.

2. **Indirect proof**
   In this method, we proceed by assuming that the conclusion is false. Then we arrive at a contradiction. This implies that the desired result must be true.

☛ *Example*

If $a + b = 0$, then $(a + b)^2 = 0$. where $a, b \in Z - \{0\}$
Given: $a + b = 0$
Conclusion: $(a + b)^2 = 0$

Proof
Let us assume that $(a + b)^2 \neq 0$
$\Rightarrow (a + b)^2 > 0$ (As $(a + b)^2$ cannot be –ve)
$\Rightarrow a + b > 0$ or $a + b < 0$
which is a contradiction to the hypothesis i.e., $a + b = 0$
$\therefore$ Our assumption i.e., $(a + b)^2 \neq 0$ is false.
Hence, if $a + b = 0$, then $(a + b)^2 = 0$.

Methods of disproof

To disprove a given statement there are two methods.

1. **Counter example method**
   In this method, we look for a counter example which disproves the given statement.

Example

(i) Every odd number is a prime number.
This statement is false, as 9 is an odd number but it is not a prime.

(ii) $x^2 - x - 6 = 0$ for all real values of x.
This statement is false, as for $x = 2$,
$x^2 - x - 6 = (2)^2 - 2 - 6 = -4 \neq 0$
$\therefore$ $x = 2$ is a counter example here.

2. **Method of contradiction**
   In this method, we assume that the given statement is true. Then we arrive at a contradiction. This implies that the given statement is false.

Example

Disprove the statement, "There can be two right angles in a triangle".

Solution

Let ABC be a triangle.
If possible, let $\angle A = 90°$ and $\angle B = 90°$.
We know that, the sum of the three internal angles of a triangle is 180°.
i.e., $\angle A + \angle B + \angle C = 180°$
$\Rightarrow 90° + 90° + \angle C = 180°$
$\Rightarrow \angle C = 0°$ which is a contradiction.
Hence, there cannot be two right angles in a triangle.

Application to switching networks

Now we consider the statements p and p^1 as switches with the property that if one is on, then the other is off and vice versa.

Further, a switch allows only two possibilities. They are

(i) it is either open (F) in which case there is no flow of current.
(or)
(ii) it is closed (T) in which case there is a flow of current.
Hence, every switch has two truth values T or F only.
Let p and q denote two switches. We can connect p and q by using a wire in a series or parallel combination as shown below.

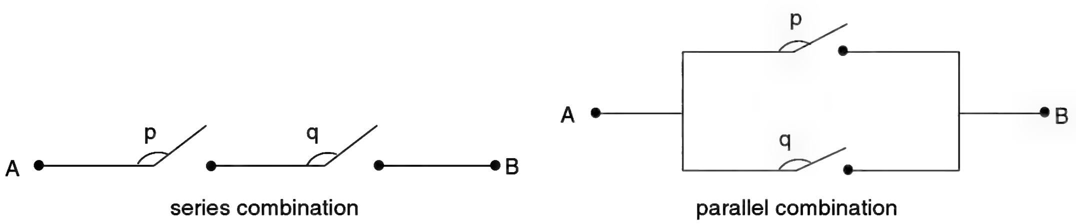


Figure 5.1

Note: p ∧ q denote the series combination and p ∨ q denote the parallel combination.

Switching network

A switching network is a repeated arrangement of wires and switches in series and parallel combinations. So, such a network can be described by using the connectives ∧ and ∨.

Example

1. Describe the behaviour of flow of current from A to B in the following circuit network.

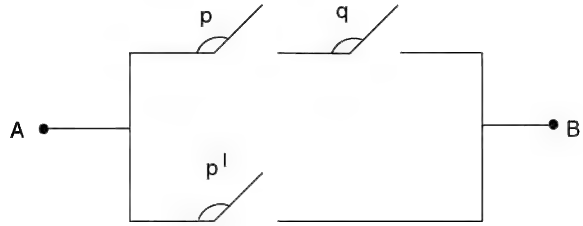


Figure 5.2

Solution

The given network can be described by the compound statement (p ∧ q) ∨ p'. Truth table of (p ∧ q) ∨ p' is:

p	q	p'	p ∧ q	(p ∧ q) ∨ p'
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

So, current flows from A to B if

(i) p is closed, q is closed,

(ii) p is open, q is closed` and

(iii) p is open, q is open.

test your concepts

Very short answer type questions

1. If truth value of (~P) is T, then truth value ~(~P) is ________.

2. ~(p ∨ ~q) ≡ ________.

3. Truth value of "If x is even, then x^2 is even" is ________.

4. Counter example which disproves "all primes are odd" is ________.

5. The negation of "no dog barks" is ________.

6. The truth value of "Hyderabad is the capital city of A.P." is ________.

7. $\sim(\sim p \Rightarrow q) = \sim(p \vee q)$ (True/False)

8. If p is F and q is T, then $\sim q \Rightarrow p$ is ________.

9. The quantifier used in negation of "every planet in the solar system has a satellite". ($\forall$ or $\exists$)

10. Truth value of "$3 \times 7 = 28$ "iff" $3 + 7 = 12$" is ________.

11. The negation of "$a > b$" is $a = b$. (True/False)

12.

The current ________ (does/does not) flow from A to B.

13. The quantifier to be used to describe the statement, "not all isosceles triangles are equilateral is ________. ($\forall$ or $\exists$)"

14. If $p \vee \sim q$ is F, then q is ________.

15. The connective used in the negation of "if the grass is green, then sky is blue" is ________.

16. Write the conjunction and implication of the statement: He is smart; He is intelligent.

17. Write the suitable quantifier for the sentence: there exists a real number x such that $x + 2 = 3$.

18. Find the truth value of "Are you attending the meeting tomorrow?".

19. The symbolic form of the statement, "If p, then neither q nor r" is

20. Find the inverse of the conditional, "If I am tired, then I will take rest".

21. The converse of converse of the statement $p \Rightarrow \sim q$ is ______.

22. Find the truth value of the statement, "The sum of any two odd numbers is an odd number".

23. Find the negation of the statement, "Some odd numbers are not prime".

24. What is the truth value of the statement, $2 \times 3 = 6$ or $5 + 8 = 10$?

25. $p \Rightarrow q$ is logically equivalent to ______.

26. The counter example of the statement, "All odd numbers are primes", is

27. Find the converse of the statement, "If ABCD is square, then it is a rectangle".

28. Write the compound statement, "If p, then q and if q, then p" in symbolic form.

29. The negation of the statement, "I go to school everyday", is

30. The contrapositive of the statement $p \Rightarrow \sim q$ is

Short answer type questions

31. What is the converse of the statement $p \Rightarrow p \vee q$?

32. Write the converse, inverse and contrapositive of the conditional, "If she is rich, then she is happy".

33. Show that $p \Rightarrow p \vee q$ is a tautology.

34. Write the converse, inverse and contra positive, of the statement "In a Δ ABC, if $AB \neq AC$, then $\angle B \neq \angle C$".

35. Show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

36. Write the truth table of $(p \vee q) \vee \sim r$

37. Show that $p \Rightarrow q \equiv \sim p \vee q$.

38. Write the truth table of $p \wedge \sim q$.

39. Show that $(p \wedge q) \vee \sim q \equiv p \vee \sim q$

40. Write the truth table of $p \Rightarrow (p \wedge q)$.

41. Write the suitable quantifier for the following sentence

(a) $x + 1 > x$ for all real values of x.

(b) there is no real number x such that $x^2 + 2x + 2 = 0$.

42. Prove that $(\sim p \wedge q) \wedge q$ is neither a tautology nor a contradiction.

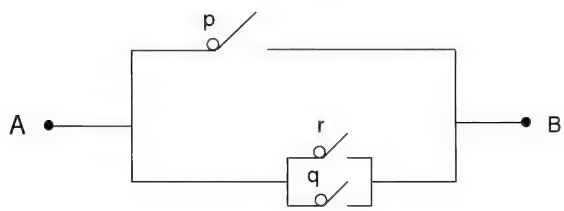
43. Write the truth table of $\sim p \vee (p \wedge q)$.

44. Show that $(p \wedge \sim p) \wedge (p \vee q)$ is a contradiction.

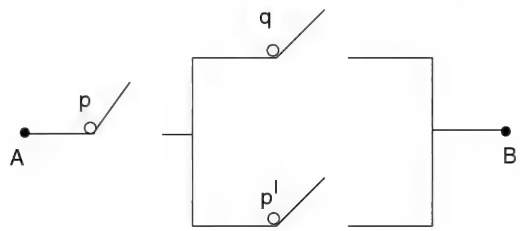
45. Negation of the compound statement $[(p \wedge q) \vee (p \wedge \sim q)]$ is

Essay type questions

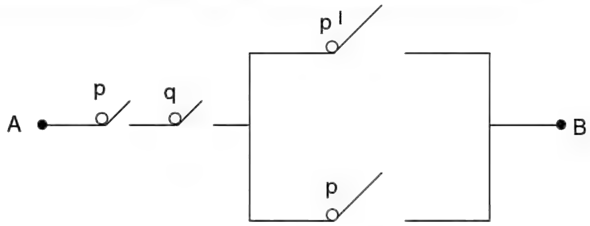
46. Discuss when does the current flow from A to B in the given network.



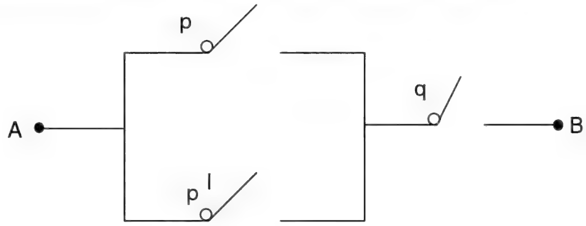
47. Discuss when does the current flow from A to B in the network given.



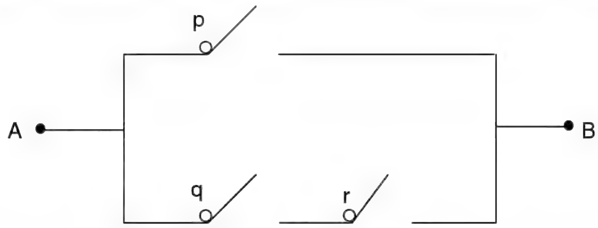
48. Discuss when does the current flow from A to B in the network given.



49. Discuss when does the current flow from A to B in the network given.



50. Discuss when does the current flow from the points A to B in the network given.



CONCEPT APPLICATION

Concept Application Level—1

1. For which of the following cases does the statement $p \wedge \sim q$ take the truth value as true?

(1) p is true, q is true
(2) p is false, q is true
(3) p is false, q is false
(4) p is true, q is false

2. Which of the following sentences is a statement?

(1) Ramu is a clever boy
(2) What are you doing?
(3) Oh! It is amazing
(4) Two is an odd number.

3. Which of the following laws does the connective $\wedge$ satisfy?

(1) Commutative law
(2) Idempotent law
(3) Associative law
(4) All the above

4. The truth value of the statement, "We celebrate our Independence day on August 15th", is

(1) T (2) F (3) neither T nor F (4) Cannot be determined

5. When does the inverse of the statement $\sim p \Rightarrow q$ results in T?

(1) $p = T, q = T$ (2) $p = T, q = F$ (3) $p = F, q = F$ (4) Both (2) and (3).

6. Which of the following is a contradiction?

(1) $p \vee q$ (2) $p \wedge q$ (3) $p \vee \sim p$ (4) $p \wedge \sim p$

7. In which of the following cases, $p \Leftrightarrow q$ is true?

(1) p is true, q is true (2) p is false, q is true (3) p is true, q is false (4) None of these

8. Find the counter example of the statement "Every natural number is either prime or composite".

(1) 5 (2) 1 (3) 6 (4) None of these

9. Which of the following pairs are logically equivalent?

(1) Conditional, Contrapositive (2) Conditional, Inverse (3) Contrapositive, Converse (4) Inverse, Contrapositive

10. The property $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ is called

(1) associative law (2) commutative law (3) distributive law (4) idempotent law

11. Which of the following is contingency?

(1) $p \vee \sim p$ (2) $p \wedge q \Rightarrow p \vee q$ (3) $p \wedge (\sim q)$ (4) None of these

12. Which of the following pairs are logically equivalent?

(1) Converse, Contrapositive (2) Conditional, Converse (3) Converse, Inverse (4) Conditional, Inverse

13. Which of the following connectives can be used for describing a switching network?

(1) $(p \wedge q) \vee p'$. (2) $(p \vee q) \wedge q'$. (3) Both (1) and (2). (4) None of these

14. Find the quantifier which best describes the variable of the open sentence $x + 3 = 5$.

(1) Universal (2) Existential (3) Neither (1) nor (2) (4) Cannot be determined

15. Which of the following is equivalent to $p \Leftrightarrow q$?

(1) $p \Rightarrow q$ (2) $q \Rightarrow p$ (3) $(p \Rightarrow q) \wedge (q \Rightarrow p)$ (4) None of these

16. The property $\sim (p \wedge q) \equiv \sim p \vee \sim q$ is called _________.

(1) associative law (2) De morgan's law (3) commutative law (4) idempotent law

17. The statement $p \vee q$ is

(1) a tautology (2) a contradiction (3) neither a tautology nor a contradiction (4) Cannot say

18. Which of the following compound statement represents a series network?

(1) $p \vee q$ (2) $p \Rightarrow q$ (3) $p \wedge q$ (4) $p \Leftrightarrow q$

19. Which of the following is a tautology?

(1) $p \wedge q$ (2) $p \vee q$ (3) $p \vee \sim p$ (4) $p \wedge \sim p$

20. Find the truth value of the compound statement, 'If 2 is a prime number, then hockey is the national game of India'.

(1) T (2) F

(3) Neither T nor F (4) Cannot be determined

21. Find the truth value of the compound statement, 4 is the first composite number and $2 + 5 = 7$.

(1) T (2) F

(3) Neither T nor F (4) Cannot be determined

22. Find the inverse of the statement, "If ΔABC is equilateral, then it is isosceles".

(1) If ΔABC is isosceles, then it is equilateral.

(2) If ΔABC is not equilateral, then it is isosceles.

(3) If ΔABC is not equilateral, then it is not isosceles.

(4) If ΔABC is not isosceles, then it is not equilateral.

23. The statement $p \Rightarrow p \vee q$ is

(1) a tautology (2) a contradiction

(3) both tautology and contradiction (4) neither a tautology nor a contradiction

24. What is the truth value of the statement 'Two is an odd number iff 2 is a root of $x^2 + 2 = 0$'?

(1) T (2) F

(3) Neither T nor F (4) Cannot be determined

25. Which of the following connectives satisfy commutative law?

(1) $\wedge$ (2) $\vee$ (3) $\Leftrightarrow$ (4) All the above

26. $\sim [\sim p \wedge (p \Leftrightarrow q)] \equiv$

(1) $p \vee q$ (2) $q \wedge p$ (3) T (4) F

27. Write the negation of the statement "If the switch is on, then the fan rotates".

(1) "If the switch is not on, then the fan does not rotate".

(2) "If the fan does not rotate, then the switch is not on".

(3) "The switch is not on or the fan rotates".

(4) "The switch is on and the fan does not rotate".

28. If p:The number of factors of 20 is 5 and q : 2 is an even prime number, then the truth values of inverse and contrapositive of $p \Rightarrow q$ respectively are

(1) T, T (2) F, F (3) T, F (4) F, T

29. "No square of a real number is less than zero" is equivalent to

(1) for every real number a, a^2 is non-negative. (2) $\forall\, a \in R, a^2 \geq 0$

(3) either (1) or (2) (4) None of these

30. If a compound statement r is contradiction, then find the truth value of $(p \Rightarrow q) \wedge (r) \wedge [p \Rightarrow (\sim r)]$.

(1) T
(2) F
(3) T or F
(4) None of these

Concept Application Level—2

31. Which of the following is/are counter example(s) of the statement $x^2 - 7x + 10 > 0$, for all real x?

(a) 2 (b) 3 (c) 4 (d) 5

(1) Only (a) and (d)
(2) Only (b) and (c)
(3) All (a), (b), (c) and (d)
(4) None of these

32. If p: 3 is an odd number and q: 15 is a prime number, then the truth value of $[\sim (p \Leftrightarrow q)]$ is equivalent to that of ________.

(a) $p \Leftrightarrow (\sim q)$ (b) $(\sim p) \Leftrightarrow q$ (c) $\sim (p \wedge q)$

(1) only (a) (2) only (c) (3) Both (a) and (b) (4) (a), (b) and (c)

33. The compound statement, "If you want to top the school, then you do not study hard" is equivalent to

(1) "If you want to top the school, then you need to study hard".
(2) "If you will not top in the school, then you study hard".
(3) "If you study hard, then you will not top the school".
(4) "If you do not study hard, then you will top in the school".

34. If p: 25 is a factor of 625 and q: 169 is a perfect square then $\sim (p \Rightarrow q)$ is equivalent to

(1) $p \wedge q$
(2) $(\sim p) \wedge q$
(3) $p \wedge (\sim q)$
(4) Both (2) and (3)

35. The compound statement, "If you won the race, then you did not run faster than others" is equivalent to

(1) "If you won the race, then you ran faster than others".
(2) "If you ran faster than others, then you won the race".
(3) "If you did not win the race, then you did not run faster than others".
(4) "If you ran faster than others, then you did not win the race".

36. "If x is a good actor, then y is bad actress" is

(1) a tautology (2) a contradiction (3) a contingency (4) None of these

37. Which of the following is negation of the statement "All birds can fly".

(1) "Some birds cannot fly".
(2) "All the birds cannot fly".
(3) "There is at least one bird which can fly"
(4) All the above.

38. What are the truth values of $(\sim p \Rightarrow \sim q)$ and $\sim (\sim p \Rightarrow q)$ respectively, when p and q always speak true in any argument?

(1) T, T (2) F, F (3) T, F (4) F, T

39. If p: 4 is an odd number and q: 4^3 is an even number, then which of the following is equivalent to $\sim (p \Rightarrow q)$?

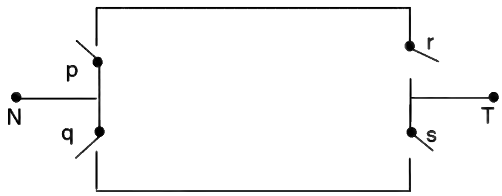
(1) "4 is an odd number and 4^3 is an even number".

(2) "The negation of the statement "4 is not an odd number or 4^3 is not an even number".

(3) Both (1) and (2)

(4) None of these

40.



In the above network, current flows from N to T when

(1) p closed, q closed, r opened and s opened.

(2) p closed, q opened, s closed and r opened.

(3) q closed, p opened, r opened and s closed.

(4) p opened, q opened, r closed and s closed.

41. If p: In a triangle, the centroid divides each median in the ratio 1 : 2 from the vertex and q: In an equilateral triangle, each median is perpendicular bisector of one of its sides. The truth values of inverse and converse of $p \Rightarrow q$ are respectively

(1) T, T (2) F, F (3) T, F (4) F, T

42. If p always speaks against q, then $p \Rightarrow p \vee \sim q$ is

(1) a tautology

(2) contradiction

(3) contingency

(4) None of these

43. If p: Every equilateral triangle is isosceles and q: Every square is a rectangle, then which of the following is equivalent to $\sim (p \Rightarrow q)$?

(1) The negation of "Every equilateral triangle is not isosceles or every square is rectangle".

(2) "Every equilateral triangle is not isosceles, then every square is not a rectangle".

(3) "Every equilateral triangle is isosceles, then every square is a rectangle".

(4) None of these

44. When does the value of the statement $(p \wedge r) \Leftrightarrow (r \wedge q)$ become false?

(1) p is T, q is F

(2) p is F, r is F

(3) p is F, q is F and r is F

(4) None of these

45. If the truth value of $p \vee q$ is true, then truth value $\sim p \wedge q$ is

(1) false if p is true

(2) true if p is true

(3) false if q is true

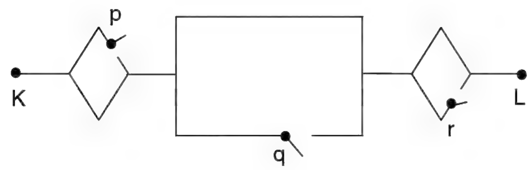
(4) true if q is true

Concept Application Level—3

46. If p and q are two statements, then $p \vee \sim(p \Rightarrow \sim q)$ is equivalent to

(1) $p \wedge \sim q$ (2) p (3) q (4) $\sim p \wedge q$

47.



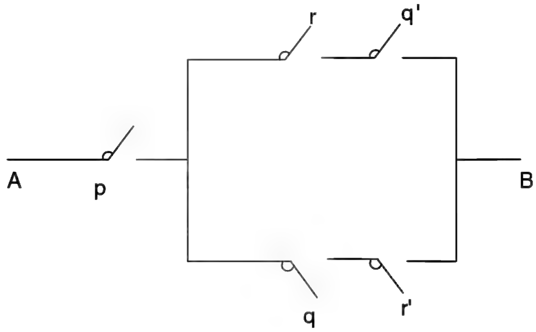
In the above network, current does not flow, when

(1) p opened, q opened and r opened.
(2) p closed, q opened and r opened.
(3) p opened, q closed and r closed.
(4) None of these

48. The contropositive of the statement $\sim p \Rightarrow (p \wedge \sim q)$ is

(1) $p \Rightarrow (\sim p \vee q)$ (2) $p \Rightarrow (p \wedge q)$ (3) $p \Rightarrow (\sim p \wedge q)$ (4) $(\sim p \vee q) \Rightarrow p$

49.



In the above circuit, the current flows from A to B when

(1) p is closed, q is open r is open.
(2) p is closed, q is closed and r is open.
(3) p is closed, q is closed and r is closed.
(4) all the above

50. If p: $5x + 6 = 8$ is an open sentence and q: 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$, then which of following is equivalent to $\sim [\sim p \vee q]$?

(1) "The negation of "If $5x + 6 = 8$ is an open sentence, then 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$".
(2) $5x + 6 = 8$ is an open sentence or 3, 4 are not roots of the equation $x^2 - 7x + 12 = 0$
(3) $5x + 6 = 8$ is not an open sentence and 3, 4 are the roots of the equation $x^2 - 7x + 12 = 0$
(4) None of these

KEY

Very short answer type questions

1. False
2. $\sim p \wedge q$
3. True
4. 2 is even prime
5. Some dogs bark.
6. True
7. True
8. True
9. $\exists$
10. True
11. False
12. does not
13. $\exists$
14. True
15. and
16. He is smart and he is intelligent.

    If he is smart, then he is intelligent.
17. $\exists$
18. Neither true nor false
19. $p \Rightarrow \sim q \wedge \sim r$.
20. "If I am not tired, then I will not take rest".
21. $p \Rightarrow \sim q$
22. False
23. All odd numbers are primes.
24. True
25. $\sim p \vee q$
26. 9
27. If ABCD is a rectangle, then it is square.
28. $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
29. Some days I do not go to school.
30. $q \Rightarrow \sim p$

Short answer type questions

31. $p \vee q \Rightarrow p$

32. Converse: If she is happy, then she is rich.

    Inverse: If she is not rich, then she is not happy.

    Contrapositive: If she is not happy, then she is not rich.

34. Converse: In a ΔABC, if $\angle B \neq \angle C$, then $AB \neq AC$.

    Inverse : In a ΔABC, if $AB = AC$, then $\angle B = \angle C$.

    Contrapositive : In a ΔABC, if $\angle B = \angle C$, then $AB = AC$.

41. $\forall$

    $\forall$

45. $\sim p \vee [-q \vee q]$

Essay type questions

46. (i) p is closed, q is closed, r is closed
    (ii) p is closed, q is closed, r is open
    (iii) p is closed, q is open, r is closed
    (iv) p is open, q is closed, r is closed
    (v) p is closed, q is open, r is open
    (vi) p is open, q is closed, r is open
    (vii) p is open, q is open, r is closed

47. p is closed, q is closed

48. p is closed, q is closed.

49. (i) p is closed, q is closed
    (ii) p is open, q is closed

50. (i) p is closed, q is closed, r is closed
    (ii) p is closed, q is closed, r is open
    (iii) p is closed, q is open, r is closed
    (iv) p is open, q is closed, r is closed
    (v) p is closed, q is open, r is open

key points for selected questions

Very short answer type questions

16. If x and y are statements, then Conjunction: x and y. Implication: If x, then y.

17. The symbol for 'for some' and 'there exists atleast one' is $\exists$.

20. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

21. Converse of converse of conditional is conditional.

24. $p \vee q$ is true if atleast one of the p, q is true.

27. Converse of $p \Rightarrow q$ is $q \Rightarrow p$.

30. Contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

Short answer type questions

31. Converse of $p \Rightarrow q$ is $q \Rightarrow p$.

32. (i) conditional: If p then q.
(ii) converse: If q then p.
(iii) inverse: If $\sim p$ then $\sim q$
(iv) contrapositive: If $\sim q$ then $\sim p$.

33. (i) First of all write possible truth values of both p and q.
(ii) Then write the truth values of $p \vee q$.
(iii) Then write truth values of $p \Rightarrow (p \vee q)$.
(iv) If each truth value in the last column of $p \Rightarrow (p \vee q)$ is T, then it is a tautology.

34. (i) conditional: If p then q.
(ii) converse: If q then p.
(iii) inverse: If $\sim p$ then $\sim q$
(iv) contrapositive: If $\sim q$ then $\sim p$.

35. (i) First of all write all the possible truth values for both p and q.
(ii) Then write the truth values of both $\sim p$ and $\sim q$.
(iii) Then write the truth values of $p \wedge q$.
(iv) Then negate the truth values of $(p \wedge q)$.
(v) Then write the truth table for $\sim p \vee \sim q$.
(vi) Then compare truth values in the last column of $\sim (p \wedge q)$ and $\sim p \vee \sim q$.

36. (i) First of all write all possible truth values of p, q and r.
(ii) Then write the truth values of $\sim r$.
(iv) Then write truth values of $(p \vee q)$ and $(p \vee q) \vee \sim r$.

37. (i) Write the possible truth values of both p and q.
(ii) Then write the truth values of $\sim p$.
(iii) Write truth values of $p \Rightarrow q$.
(iv) Write truth values of $(\sim p \vee q)$.
(v) Compare the truth values of the last columns of $p \Rightarrow q$ and $(\sim p \vee q)$.

38. (i) First of all write possible truth values of p and q.
(ii) Then write truth values of $\sim q$.
(iii) Then write truth values for $p \wedge \sim q$.

39. (i) First of all write all the possible truth values of both p and q.
(ii) Then write the truth values for $\sim q$.
(iii) Then write the truth values for $(p \wedge q)$.
(iv) Then write the truth values for both $(p \wedge q) \vee \sim q$ and $p \vee q$.
(v) Then compare the truth values of both $(p \wedge q) \vee \sim q$ and $(p \vee \sim q)$.

40. (i) Write the possible truth values of both p and q.
(ii) Write truth values of $p \wedge q$.
(iii) Then write truth values of $p \Rightarrow (p \wedge q)$

41. The symbol for 'for all' and 'for every' is $\forall$.

42. (i) First of all write all the possible truth values of both p and q.
(ii) Write the truth values of $\sim p$.
(iii) Write the truth values of $(\sim p \wedge q)$.
(iv) Then write the truth values of $(\sim p \wedge q) \wedge q$.
(v) If the truth values in the last column of $(\sim p \wedge q) \wedge q$ are different, then it is contingency.

43. (i) First of all write possible truth values of p and q.
(ii) Then write truth values of $\sim p$ and $p \wedge q$.
(iii) Write the truth values of $(\sim p) \vee (p \wedge q)$.

44. (i) First of all write all the possible truth values of both p and q.
(ii) Then write truth values of $\sim p$.
(iii) As $p \wedge \sim p$ is always false, then $(p \wedge \sim q) \wedge (p \vee q)$ i.e., $p \wedge (p \vee q)$ is also false.

45. Use the laws $\sim(p \vee q) = \sim p \wedge \sim q$ and $\sim(p \wedge q) = \sim p \vee \sim q$.

Essay type questions

46. (i) Parallel connection is represented by $x \wedge y$.
(ii) Series connection is represented by $x \vee y$.
(iii) Given network is represented by $p \vee (r \vee q)$
(iv) Write the truth values for $p \vee (r \vee q)$.
(v) When the truth value of s is T, current flows from A to B.

47. (i) Parallel connection represented by $x \wedge y$.
(ii) Series connection represented by $x \vee y$.
(iii) Given network is represented by $p \wedge (q \vee \sim p)$
(iv) Write the truth values of $p \wedge (q \vee \sim p)$ (say r)
(v) When the truth value of r is T, current flows from A to B.

48. (i) Parallel connection is represented by $x \wedge y$.
(ii) Series connection is represented by $x \vee y$.
(iii) Given network is represented by $(p \wedge q) \wedge (\sim p \vee p)$
(iv) Write the truth values for $(p \wedge q) \wedge (\sim p \vee p)$
(v) When the truth value of r is T, correct flows from A to B.

49. (i) Parallel connection represented by $x \wedge y$.
(ii) Series connection represented by $x \vee y$.
(iii) Given network is represented by $(p \vee \sim p) \wedge q$.
(iv) Write the truth values of $(p \vee \sim q) \wedge q$ (say r)
(v) When the truth value of r is T, then current flows from A to B.

50. (i) Parallel connection is represented by $x \wedge y$.
(ii) Series connection is represented by $x \vee y$.
(iii) Given network is represented by $p \vee (q \wedge r)$.
(iv) Write the truth values of $p \vee (q \wedge r)$ (say r)
(v) When the truth value of r is T, current flows from A to B.

KEY

Concept Application Level—1,2,3

1. 4	**14.** 3	**27.** 4	**39.** 4
2. 4	**15.** 3	**28.** 4	**40.** 3
3. 4	**16.** 2	**29.** 3	**41.** 2
4. 1	**17.** 3	**30.** 2	**42.** 1
5. 4	**18.** 3	**31.** 3	**43.** 1
6. 4	**19.** 3	**32.** 4	**44.** 4
7. 1	**20.** 1	**33.** 3	**45.** 1
8. 2	**21.** 1	**34.** 4	**46.** 2
9. 1	**22.** 3	**35.** 4	**47.** 4
10. 3	**23.** 1	**36.** 4	**48.** 4
11. 3	**24.** 1	**37.** 1	**49.** 2
12. 3	**25.** 4	**38.** 3	**50.** 1
13. 3	**26.** 1		

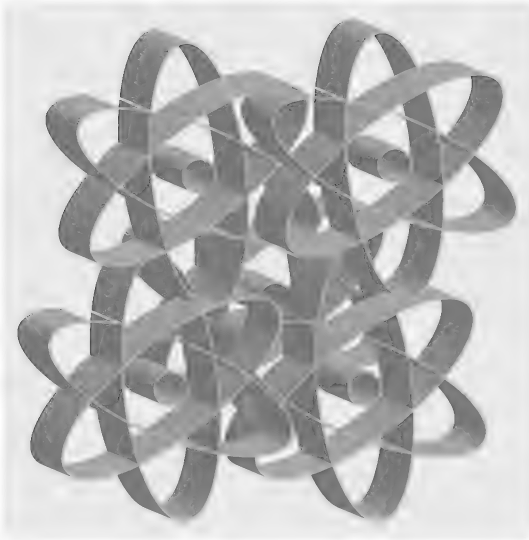
Concept Application Level—1,2,3

Key points for select questions

1. Check from options.
2. Recall the definition of statement.
3. Recall the properties.
5. Check from options.
6. Verify through truth tables.
7. Check from truth tables.
10. Recall the properties.
11. Converse of $p \Rightarrow q$ is $q \Rightarrow p$.
13. Recall the concept of switching network.
14. For only one value of 'x' the equation is true.
15. Check from truth tables.
16. Recall the properties.
17. Verify through truth tables.
18. Recall the concept of switching networking.
19. Check from truth tables.
20. If $p \Rightarrow q$ is false only p is true, q is false.
21. Conjunction is true only when both the statements are true.
22. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$
23. Use truth table.
24. Recall Biconditinal truth table.
25. Recall the properties.
26. Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$
27. $\sim(p \Rightarrow q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$.
28. (i) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$. The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.
    (ii) $p \Rightarrow q$ is false only when p is true and q is false.
29. Square of a real number is always non-negative.
30. Conjunction is false if atleast one of the statements is false.
31. (i) Check from the options.
    (ii) Go contrary to the given statement by substituting the value of x in it.
32. (i) Use truth table of double implication.
    (ii) Apply the identity, $\sim(p \Leftrightarrow q) \equiv (\sim p \Leftrightarrow q) \equiv (p \Leftrightarrow \sim q)$
33. $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$.
34. (i) Both p and q are true.
    (ii) $\sim(p \Rightarrow q) \equiv p \wedge \sim q$.
35. A conditional is equivalent to its contrapositive. i.e., $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$.
36. Given sentence is not a statement.
37. Contradict the given statement by using different quantifiers.
38. $p \Rightarrow q$ is false only when p is true and q is false.
39. $\sim(p \Rightarrow q) \equiv p \wedge \sim q \equiv \sim(p \vee q)$
40. Current flows only when p and r are closed or q and s are closed.
41. (i) p is false and q is true.
    (ii) The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$. The converse of $p \Rightarrow q$ is $q \Rightarrow p$
    (iii) $p \Rightarrow q$ is false only when p is true and q is false.
42. Verify through truth table.
43. $\sim (p \Rightarrow q) \equiv p \wedge \sim q \equiv \sim (p \vee q)$
44. (i) $x \wedge y$ is always false when y is false.
    (ii) Compare with truth table of double implication
45. Cheek from options.
46. (i) $\sim (p \Rightarrow q) = \sim p \vee q$.
    (ii) $p \vee (q \vee r) = (p \vee q) \vee r$
47. Current flows irrespective of p, q and r whether closed or open.
48. The contrapositive of $p \Rightarrow q \Rightarrow \sim q \Rightarrow \sim p$.
49. (i) Parallel connection is represented by $x \wedge y$.
    (ii) Series connection is represented by $x \vee y$.
50. $\sim(\sim p \vee q) = p \wedge \sim q$.

CHAPTER 6

Sets, Relations and Functions



INTRODUCTION

In everyday life we come across different collections of objects. For example: A herd of sheep, a cluster of stars, a posse of policemen etc. In mathematics, we call such collections as sets. The objects are referred to as the elements of the sets.

Set

A set is a well-defined collection of objects.

Let us understand what we mean by a well-defined collection of objects.

We say that a collection of objects is well-defined if there is some reason or rule by which we can say whether a given object of the universe belongs to or does not belong to the collection.

We usually denote the sets by capital letters A, B, C or X, Y, Z etc.

To understand the concept of a set, let us look at some examples.

Examples

1. Let us consider the collection of odd natural numbers less than or equal to 15.
   In this example, we can definitely say what the collection is. The collection comprises the numbers 1, 3, 5, 7, 9, 11, 13 and 15.
2. Let us consider the collection of students in a class who are good at painting. In this example, we cannot say precisely which students of the class belong to our collection. So, this collection is not well defined.

   Hence, the first collection is a set where as the second collection is not a set. In the first example given above, the set of the odd natural numbers less than or equal to 15 can be represented as set $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$

Elements of a set

The objects in a set are called its elements or members.

If a is an element of a set A, then we say that a belongs to A and we write it as, $a \in A$.

If a is not an element of A, then we say that a does not belong to A and we write it as, $a \notin A$.

Some sets of numbers and their notations

N = Set of all natural numbers = $\{1, 2, 3, 4, 5, \ldots\ldots\ldots\}$
W = Set of all whole numbers = $\{0, 1, 2, 3, 4, 5, \ldots\ldots\ldots\}$
Z or I: Set of all integers = $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

Q = Set of all rational numbers = $\left\{\frac{p}{q} \text{ where } p, q \in Z \text{ and } q \neq 0\right\}$

Cardinal number of a set

The number of elements in a set A is called its cardinal number. It is denoted by n(A). A set which has finite number of elements is a finite set and a set which has infinite number of elements is an infinite set.

Examples

1. Set of English alphabets is a finite set.
2. Set of number of days in a month is a finite set.
3. The set of all even natural numbers is an infinite set.
4. Set of all the lines passing through a point is an infinite set.
5. The cardinal number of the set $X = \{a, c, c, a, b, a\}$ is $n(X) = 3$ as in sets distinct elements only are counted.
6. If $A = \{a, \{a, b\}, b, c, \{c, d\}\}$, then $n(A) = 5$.

Representation of sets

We represent sets by the following methods:

1. **Roster or list method**
   In this method, a set is described by listing out all the elements in the set.

Examples

(i) Let W be the set of all letters in the word JANUARY.
Then we represent W as, $W = \{A, J, N, R, U, Y\}$.

(ii) Let M be the set of all multiples of 3 less than 20. Then we represent the set M as,
$M = \{3, 6, 9, 12, 15, 18\}$.

2. **Set–Builder method**
   In this method, a set is described by using a representative and stating the property or properties which the elements of the set satisfy, through the representative.

☛ *Examples*

(i) Let D be the set of all days in a week. Then we represent D as,
$D = \{x/x \text{ is a day in a week}\}$.

(ii) Let N be the set of all natural numbers between 10 and 20, then we represent the set N as,
$N = \{x/10 < x < 20 \text{ and } x \in N\}$.

Types of sets

1. **Empty set or Null set or Void set**
A set with no elements in it is called an empty set (or) void set (or) null set. It is denoted by { } or ϕ. (read as phi)

NOTE: $n(\phi) = 0$

☛ *Examples*

(i) Set of all positive integers less than 1 is an empty set.
(ii) Set of all mango trees with apples.

2. **Singleton set**
A set consisting of only one element is called a singleton set.

☛ *Examples*

(i) The set of all vowels in the word MARCH is a singleton as A is the only vowel in the word.
(ii) The set of whole numbers which are not natural numbers is a singleton as 0 is the only whole number which is not a natural number.
(iii) The set of all SEVEN WONDERS in India is a singleton as Tajmahal is the only wonder in the set.

3. **Equivalent sets**
Two sets A and B are said to be equivalent if their cardinal numbers are equal.
We write this symbolically as $A \sim B$ or $A \leftrightarrow B$.

☛ *Examples*

(i) Sets, $X = \{2, 4, 6, 8\}$ and
$Y = \{a, b, c, d\}$ are equivalent as $n(X) = n(Y) = 3$

(ii) Sets, X = {Dog, Cat, Rat}
$Y = \{\Delta, \bigcirc, \square\}$ are equivalent.

(iii) Sets, $X = \{-1, -7, -5\}$ and
B = {Delhi, Hyderabad} are not equivalent as $n(X) \neq n(B)$

NOTE: If the sets A and B are equivalent, we can establish a one-to-one correspondence between the two sets. i.e., we can pair up elements in A and B such that every element of A is paired with a distinct element of set B and every element of set B is paired with a distinct element of set A.

4. **Equal Sets**
Two sets A and B are said to be equal if they have the same elements.

Examples

(i) Sets, $A = \{a, e, i, o, u\}$ and
$B = \{x/x$ is a vowel in the English alphabet$\}$ are equal sets.

(ii) Sets, $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$ are not equal sets.

(iii) Sets, $A = \{1, 2, 3, 4, \ldots.\}$ and
$B = \{x/x$ is a natural number$\}$ are equal sets.

NOTE: If A and B are equal sets, then they are equivalent but the converse need not be true.

5. **Disjoint sets**
Two sets A and B are said to be disjoint, if they have no elements in common.

Examples

(i) Sets $X = \{3, 6, 9, 12\}$ and
$Y = \{5, 10, 15, 20\}$ are disjoint as they have no element in common.

(ii) Sets $A = \{a, e, i, o, u\}$ and
$B = \{e, i, j\}$ are not disjoint as they have common elements e and i.

6. **Subset and Superset**
Let A and B be two sets. If every element of set A is also an element of set B, then A is said to be a subset of B or B is said to be a superset of A. If A is a subset of B, then we write. $A \subseteq B$ or $B \supseteq A$.

Examples

(i) Set $A = \{2, 4, 6, 8\}$ is a subset of set $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(ii) Set of all primes except 2 is a subset of the set of all odd natural numbers.

(iii) Set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is a superset of set $B = \{1, 3, 5, 7\}$.

NOTE:

(i) Empty set is a subset of every set.

(ii) Every set is a subset of itself.

(iii) If a set A has n elements, then the number of subsets of A is 2^n.

(iv) If a set A has n elements, then the number of non-empty subsets of A is $2^n - 1$.

7. **Proper subset**
If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B and is denoted by $A \subset B$. When $A \subset B$ then B is called a superset of A and is denoted as $B \supset A$ if $A \subset B$ then $n(A) < n(B)$ and if $B \supset A$ then $n(B) > n(A)$.

8. **Power set**
The set of all subsets of a set A is called its power set. It is denoted by P(A).

Examples

(i) Let $A = \{x, y, z\}$. Then the subsets of A are ϕ, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, $\{x, y, z\}$
So, $P(A) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$.
We observe that the cardinality of P(A) is $8 = 2^3$.

NOTE:

(i) If a set A has n elements, then the number of subsets of A is 2^n i.e., the cardinality of the power set is 2^n.

(ii) If a set A has n elements, then the number of proper subsets of A is $2^n - 1$.

9. **Universal set**

A set which consists of all the sets under consideration or discussion is called the universal set. It is usually denoted by $\cup$ or μ.

Examples

(i) Let $A = \{a, b, c\}$, $B = \{c, d, e\}$ and
$C = \{a, e, f, g, h\}$. Then, the set $\{a, b, c, d, e, f, g, h\}$ can be taken as the universal set here.
$\therefore\ \mu = \{a, b, c, d, e, f, g, h\}$.

10. **Complement of a set**

Let μ be the universal set and $A \subseteq \mu$. Then, the set of all those elements of μ which are not in set A is called the complement of the set A. It is denoted by A' or A^c.
$A' = \{x/x \in \mu \text{ and } x \notin A\}$

Examples

(i) Let $\mu = \{3, 6, 9, 12, 15, 18, 21, 24\}$ and $A = \{6, 12, 18, 24\}$. Then, $A' = \{3, 9, 15, 21\}$.

(ii) Let $\mu = \{x/x$ is a student and $x \in$ class X$\}$
And $B = \{x/x$ is a boy and $x \in$ class X$\}$.
Then, $B' = \{x/x$ is a girl and $x \in$ class X$\}$.

NOTE:

1. A and A' are disjoint sets.
2. $\mu' = \phi$ and $\phi' = \mu$.

Operations on sets

1. **Union of sets**

Let A and B be two sets. Then, the union of A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A and B.
i.e., $A \cup B = \{x/x \in A \text{ or } x \in B\}$

Examples

(i) Let $A = \{-1, -3, -5, 0\}$ and
$B = \{-1, 0, 3, 5\}$. Then,
$A \cup B = \{-5, -3, -1, 0, 3, 5\}$.

(ii) Let $A = \{x/5 \le 5x < 25 \text{ and } x \in N\}$ and
$B = \{x/5 \le (10x) \le 20 \text{ and } x \in N\}$
Then, $A \cup B = \{x/5 \le 5x \le 20 \text{ and } x \in N\}$.

NOTE:

1. If $A \subseteq B$, then $A \cup B = B$.
2. $A \cup \mu = \mu$ and $A \cup \phi = A$
3. $A \cup A' = \mu$

2. **Intersection of sets**
   Let A and B be two sets. Then the intersection of A and B, denoted by $A \cap B$, is the set of all those elements which are common to both A and B.
   i.e., $A \cap B = \{x/x \in A \text{ and } x \in B\}$

Examples

(i) Let $A = \{1, 2, 3, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 7\}$. Then,
$A \cap B = \{1, 3, 5, 7\}$.

(ii) Let A be the set of all English alphabets and B be the set of all consonants, then, $A \cap B$ is the set of all consonants in the English alphabet.

(iii) Let E be the set of all even natural numbers and O be the set of all odd natural numbers. Then $E \cap O = \{\ \}$ or ϕ.

NOTE:

1. If A and B are disjoint sets, then $A \cap B = \phi$.
2. If $A \subseteq B$, then $A \cap B = A$.
3. $A \cap \mu = A$ and $A \cap \phi = \phi$.
4. $A \cap A' = \phi$

3. **Difference of sets**
   Let A and B be two sets. Then the difference $A - B$ is the set of all those elements which are in A but not in B.
   i.e., $A - B = \{x/x \in A \text{ and } x \notin B\}$

Example

(i) Let $A = \{3, 6, 9, 12, 15, 18\}$ and $B = \{2, 6, 8, 10, 14, 18\}$.
$A - B = \{3, 9, 12, 15\}$ and
$B - A = \{2, 8, 10, 14\}$.

NOTE:

1. $A - B \neq B - A$ unless $A = B$
2. For any set A, $A' = \mu - A$.

4. **Symmetric difference of sets**
   Let A and B be two sets. Then the symmetric difference of A and B, denoted by $A \Delta B$, is the set of all those elements which are either in A or in B but not in both. i.e., $A \Delta B = \{x/x \in A \text{ and } x \notin B \text{ or } x \in B \text{ and } x \notin A\}$

NOTE: $A \Delta B = (A - B) \cup (B - A)$ (or) $A \Delta B = (A \cup B) - (A \cap B)$

Example

Let $A = \{1, 2, 4, 6, 8, 10, 12\}$ and
$B = \{3, 6, 12\}$. Then,

$$A \Delta B = (A - B) \cup (B - A) = \{1, 2, 4, 8, 10\} \cup \{3\} = \{1, 2, 3, 4, 8, 10\}.$$

Some results

For any three sets A, B and C, we have the following results.

1. **Commutative law**
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
   (c) $A \Delta B = B \Delta A$

2. **Associative law**
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$
   (c) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

3. **Distributive law**
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **De–Morgan's Law**
   (a) $(A \cup B)' = A' \cap B'$
   (b) $(A \cap B)' = A' \cup B'$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $A - (B \cap C) = (A - B) \cup (A - C)$

5. **Identity law**
   (a) $A \cup \phi = \phi \cup A = A$
   (b) $A \cap \mu = \mu \cap A = A$

6. **Idempotent law**
   (a) $A \cup A = A$
   (b) $A \cap A = A$

7. **Complement law**
   (a) $(A')' = A$
   (b) $A \cup A' = \mu$
   (c) $A \cap A' = \phi$

Dual of an identity

An identity obtained by interchanging $\cup$ and $\cap$, and φ and μ in the given identity is called the dual of the identity.

Examples

1. Consider the identity, $A \cup B = B \cup A$
   Dual of the identity is, $A \cap B = B \cap A$
2. Consider the identity, $A \cup \mu = \mu$.
   Dual of the identity is, $A \cap \varphi = \varphi$

Venn diagrams

We also represent sets pictorially by means of diagrams called Venn diagrams. In Venn diagrams, the universal set is usually represented by a rectangular region and its subsets by closed regions inside the rectangular region. The elements of the set are written in the closed regions and the elements which belong to the universal set are written in the rectangular region.

Example

1. Let $\mu = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 7, 8\}$ and $B = \{2, 3, 4, 5, 9\}$.
   We represent these sets in the form of Venn diagram as follows:

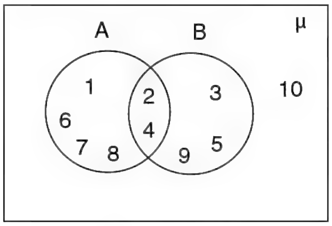


Figure 6.1

We can also represent the sets in Venn diagrams by shaded regions:

Examples

1. Venn diagram of $A \cup B$, where A and B are two overlapping sets, is

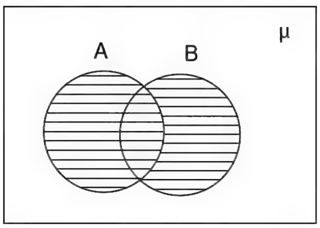


Figure 6.2

2. Let A and B be two overlapping sets. Then, the Venn diagram of $A \cap B$ is

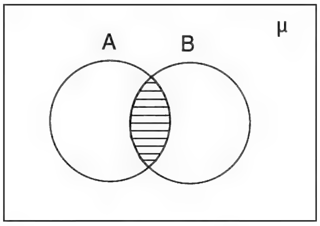


Figure 6.3

3. For a non-empty set A, Venn diagram of A' is

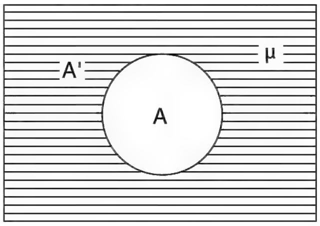


Figure 6.4

4. Let A and B be two overlapping sets. Then, the Venn diagram of A − B is

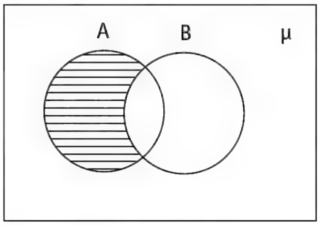


Figure 6.5

5. Let A and B be two sets such $A \subseteq B$. We can represent this relation using Venn diagram as follows.

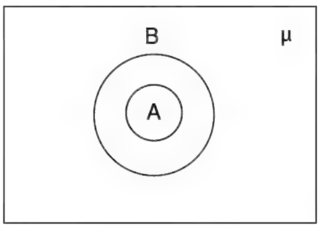


Figure 6.6

Some formulae on the cardinality of sets

Let $A = \{1, 2, 3, 5, 6, 7\}$ and $B = \{3, 4, 5, 8, 10, 11\}$.
Then, $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$ and $A \cap B = \{3, 5\}$.

In terms of the cardinal numbers, $n(A) = 6$, $n(B) = 6$, $n(A \cap B) = 2$ and $n(A \cup B) = 10$.
So, $n(A) + n(B) - n(A \cap B) = 6 + 6 - 2 = 10 = n(A \cup B)$
We have the following formulae:

For any three sets A, B and C

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

Examples

1. If $n(A) = 7$, $n(B) = 5$ and $n(A \cup B) = 10$, then find $n(A \cap B)$.

Solution

Given, $n(A) = 7$, $n(B) = 5$ and $n(A \cup B) = 10$.

We know that,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

So, $10 = 7 + 5 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 2$.

2. If $n(A) = 8$ and $n(B) = 6$ and the sets A and B are disjoint, then find $n(A \cup B)$.

Solution

Given, $n(A) = 8$, $n(B) = 6$.

A and B are disjoint

$\Rightarrow A \cap B = \phi \Rightarrow n(A \cap B) = 0$

$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 8 + 6 - 0 = 14$.

NOTE: If A and B are two disjoint sets then $n(A \cup B) = n(A) + n(B)$

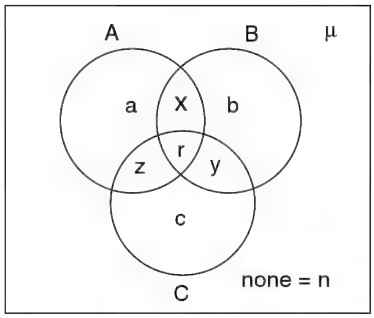


Figure 6.7

The Venn diagram and the summary of it when three overlaping sets are given are as follows.

$A = a + x + r + z$

Only A (or) exactly $A = a$

$B = b + x + r + y$

Only B (or) exactly $B = b$

$C = c + y + r + z$

Only C (or) exactly $C = c$

$A \cap B = x + r$

$A \cap B \cap C'$ (or) only $A \cap B = x$

$B \cap C = y + r$

$B \cap C \cap A'$ (or) only $B \cap C = y$; $C \cap A = z + r$; $C \cap A \cap B'$ (or) only $C \cap A = z$

Only two sets (or) exactly two sets $= x + y + z$

$A \cap B \cap C$ (or) all the three $= r$

Atleast one set = Exactly one + exactly two + exactly three

$= a + b + c + x + y + z + r$

$=$ Total $-$ none

$$\begin{aligned}
\text{Atleast two sets} &= \text{Exactly two} + \text{exactly three} \\
&= x + y + z + r \\
\text{Atleast three sets} &= \text{exactly three} = r \\
\text{Atmost two sets} &= \text{exactly two} + \text{exactly one} + \text{exactly zero} \\
&= x + y + z + a + b + c + n \\
&= n(\mu) - r \\
\text{Atmost three sets} &= \text{Exactly three} + \text{exactly two} + \text{exactly one} + \text{exactly zero} \\
&= r + x + y + z + a + b + c + n \\
&= n(\mu)
\end{aligned}$$

Ordered pair

Let A be a non-empty set and $a, b \in A$. The elements a and b written in the form (a, b) is called an ordered pair. In the ordered pair (a, b), a is called the first co-ordinate and b is called the second co-ordinate.

Note: Two ordered pairs are said to be equal only when their first as well as the second co-ordinates are equal ie., $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$
so, $(1, 2) \neq (2, 1)$ and if $(a, 5) = (3, b) \Rightarrow a = 3$ and $b = 5$

Cartesian product of sets

Let A and B be two non-empty sets. The Cartesian product of A and B, denoted by $A \times B$ is the set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$.
i.e., $A \times B = \{(a, b)/a \in A, b \in B\}$

Note:

1. $A \times B \neq B \times A$, unless $A = B$
2. For any two sets A and B, $n(A \times B) = n(B \times A)$
3. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

☛ *Example*

Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$.
$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$ and
$B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$
We observe that, $A \times B \neq B \times A$ and
$n(A \times B) = 6 = n(B \times A)$.

Some results on cartesian product

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (or) $(A \cup B) \times (A \cup C)$
(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (or) $(A \cap B) \times (A \cap C)$
(c) If $A \times B = \phi$, then either (i) $A = \phi$ or
(ii) $B = \phi$ or
(iii) both $A = \phi$ and $B = \phi$

Cartesian product of sets can be represented in following ways.

1. Arrow diagram
2. Tree diagram and
3. Graphical representation

1. **Representation of A × B using arrow diagram**

Example

If A = {a, b, c} and B = {1, 2, 3}, then find A × B.

In order to find A × B, represent the elements of A and B as shown below.

Now draw an arrow from each element of A to each element of B, as shown in the above figure.

Now represent all the elements related by arrows in ordered pairs in a set, which is the required A × B.

i.e., A × B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)}

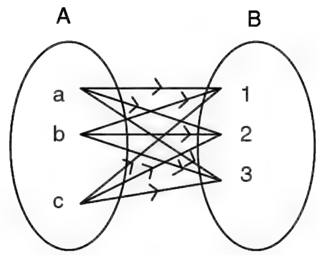


Figure 6.8

2. **Representation of A × B using a tree diagram**

Example

If A = {a, b, c} and B = {1, 2, 3}, then find A × B.

To represent A × B using tree diagram, write all the elements of A vertically and then for each element of A, write all the elements of B as shown below and draw arrows as shown in the figure below.

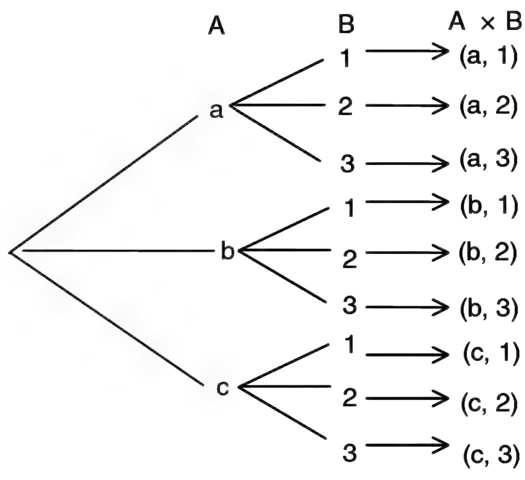


Figure 6.9

∴ A × B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c 1), (c, 2), (c, 3)}

3. **Graphical representation of A × B:**

Example

If A = {1, 2, 3} and B = {3, 4, 5}, then find A × B.

Consider the elements of A on the X-axis and the elements of B on the Y-axis and mark the points.

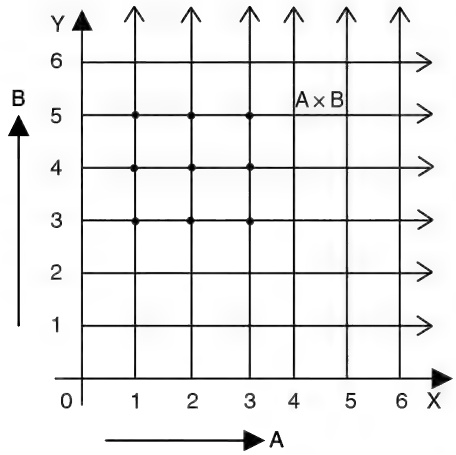


Figure 6.10

$\therefore$ A × B = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)}

Relation

We come across certain relations in real life and also in basic geometry, like *is father of*, *is a student of*, *is parallel to*, *is similar to* etc., We now define the mathematical form of the sum.

Definition

Let A and B be two non-empty sets and $R \subseteq A \times B$. R is called a relation from the set A to B. (Any subset of A × B is called a relation from A to B).
$\therefore$ A relation contains ordered pairs as elements. Hence "A relation is a set of ordered pairs".

Examples

1. Let A = {1, 2, 4} and B = {2, 3}
   Then, A × B = {(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3)}
   Let R_1 = {(1, 2), (1, 3), (2, 3)}
   Clearly, $R_1 \subseteq A \times B$ and we also notice that, for every ordered pair $(a, b) \in R_1$, $a < b$.
   So, R_1 is the relation *is less than* from A to B.

2. In the previous example, let R_2 = {(4, 2), (4, 3)}.
   Clearly, $R_2 \subseteq A \times B$ and we also notice that, for every ordered pair $(a, b) \in R_2$, $a > b$.
   So, R_2 is the relation *is greater than* from A to B.

NOTE:

1. If n(A) = p and n(B) = q, then the number of relations possible from A to B is 2^{pq}.
2. If $(x, y) \in R$, then we write x Ry and read as x is related to y.

Domain and range of a relation

Let A and B be two non-empty sets and R be a relation from A to B, we note that

(a) the set of first co-ordinates of all ordered pairs in R is called the domain of R.
(b) the set of second co-ordinates of all ordered pairs in R is called the range of R.

Example

Let $A = \{1, 2, 4\}$, $B = \{1, 2, 3\}$ and

$R = \{(1, 1), (1, 2), (2, 1), (2, 3), (4, 3)\}$ be a relation from A to B.

Then, domain of $R = \{1, 2, 4\}$ and range of $R = \{1, 2, 3\}$

Representation of relations

We represent the relations by the following methods:

(i) **Roster–method (or) List method**

In this method we list all the ordered pairs that satisfy the rule or property given in the relation.

Example

Let $A = \{1, 2, 3\}$

If R is a relation on the set A having the property *is less than*, then the roster form of R is,

$R = \{(1, 2), (1, 3), (2, 3)\}$

(ii) **Set–builder method**

In this method, a relation is described by using a representative and stating the property or properties, which the first and second co-ordinates of every ordered pair of the relation satisfy, through the representative.

Example

Let $A = \{1, 2, 3\}$.

If R is a relation on the set A having the property *is greater than or equal to*, then the set builder form of R is,

$R = \{(x, y) / x, y \in A \text{ and } x \geq y\}$

(iii) **Arrow diagram**

In this method, a relation is described by drawing arrows between the elements which satisfy the property or properties given in the relation.

Example

Let $A = \{1, 2, 4\}$ and $B = \{2, 3\}$

Let R be a relation from A to B with the property *is less than*.

Then, the arrow diagram of R is:

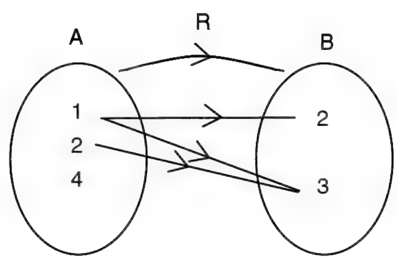


Figure 6.11

Inverse of a relation

Let R be a relation from A to B. The inverse relation of R, denoted by R^{-1}, is defined as, $R^{-1} = \{(y, x)/(x, y) \in R\}$

Example

Let $R = \{(1, 1), (1, 2), (2, 1), (2, 3), (4, 3)\}$ be a relation from A to B, where $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$

Then, $R^{-1} = \{(1, 1), (1, 2), (2, 1), (3, 2), (3, 4)\}$

NOTE:

1. Domain of R^{-1} = Range of R
2. Range of R^{-1} = Domain of R
3. If R is a relation from A to B, then R^{-1} is a relation from B to A.
4. If $R \subseteq A \times A$, then R is called a binary relation or simply a relation on the set A.
5. For any relation R, $(R^{-1})^{-1} = R$.

Types of relations

1. **One-One relation:** A relation $R : A \to B$ is said to be one-one relation if different elements of A are paired with different elements of B. i.e., $x \neq y$ in $A \Rightarrow f(x) \neq f(y)$ in B.

Example

Here, the relation $R = \{(1, 3), (2, 4), (3, 5)\}$

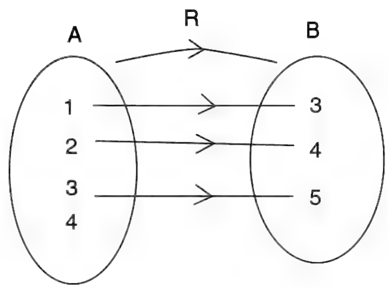


Figure 6.12

2. **One-many relation:** A relation $R : A \to B$ is said to be one-many relation if at least one element of A is paired with two or more elements of B.

Example

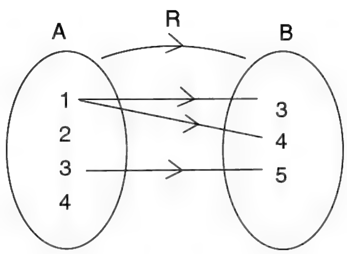


Figure 6.13

Here, the relation $R = \{(1, 3), (1, 4), (3, 5)\}$.

3. **Many-one relation:** A relation $R : A \to B$ is said to be many-one relation if two or more elements of A are paired with an element of B.

Example

Here, the relation $R = \{(1, 2), (2, 4), (3, 2), (5, 3)\}$

4. **Many-many relation:** A relation $R : A \to B$ is said to be many-many relation if two or more elements of A are paired with two or more elements of B.

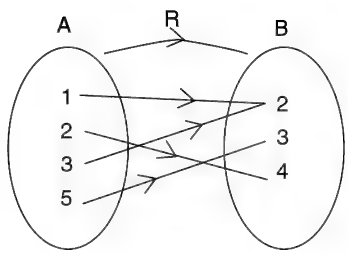


Figure 6.14

Example

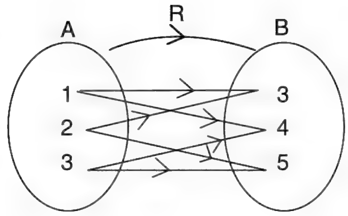


Figure 6.15

Here, the relation R = {(1, 3), (1, 4), (2, 3), (2, 5), (3, 4), (3, 5)}

Properties of relations

(i) **Reflexive relation:** A relation R on a set A is said to be reflexive if for every x ∈ A, (x, x) ∈ R.

Example

1. Let A = {1, 2, 3} then, R = {(1, 1), (1, 2), (2, 2), (3, 3), (2, 3)} is a reflexive on A.
2. Let A = {1, 2, 3} then, R = {(1, 1), (2, 3), (1, 2), (1, 3), (2, 2)} is not a reflexive relation as (3, 3) ∉ R.

NOTE: Number of reflexive relations defined on set having *n* elements is $2^{n^2 - n}$

(ii) **Symmetric relation:** A relation R on a set A is said to be symmetric, if for every (x, y) ∈ R, (y, x) ∈ R.

Examples

1. Let A = {1, 2, 3}. Then, R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} is a symmetric relation on A.
2. Let A = {1, 2, 3}. Then, R = {(1, 1), (1, 2), (3, 1), (2, 2), (3, 3)} is not a symmetric relation as (1, 2) ∈ R but (2, 1) ∉ R.

NOTE:

A relation R on a set A is symmetric iff $R = R^{-1}$ i.e., R is symmetric iff $R = R^{-1}$.

(iii) **Transitive relation:** A relation R on a set A is said to be transitive if (x, y) ∈ R and (y, z) ∈ R, then (x, z) ∈ R.
i.e., R is said to be transitive, whenever (x, y) ∈ R and (y, z) ∈ R ⇒ (x, z) ∈ R.

Examples

1. If A = {1, 2, 3}, then the relation on set A, R = {(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (2, 2), (3, 3)} is a transitive relation.
2. Let A = {1, 2, 3}, then the relation on set A, R = {(1, 2), (2, 2), (2, 1), (3, 3), (1, 3)} is not a transitive relation as (1, 2) ∈ R and (2, 1) ∈ R but (1, 1) ∉ R.

(iv) **Anti-symmetric relation:** A relation R on a set A is said to be anti-symmetric if (x, y) ∈ R and (y, x) ∈ R, then x = y.
i.e., R is said to be anti-symmetric if for x ≠ y, (x, y) ∈ R ⇒ (y, x) ∉ R.

Examples

1. let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (1, 2), (3, 3)\}$ is an anti-symmetric relation.
2. Let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 3)\}$ is not an anti-symmetric relation as $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

(v) **Equivalence relation:** A relation R on a set A is said to be an equivalence relation if it is,

(i) reflexive
(ii) symmetric and
(iii) transitive.

NOTE: For any set A, $A \times A$ is an equivalence relation. In fact it is the largest equivalence relation.

Identity relation

A relation R on a set A defined as, $R = \{(x, x)/x \in A\}$ is called an identity relation on A. It is denoted by I_A.

Example

Let $A = \{1, 2, 3\}$. Then, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

NOTE:

Identity relation is the smallest equivalence relation on a set A.

Function

Let A and B be two non-empty sets. f is a relation from A to B. If f is such that

(i) for every $a \in A$, there is $b \in B$ such that $(a, b) \in f$ and
(ii) no two ordered pairs in f have the same first element, then f is called a function from set A to set B and is denoted as $f : A \to B$.

NOTE:

1. If $(a, b) \in f$ then $f(a) = b$ and b is called the f image of a. a is called the preimage of b.
2. If $f: A \to B$ is a function, then A is called the domain of f and B is called the co-domain of f.
3. The set f(A) which is all the images of elements of A under the mapping f is called the range of f.

Few examples of functions are listed below:

(1)

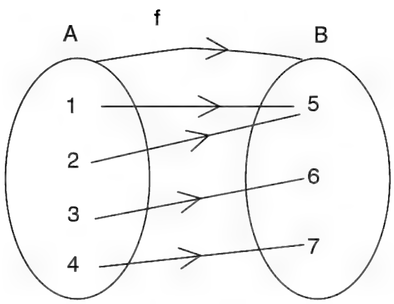


Figure 6.16

(2)

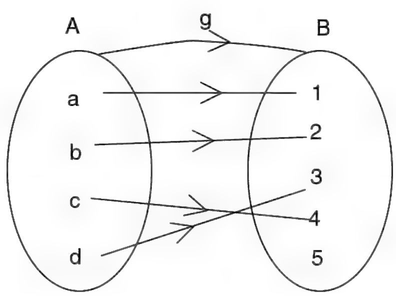


Figure 6.17

Function

Let A and B be two non-empty sets. A relation f from A to B is said to be a function, if every element in A is associated with exactly one element in B. It is denoted by $f: A \to B$ (read as f is mapping from A to B). If $(a, b) \in f$, then b is called the f image of a and is written as $b = f(a)$. a is called the pre image of b. Also in $f(a) = b$, a is called the independent variable and b is called the dependent variable.

Domain and co-domain

If $f: A \to B$ is a function, then A is called domain and B is the co-domain of the function.

Range

If $f: A \to B$ is a function, then the set of all images of elements in its domain is called the range of f and is denoted by f(A)
i.e., $f(A) = \{f(a) / a \in A\}$

NOTE: Range of a function is always subset of its co-domain i.e., $f(A) \subseteq B$.

(i) If $f: A \to B$ is a function, and $n(A) = m$, $n(B) = p$, then the number of functions that can be defined from A to B is p^m

Examples

(i) $A = \{1, 2, 3, 4\}$; $B = \{2, 3, 4, 5, 6\}$ are two sets. A relation f is defined as $f(x) = x + 2$.

The relation $f: A \to B$ is a function and $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$.

(ii) $A = \{-2, 2, 3, 4\}$, $B = \{4, 9, 16\}$ are two sets. The relation f, defined as $f(x) = x^2$, is a function from A to B, since every element in A is associated with exactly one element in B. The function $f = \{(-2, 4), (2, 4), (3, 9), (4, 16)\}$.

(iii) $A = \{-1, 1, 2, 5\}$, $B = \{1, 8, 125\}$ are two sets. The relation f defined as $f(x) = x^3$ is not a function from A to B. The relation $f = \{(1, 1), (2, 8), (5, 125)\}$. The number -1 is an element in A but it has no image in B.

(iv) $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, t, u, v\}$ are two sets. A relation f is defined as follows.

$f = \{(1, x), (2, y)\ (3, z), (4, t)\ (1, u)\}$. Here f is not a function, because the element 1 in A is associated with two elements x, u in B. Therefore, f is not a function.

Arrow diagram

Functions can be represented by arrow diagrams.

Example $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 4, 9, 16, 25, 36\}$ are two sets. A relation f is defined as $f(n) = n^2$

The arrow diagram of this function is given below.

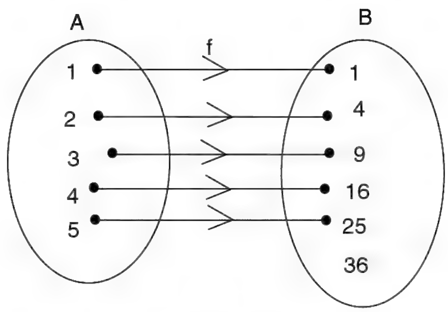


Figure 6.18

Example

Every element in A is associated with exactly one element in B. So f :A→B is a function.

Example

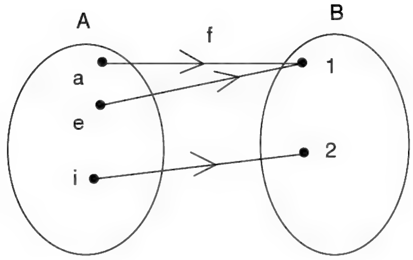


Figure 6.19

Every element in A is associated with exactly one element in B. So, f :A→B is a function.

Example

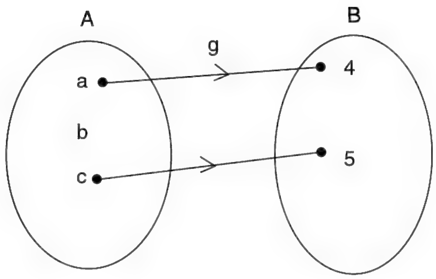


Figure 6.20

b is an element in A and it is not associated with any element in B. So g: A→B is not a function.

Example

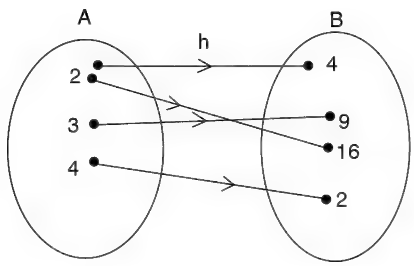


Figure 6.21

2 is an element in A. It is associated with two different elements in B. i.e., 2 has two different images. So h :A → B is not a function.

Difference between relations and functions

Every function is a relation but every relation need not be a function. A relation f from A to B is called a function if

(i) Dom (f) = A,

(ii) no two different ordered pairs in f have the same first coordinate.

Example

Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d, e\}$

Some relations f, g, h are defined as follows.

$f = \{(1, a), (2, b), (3, c), (4, d)\}$

$g = \{(1, a), (2, b), (3, c)\}$

$h = \{(1, a), (1, b), (2, c), (3, d), (4, e)\}$.

In the relation f the domain of f is A and all first coordinates are different. So f is a function. In the relation g the domain of g is not A. So g is not a function. In the relation h the domain of h is A but two of the first coordinates are equal i.e., 1 has two different images. So h is not a function.

Types of functions

One-one function or injection

Let $f : A \to B$ be a function. If different elements in A are assigned to different elements in B, then the function $f : A \to B$ is called a one-one function or an injection.

i,e., $a_1, a_2 \in A$ and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ then $f : A \to B$ is a one-one function (or) if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ then $f : A \to B$ is a one-one function.

Example

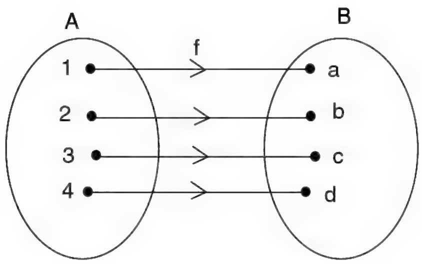


Figure 6.22

Different elements in A are assigned to different elements in B. $f : A \to B$ is a one-one function.

Example

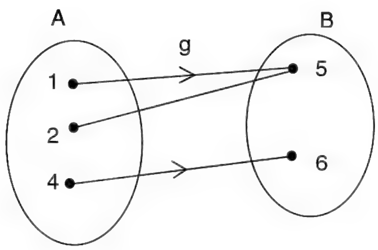


Figure 6.23

1 and 2 are two different elements in A, but they are assigned to the same element in B. So $g : A \to B$ is not a one-one function.

Note: A and B are finite sets and $f : A \to B$ is one-one. Then $n(A) \leq n(B)$.

Many to one function

If the function $f: A \to B$ is not one-one, then it is called a many to one function; i.e., two or more elements in A are assigned to the same element in B.

Example

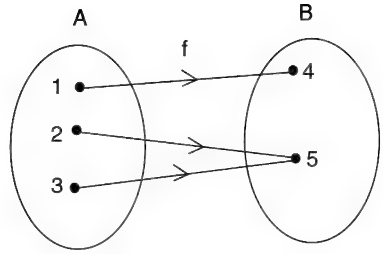


Figure 6.24

2, 3 are different elements in A, and they are assigned to the same element i.e., 5 in B. So $f: A \to B$ is a many to one function.

Onto function or surjection

$f: A \to B$ is said to be an onto function, if every element in B is the image of at least one element in A. i.e., for every $b \in B$, there exists at least one element $a \in A$, such that $f(a) = b$.

NOTE: If $f: A \to B$ is an onto function, then the co-domain of f must be equal to the range of f. $f(A) = B$

Example

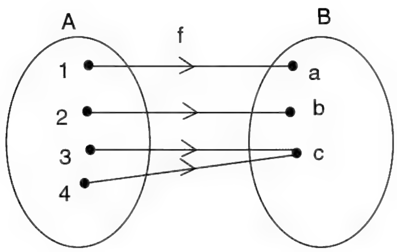


Figure 6.25

Range = {a, b, c}

Co-domain = {a, b, c}

Range = co-domain

In the diagram, every element in B is the image of one element in A. Therefore. it is an onto function.

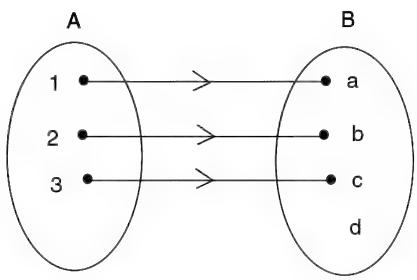


Figure 6.26

Example

In the above diagram, d is an element in B, but it is not the image of any element in A. Therefore, it is not an onto function.

NOTE: A and B are finite sets and $f: A \to B$ is onto. Then, $n(B) \le n(A)$.

Into function

If a function is not onto, then it is an into function, i.e., at least one element in B is not the image of any element in A, or the range is a subset of the co-domain.

Bijective function

If the function $f:A \rightarrow B$ is both one-one and onto then it is called a bijective function.

Example

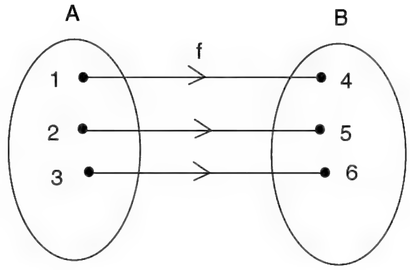


Figure 6.27

In the diagram, $f:A\rightarrow B$ is both one-one and onto. So f is a bijective function.

Example

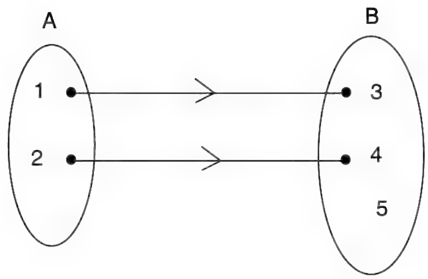


Figure 6.28

In the diagram, it is only one-one but not onto, so it is not bijective.

NOTE: A and B are finite sets and $f:A \rightarrow B$ is one-one and onto then $n(A) = n(B)$.

Inverse of a function

If $f:A\rightarrow B$ is function, then the set of ordered pairs obtained by interchanging the first and second coordinates of each ordered pair in f is called the inverse of f and is denoted by f^{-1}. i.e., $f:A\rightarrow B$ is a function then its inverse is $f^{-1}: B\rightarrow A$

Examples

(i) $f = \{(1, 2), (2, 3), (3, 4)\}$
$f^{-1} = \{(2, 1), (3, 2), (4, 3)\}$

(ii) $g = \{(1, 4), (2, 4), (3, 5), (4, 6)\}$
$g^{-1} = \{(4, 1), (4, 2), (5, 3), (6, 4)\}$

(iii) $A = \{1, 2, 3, 4\}, B = \{x, y, z, a, b\}$ are two sets the function $h: A \to B$ is defined as follows.
$h = \{(1, a), (2, b), (3, x), (4, z)\}$
$h^{-1} = \{(a, 1), (b, 2), (x, 3), (z, 4)\}$ in the above examples only f^{-1} is a function, but g^{-1}, h^{-1} are not functions
$\therefore$ $f: A \to B$ a function $f^{-1}: B \to A$ need not be a function

Inverse function

If $f: A \to B$ is a bijective function then $f^{-1}: B \to A$ is also a function.
i.e., the inverse of a function is also a function, only when the given function is bijective.

☛ *Example* $A = \{1, 2, 3, 4. 5\}; B = \{a, e, i, p, u\}$ A function f is defined as follows.

$f = \{(1, a), (2, e), (3, i), (4, u), (5, p)\}$. Clearly, f is a bijective function
Now $f^{-1} = \{(a, 1), (e, 2), (i, 3), (u, 4), (p, 5)\}$. Clearly f^{-1} is also a function and it is also bijective.

Identity function

$f: A \to A$ is said to be an identity function on A if $f(a) = a$ for every $a \in A$ it is denoted by I_A

☛ *Example* $A = \{1, 2, 3, 4\}$ The identity function on A is $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

NOTE:

(i) Identity function is always bijective function.
(ii) The inverse of the identity function is the identify function itself.

Constant function

A function $f: A \to B$ is a constant function if there is an element $b \in B$, such that $f(a) = b$, for all $a \in A$.
i.e., in a constant function the range has only one element.

☛ *Example* $A = \{1, 2, 3, 4\}; B = \{a, e, i, u\}$ are two sets and a function from A to B is defined as follows
$f = \{(1, a), (2, a), (3, a), (4, a)\}$
f is a constant function.

NOTE: The range of a constant function is a singleton.

Equal function

Two functions f and g, defined on the same domain D are said to be equal if $f(x) = g(x)$ for all $x \in D$.

☛ *Example* Let $f R - \{2\} \to R$ be defined by $f(x) = x + 2$;

and $g: R - \{2\} \to R$ be defined by $g(x) = \dfrac{x^2 - 4}{x - 2}$;

$\therefore$ f and g have the same domain $R - \{2\}$
Given $f(x) = x + 2$;

$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2$$

$\therefore$ $f(x) = g(x)$ for all $x \in R - \{2\}$

Composite function (or) product function

Let f and g be two functions such that $f: A \to B$ and $g: B \to C$. Let a be an orbitary element in a.

Since f is a function from A to B, there exists an element $b \in B$ such that $f(a) = b$.

Since g is a function from B to C, there exists an element $c \in C$ such that $g(b) = c$.

$\therefore g(f(a)) = c \Rightarrow gof(a) = c$

$\therefore$ gof is a function from A to C.

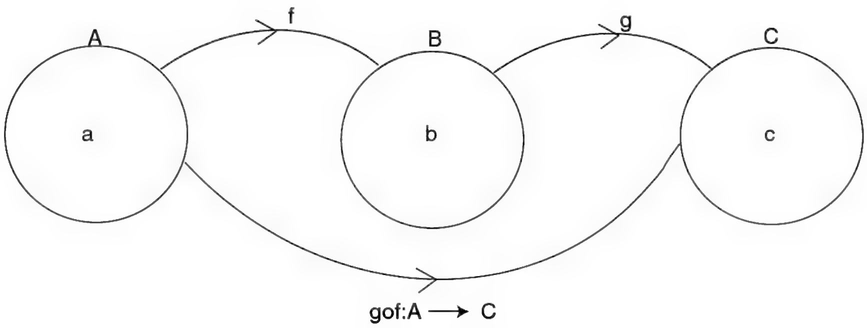


Figure 6.29

If $f: A \to B$ and $g: B \to C$ are two functions, then the function $g[f(x)] = gof$ from A to C, denoted by gof is called the composite function of f and g.
In the composite function gof,

(i) the co-domain of f is the domain of g.

(ii) the domain of gof is the domain of f, the co-domain of gof is the co-domain of g.

(iii) composite function does not satisfy commutative property i.e $gof \neq fog$.

(iv) if $f: A \to B$; $g: B \to C$; $h: C \to D$ are three functions, $ho(gof) = (hog)of$

i.e., the composite function satisfies associative property.

Real function

If $f: A \to B$ such that $A \subseteq R$ then f is said to be a real variable function.

If $f: A \to B$ such that $B \subseteq R$ then f is said to be a real valued function.

If $f: A \to B$, and A and B are both subsets of the set of real numbers (R), then f is called a real function.

Even and odd functions

1. If $f(-x) = f(x)$, then the function $f(x)$ is called an even function.

Example

$f(x) = x^2$

$f(-x) = x^2$

Here $f(-x) = f(x)$

$f(x) = x^2$ is an even function.

2. If $f(-x) = -f(x)$, then the function $f(x)$ is called an odd function.

Example $f(x) = x^3$

$f(-x) = -x^3$
Here, $f(x) = -f(x)$
$f(x) = x^3$ is an odd function.

NOTE:

1. There are functions which are neither even nor odd.

Example 1. $2x + 3$, a^x etc.

2. If $f(x)$ is a real function, then $\frac{f(x)+f(-x)}{2}$ is always even and $\frac{f(x)-f(-x)}{2}$ is always odd.
3. Product of two even functions is even.
4. Product of two odd functions is even.
5. Product of an even function and an odd function is odd.
6. $f(x) = 0$ is both even and odd.

Domain and range of some functions are listed below

[redacted]		
$\frac{1}{x}$	$R - \{0\}$	R
$\sqrt{x}$	$[0, \infty)$	$(0, \infty)$
$\lvert x \rvert$	R	$(0, \infty)$
$\log x$	$(0, \infty)$	R
a^x $(a > 0)$	R	$(0, \infty)$

Graphs of functions

A graph does not represent a function, if there exists a vertical line which meets the graph in two or more points, i.e., a vertical line meets the graph at only one point, then the graph represents a function.

Example

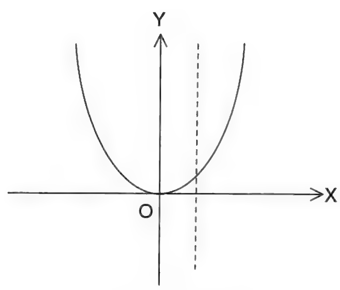


Figure 6.30

In this diagram, the vertical dotted line meets the graph at only one point. So the graph represents a function.

Example

In the above diagram, the vertical dotted line meets the graph at two points. So it is not a function.

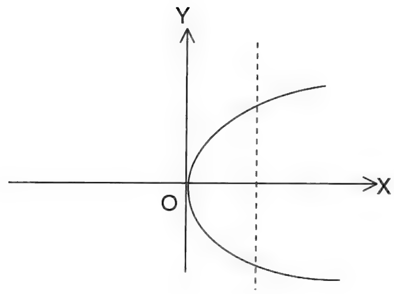


Figure 6.31

NOTE:

(i) The Y-axis does not represent a function.

(ii) The X-axis represents a many-one function.

Zeroes of a function

If $f : A \to R$ $(A \subset R)$, then the points $k \in A$ such that $f(k) = 0$ are called the zeroes of the function f.

If k is a zero of $f : A \to R$ then $(k, 0)$ is the corresponding point on the graph of f : k is called the x-intercept of the graph.

Example

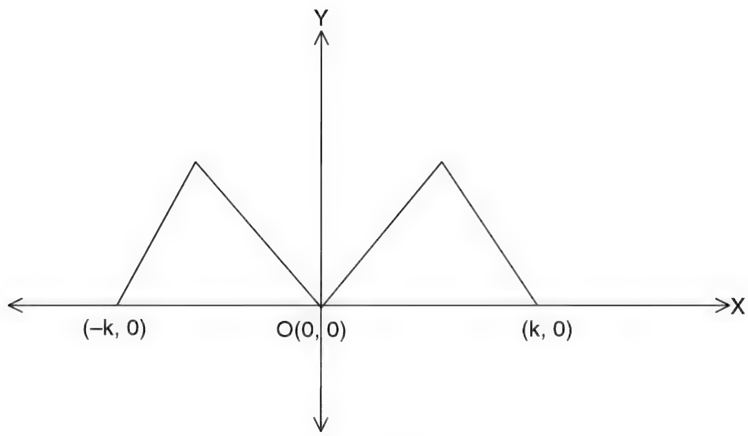


Figure 6.32

Above graph represents a function, zeroes of the graph are $-k$, 0 and k.

test your concepts

Very short answer type questions

1. If A is a non-empty set, then $(((A')')')'$ is ________.
2. The number of non-empty proper subsets of a set A is 0, then $n(A) =$ __________.
3. The number of non-empty proper subsets of a set containing 7 elements is __________.

4. If A and B are disjoint sets, then $A \Delta B$ = _______.

5. $n(A \cup B \cup C)$ = ___________.

6. If A and B are disjoint, then $(A \cap B)'$ = ____________.

7. In the following figure, if A and B are any two non-empty sets and μ is an universal set, then the shaded region represents ___________.

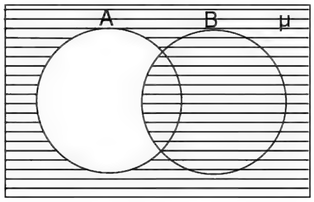


Figure 6.33

8. If any two of the sets $A_1, A_2, \ldots\ldots\ldots A_n$ are disjoint, then $A_1 \cap A_2 \cap \ldots\ldots \cap A_n$ = ________.

9. If $n(A) = 25$, $n(B) = 10$ and also $B \subset A$, then $n(B - A)$ = ___________.

10. If $n(A) = 15$, $n(B) = 13$ and $n(A \cap B) = 10$, then the symmetric difference of A and B is __________.

11. The ordered pair (x, y) is a subset of {x, y}. (True/False)

12. If A = {a, b, c, d} and B = {1, 2, 3}, then R = {(a, 2), (b, 1), (d, 3), (2, c)} is a relation from A to B. (True/False)

13. A = {a, b, c} and R is an identity relation on set A. Then the ordered pairs of R are ________.

14. $n(P \times Q) = 200$ and $n(P) = 100$, then $n(Q)$ = _______.

15. Relation $R = \{(x, y) : x > y \text{ and } x + y = 8 \text{ where } x, y \in N\}$, then write R^{-1} in roster form.

16. If $R = R^{-1}$, then the relation R is _____.

17. If A = {1, 2, 3, 4, 5, 6} and B = {1, 3, 5, 7}, then $n(A \times B)$ = ___________.

18. If A and B are two equivalent sets and $n(B) = 6$, then $n(A \times B)$ = ____________.

19. The domain of the relation $R = \{(x, y) : x, y \in N \text{ and } x + y \leq 9\}$ is________.

20. The range of the relation $R = \{(x, y) : 3x + 2y = 15 \text{ and } x, y \in N\}$ is __________.

21. A = {x, y, z, p}; B = {7, 8, 9, 10} and a rule f is given by f(x) = 7, f(y) = 7, f(z) = 7, f(p) = 7. The relation f is a function. [True/False]

22. Range of the function $|x - 5|$ is _______.

23. If $f(x) = 2x + \frac{3}{2}$ then find f(3) and $f\left(\frac{3}{2}\right)$.

24. If the function $f : A \rightarrow \{a, b, c, d\}$ is an onto function, then the minimum number of elements in A must be equal to __________.

25. In a bijection, the number of elements of the domain is equal to the number of elements of the co-domain. [True/False]

26. Domain of the function $\frac{1}{\sqrt{x}}$ is ___________.

27. Number of elements of an identity function defined on a set containing four elements is _________.

28. If f is a constant function and f (100) = 100, then f(2007) = ___________.

29. $f : R \to R$ be defined by $f(x) = 7x + 6$. What is f^{-1}?

30. If $f : A \to B$ and $g : B \to C$ are such that g of is onto then g is necessarily onto. [True/false]

Short answer type questions

31. If A = {1, 2, 3, 6, 8, 9},
B = {3, 4, 5, 6}, and μ = {1, 2, 3, ... 10}, then find $(A \cup B)'$

32. Write the following sets in the roster form.
$X = \{a/30 \le a \le 40 \text{ and a is a prime}\}$

33. If $n(P - Q) = x + 37$, $n(Q - P) = 30 + 3x$, $n(P \cup Q) = 120 + 2x$ and $n(P \cap Q) = 35$, then find x.

34. In a club of 70 members 30 play Tennis but not cricket and 55 play Tennis. How many members play cricket but not Tennis? (Each member plays either Tennis or Cricket).

35. Find the value of $n(A \cap B \cap C)$, if $n(A) = 35$, $n(A \cap B \cap C') = 8$, $n(A \cap C \cap B') = 10$ and $n(A \cap B' \cap C') = 6$.

36. $R = \{(a, 2a - b) / a, b \in N \text{ and } 0 < a, b < 3\}$, then find the domain and range of the relation R.

37. If $n(X \cap Y') = 9$, $n(Y \cap X') = 10$ and $n(X \cup Y) = 24$, then find $n(X \times Y)$.

38. In a gathering, two persons are related "if they have the same bike", then find the properties that are satisfied by the relation.

39. If A = {3, 5, 6, 9} and R is a relation in A defined as $R = \{(x, y) \in R \text{ and } x + y < 18\}$, then write R in roster form.

40. Given R = {(a, a), (a, b), (b, c), (a, c), (b, b), (b, a), (c, a), (c, b)} on set A = {a, b, c}. What are the properties that R satisfies?

41. If $f(x) = \frac{2x+3}{4}$, then find $f^{-1}\left(\frac{3}{4}\right)$.

42. If f(x) is a polynomial function of 4th degree, $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and f (2) = 17, then find f(3)

43. If $f(x) = (1 - x^3)^{\frac{1}{3}}$, then find fof(x).

44. If $f(x) = \frac{x^2 - 1}{3}$ for $x \in \{-2, -1, 0, 1, 2\}$, then find $f^{-1}(x)$.

45. Find the range of the function $f(x) = \frac{1}{2x^2 + 1}$.

Essay type questions

46. In a colony of 125 members, 70 members watch Telugu channel, 80 members watch Hindi channel and 95 watch English channel, 20 watch only Telugu and Hindi, 35 watch only English and Hindi and 15 watch only Telugu and English. How many members watch all the three channels, if each watches either of the channels?

47. If $f(a) = \log\left(\frac{1+a}{1-a}\right)$, then find $f\left(\frac{a_1 + a_2}{1 + a_1a_2}\right)$ in terms of $f(a_1)$, $f(a_2)$.

48. If $f(x) = 2x + 1$ and $g(x) = 3x - 5$, then find $(fog)^{-1}(0)$.

49. If $f(x) = x^3 - 1, x < 0 = x^2 - 1, x \geq 0$ and $g(x) = (x+1)^{\frac{1}{3}}, x < 1 = (x+1)^{\frac{1}{2}}, x \geq 1$, then find gof (x).

50. If f = {(1, 3), (2, 1), (3, 4), (4, 2)} and g = {(1, 2), (2, 3), (3, 4), (4, 1)}, then find fog.

CONCEPT APPLICATION

Concept Application Level—1

1. A, B and C are three non-empty sets. If $A \subset B$ and $B \subset C$, then which of the following is true?

(1) $B - A = C - B$ (2) $A \cap B \cap C = B$ (3) $A \cup B = B \cap C$ (4) $A \cup B \cup C = A$

2. If S is the set of squares and R is the set of rectangles, then $(S \cup R) - (S \cap R)$ is

(1) S.
(2) R.
(3) set of squares but not rectangles.
(4) set of rectangles but not squares.

3. If A = { 1, 2, 3, 4, 5, 6}, then how many subsets of A contain the elements 2, 3 and 5?

(1) 4 (2) 8 (3) 16 (4) 32

4. If A = {a, b, c, d, e}, B = {a, c, e, g} and C = {b, d, e, g}, then which of the following is true?

(1) $C \subset (A \cup B)$
(2) $C \subset (A \cap B)$
(3) $A \cup B = A \cup C$
(4) Both (1) and (3)

5. If $A_1 \subset A_2 \subset A_3 \subset \ldots \subset A_{50}$ and $n(A_x) = x - 1$, then find $n\left[\bigcap_{x=11}^{50} A_x\right]$.

(1) 49 (2) 50 (3) 11 (4) 10

6. A group of 30 people take either tea or coffee. If 12 people do not take tea and 15 people take coffee, then how many people take tea?

(1) 18 (2) 16 (3) 15 (4) 12

7. If P is the set of parallelograms, and T is the set of trapeziums, then $P \cap T$ is

(1) P. (2) T.
(3) ϕ. (4) None of these

8. If X, Y and Z are three sets such that $X \supset Y \supset Z$, then $(X \cup Y \cup Z) - (X \cap Y \cap Z) =$ ____.

(1) $X - Y$ (2) $Y - Z$ (3) $X - Z$ (4) None of these

9. If $n(A_x) = x + 1$ and $A_1 \subset A_2 \subset A_3 \subset \dots\dots\dots \subset A_{99}$, then $n\left[\bigcup_{x=1}^{99} A_x\right] =$ _____ .

(1) 99 (2) 98 (3) 100 (4) 101

10. In a class every student can speak either English or Telugu. The number of students who can speak only English, the number of students who can speak only telugu and the number of students who can speak both English and Telugu are equal. Then which of the following can represent the number of students of the class?

(1) 20 (2) 25 (3) 45 (4) 50

11. If $(2x - y, x + y) = (1, 11)$, then the values of x and y respectively are

(1) 6, 5 (2) 7, 4 (3) 4, 7 (4) 7, 3

12. A relation between two persons is defined as follows:
a R b 'if a and b born in different months', R is

(1) reflexive. (2) symmetric. (3) transitive. (4) equivalence.

13. If A is a non-empty set, then which of the following is false?
p: There is atleast one reflexive relation on A
q: There is atleast one symmetric relation on A

(1) p alone (2) q alone (3) Both p and q (4) Neither p nor q

14. In a set of teachers of a school, two teachers are said to be related if 'they teach the same subject', then the relation is

(1) reflexive and symmetric. (2) symmetric and transitive.
(3) reflexive and transitive. (4) equivalence.

15. If $A = \{x, y, z\}$, then the relation $R = \{(x, x), (y, y), (z, z), (z, x), (z, y)\}$ is

(1) symmetric. (2) anti symmetric. (3) transitive. (4) both (2) and (3).

Direction for questions 15 and 16: In the set of animals, a relation R is defined in each question.

16. a R b if "a and b are in different zoological parks," then R is

(1) only reflexive. (2) only symmetric. (3) only transitive. (4) equivalence.

17. On the set of human beings a relation R is defined as follows:
a R b if "a and b have the same brother", then R is

(1) only reflexive. (2) only symmetric. (3) only transitive. (4) equivalence.

18. Consider the following statements:

p: Every reflexive relation is a symmetric relation.
q: Every anti-symmetric relation is reflexive.

Which of the following is/are true?

(1) p alone (2) q alone
(3) Both p and q (4) Neither p nor q

19. In a set of ants in a locality, two ants are said to be related if they walk on a same straight line, then the relation is

(1) reflexive and symmetric. (2) symmetric and transitive.
(3) reflexive and transitive. (4) equivalence.

20. If n (A) = 4 and n (B) = 4, then find the number of subsets of $A \times B$.

(1) 65636 (2) 65536 (3) 65532 (4) None of these

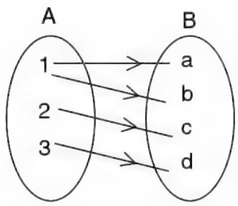
21. A function f is constant from set A = {1, 2, 3} onto set B = {a, b, c} such that f (1) = a, then the range of f is

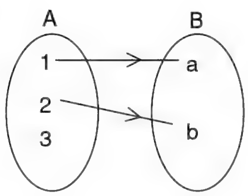
(1) {a, c} (2) {a} (3) {a, b} (4) {a, b, c}

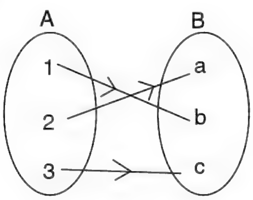
22. Which of the following is an odd function?

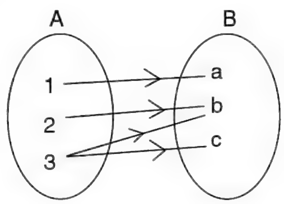
(1) $x + x^3$ (2) $x^3 - x^2 - 5$ (3) $x^2 + x^4$ (4) $\dfrac{3x^2}{x^2+1}$

23. Which of the following relations is a function?

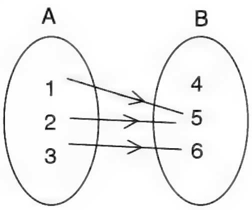
(1)


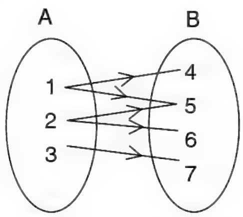
(2)


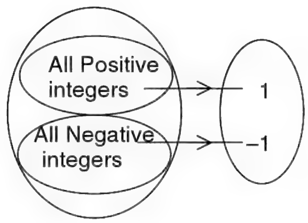
(3)


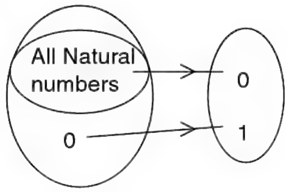
(4)


24. Which of the following relation is not a function?

(1)


(2)


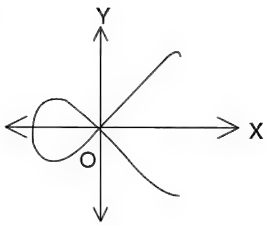
(3) 

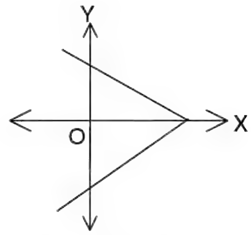
(4) 

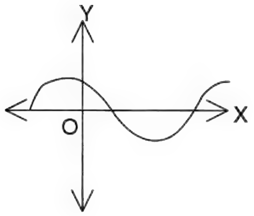
25. If $f : A \to B$ is an onto function defined by $f(x) = 3x - 4$ and $A = \{0, 1, 2, 3\}$, then the co-domain of f is

(1) $\{-4, 0, 2, 5\}$. (2) $\{-1, 2, 5, 6\}$. (3) $\{-4, -1, 2, 5\}$. (4) None of these

26. Which of the following graphs represents a function?

(1) 

(2) 

(3) 

(4) None of these

27. If $f(x) = 2x - 3x^2 - 5$ and $g(x) = \dfrac{f(x) + f(-x)}{2}$, then g(x) is

(1) odd. (2) even.
(3) even as well as odd. (4) neither even nor odd.

28. Domain of the function $f(x) = \dfrac{5 - x}{|3 - x|}$ is

(1) $x \in R$ (2) $x \in Z$ (3) $R - \{3\}$ (4) $R - \{5\}$

29. $A = \{-1, 0, 1, 2\}$, $B = \{0, 1, 2\}$ and
$f : A \to B$ defined by $f(x) = x^2$, then f is

(1) only one-one function. (2) only onto function.
(3) bijective. (4) not a function.

30. If two sets A and B have p and q no. of elements respectively and $f : A \to B$ is one-one, then the relation between p and q is

(1) $p \geq q$ (2) $p > q$
(3) $p \leq q$ (4) $p = q$

Concept Application Level—2

31. If, $X = \left\{ p : \text{where } p = \dfrac{(n+2)(2n^5 + 3n^4 + 4n^3 + 5n^2 + 6)}{n^2 + 2n} \text{ and } n, p \in Z^+ \right\}$, then find the number of elements in the set X.

(1) 2 (2) 3 (3) 4 (4) 6

32. If n (A ∩ B) = 10, n (B ∩ C) = 20 and n (A ∩ C) = 30, then find the greatest possible value of n (A ∩ B ∩ C).

(1) 15 (2) 20 (3) 10 (4) 4

33. If X, Y and Z are any three non empty sets such that any two of them are disjoint, then $(X \cup Y \cup Z) \cap (X \cap Y \cap Z)$ is

(1) X (2) Y (3) Z (4) ϕ

34. A and B are any two non empty sets and A is proper subset of B. If n (A) = 5, then find the minimum possible value of n (A Δ B).

(1) 1
(2) 5
(3) Cannot be determined
(4) None of these

35. If n (A ∩ B) = 5, n (A ∩ C) = 7 and n (A ∩ B ∩ C) = 3, then the minimum possible value of n (B ∩ C) is ______.

(1) 0 (2) 1 (3) 3 (4) 2

36. If a set contains n elements, then which of the following cannot be the number of reflexive relations in the set?

(1) 2^n (2) 2^{n-1} (3) 2^{n^2-1} (4) 2^{n+1}

37. If A = {4, 6, 10, 12} and R is a relation defined on A as "two elements are related if they have exactly one common factor other than 1". Then the relation R is

(1) anti symmetric.
(2) only transitive.
(3) only symmetric.
(4) equivalence.

38. X is the set of all members in a colony and R is a relation defined on X as "two persons are related if they speak same language". The relation R is

(1) only symmetric.
(2) only reflexive.
(3) both symmetric and reflexive but not transitive.
(4) equivalence.

39. If A = {a, b, c, x, y, z}, then the maximum number of elements in any relation on A is

(1) 12 (2) 16 (3) 32 (4) 36

40. The relation 'is a factor of' on the set of natural numbers is not

(1) reflexive
(2) symmetric
(3) anti symmetric
(4) transitive

41. If $f(x) = \log x$, then $\dfrac{f(xy)+f(x/y)}{f(x)f(y)} =$

(1) $\dfrac{2}{\log x}$ (2) $2\log y$ (3) $2\log x$ (4) $\dfrac{2}{\log y}$

42. If $f(x) = \dfrac{x-1}{x+1}, x \neq -1$; then find $f\left(\dfrac{x-1}{x+1}\right)$.

(1) x (2) $-\dfrac{1}{x}$ (3) $f(x)$ (4) $f\left(\dfrac{1}{x}\right)$

43. If $f: R \to R$ defined by $f(x) = 3x - 5$, then $f^{-1}(\{-1, -2, 1, 2\}) =$

(1) $\left\{1, \dfrac{4}{3}, \dfrac{7}{3}\right\}$ (2) $\left\{-1, 2, \dfrac{-4}{3}\right\}$ (3) $\left\{1, 2, \dfrac{4}{3}, \dfrac{7}{3}\right\}$ (4) $\{1, 2, -1, -2\}$

44. If $f: R \to R$ is a function defined as $f(\alpha - f(\alpha)) = 5f(\alpha)$ and $f(1) = 7$, then find $f(-6)$.

(1) 37 (2) 35 (3) 7 (4) 21

45. The domain of the function $f(x) = \dfrac{1}{x} + \dfrac{1}{\log(2-x)}$ is

(1) $x > 2$ (2) $x \in R - \{2\}$ (3) $x < 2, x \neq 0, x \neq 1$ (4) $x < 2, x \neq 0$

Concept Application Level—3

46. All the students of a class like Horlicks, Maltova or Viva. Number of students who like only Horlicks and Moltova, only Maltova and Viva and only Horlicks and Viva are all equal to twice the number of students who like all the three foods. Number students who like only Horlicks, only Maltova and only Viva are all equal to thrice the number of students who like all the three foods. If four students like all the three, then find the number of students in the class.

(1) 64 (2) 48 (3) 68 (4) 52

47. If $f(x) = 2x + 3$ and $g(x) = 3x - 1$ then find $f^{-1}og^{-1}$.

(1) $\dfrac{x+8}{6}$ (2) $\dfrac{x-8}{6}$ (3) $\dfrac{8-x}{6}$ (4) $\dfrac{x-8}{2}$

48. The inverse of the function $f(x) = (x^3-1)^{1/4} - 12$ is

(1) $[1 + (x + 12)^3]^{1/4}$ (2) $[1 - (x + 12)^4]^{1/4}$
(3) $[(x + 12)^3 - 1]^{1/4}$ (4) $[1 + (x + 12)^4]^{1/4}$

49. The relation, $R = \{(1, 3), (3, 5)\}$ is defined on the set with minimum number of elements of natural numbers. The minimum number of elements to be included in R so that R is equivalence is

(1) 5 (2) 6 (3) 7 (4) 8

50. If $f(2x+3) = 4x^2 + 12x + 15$, then the value of $f(3x+2)$ is

(1) $9x^2 - 12x + 36$ (2) $9x^2 + 12x + 10$ (3) $9x^2 - 12x + 24$ (4) $9x^2 - 12x - 5$

KEY

Very short answer type questions

1. A
2. 1
3. 126
4. $A \cup B$
5. $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.
6. μ
7. A'
8. ϕ
9. 0
10. 8
11. False
12. False
13. $R = \{(a, a), (b, b), (c, c)\}$
14. 2
15. $R^{-1} = \{(1, 7), (2, 6), (3, 5)\}$
16. Symmetric
17. 24
18. $n(A \times B) = 36$
19. $\{1, 2, 3, 4, 5, 6, 7, 8\}$
20. $\{3, 6\}$
21. True
22. $(0, \infty)$
23. $\frac{15}{2}, \frac{9}{2}$
24. 4
25. True
26. R^+
27. 4
28. 100
29. $f^{-1}(y) = \frac{y-6}{7}$
30. true

Short answer type questions

31. $\{7, 10\}$
32. $X = \{31, 37\}$
33. 9
34. 15
35. 11
36. Domain = $\{1, 2\}$, Range = $\{1, 2, 0, 3\}$
37. 210.
38. it is reflexive, symmetric, transitive.
39. $R = \{(3, 3), (3, 5), (3, 6), (3, 9), (5, 3), (5, 5), (5, 6), (5, 9), (6, 3), (6, 5), (6, 6), (6, 9), (9, 3), (9, 5), (9, 6)\}$
40. R is only symmetric.
41. 0
42. 82
43. x
44. Inverse does not exist
45. $(0, 1)$

Essay type questions

46. 25
47. $f(a_1) + f(a_2)$
48. $\frac{3}{2}$.
49. When $x < 0$ or $x \geq 1$, $gof(x) = x$ and when $0 \leq x < 1$, $gof(x) = x^{2/3}$
50. $\{(1, 1), (2, 4), (3, 2), (4, 3)\}$

key points for selected questions

Short answer type questions

31. (a) Find $(A \cup B)$

(b) Use, $(A \cap B)^1 = \mu - (A \cup B)$

32. X is the set of all the primes between 30 and 40

33. Use, $n(P \cup Q) = n(P - Q) + n(Q - P) + n(P \cap Q)$ and find x

34. (i) Draw the Venn diagram according to the given data.

(ii) Use, Total strength = Only cricket + Only Tennis + both cricket and tennis and find only tennis.

35. $n(A) = n(A \cap C \cap B^1) + n(A \cap B \cap C) + n(A \cap C \cap B^1) + n(A \cap B^1 \cap C^1)$

36. Substitute the values of b as 1, 2 and a = 1, 2, 3 and find (a, 2a – b).

37. (i) Draw the Venn diagrams according the given data.

(ii) Find $n(X \cap Y)$

(iii) $n(X) = n(X \cap Y^1) + n(X \cap Y)$ and $n(Y) = n(X^1 \cap Y) + n(X \cap Y)$

(iv) Use, $n(X \times Y) = n(X) \times n(Y)$

38. Recall the properties of relation.

39. (i) $x, y \in \{3, 5, 6, 9\}$

(ii) Find all the possibilities of (x, y) which satisfy $x + y < 18$.

40. Check whether the elements satisfy the properties or not.

41. (i) Let, $\frac{2x+3}{4} = y$

(ii) Then write x interms of y i.e., $f^{-1}(y)$

(iii) Then substitute $y = \frac{3}{4}$.

42. Assume $f(x) = x^4 + 1$ and proceed.

43. Apply the concept of composite function.

44. (i) First of all assume, $\frac{x^2-1}{3} = y$

(ii) Then write x interms of y, i.e., $f^{-1}(y)$.

(iii) Then substitute y = –2, –1, 0, 1, 2.

45. (i) Find $f^{-1}(x)$,

(ii) The domains of $f^{-1}(x)$ is the range of f(x).

Essay type questions

46. (i) Draw the Venn diagram according to the given data

(ii) Use the formula, $n(A \cup B \cup C) + n(A \cap B \cap C') + n(A \cap B' \cap C) + n(A' \cap B \cap C) + 2n(A \cap B \cap C) = n(A) + n(B) + n(C)$

47. (i) Put $a = \frac{a_1 + a_2}{1 + a_1 a_2}$ in f(a).

(ii) Then simplify and express $f\left(\frac{a_1 + a_2}{1 + a_1 a_2}\right)$ in terms of $f(a_1)$ and $f(a_2)$

48. (i) First of all find fog (x).

(ii) Then find $(fog)^{-1}(x)$.

(iii) And then put x = 0.

49. Apply the concept of composite function.

50. If $(a, b) \in g$ and $(b, c) \in f$, then $(a, c) \in (fog)$.

Concept Application Level—1,2,3

1. 3
2. 4
3. 2
4. 4
5. 4
6. 1
7. 1
8. 3
9. 3
10. 3
11. 3
12. 2
13. 4
14. 4
15. 4
16. 2
17. 4
18. 4
19. 4
20. 2
21. 2
22. 1
23. 3
24. 2
25. 3
26. 3
27. 2
28. 3
29. 4
30. 3
31. 3
32. 3
33. 4
34. 1
35. 3
36. 4
37. 3
38. 4
39. 4
40. 2
41. 4
42. 2
43. 3
44. 2
45. 3
46. 1
47. 2
48. 4
49. 3
50. 3

Concept Application Level—1,2,3

Key points for select questions

1. Using the concept of subset, verify the options.
2. Every square is a rectangle.
4. Find $A \cup B, A \cap B$ and $A \cup C$.
5. If $A_1 \subset A_2 \subset A_3$, ..., then $n(A_1 \cap A_2 \cap A_3 \, ...) = n(A_1)$.
6. Use venn diagram concept.
7. Recall the properties of parallelogram and trapezium
8. Recall the concept of superset, union of sets and intersection.
9. Recall the concepts of subset and union of sets.
10. $n(A \cup B) = n(\text{only } A) + n(\text{only } B) + n(A \cap B)$
11. Equate the corresponding co-ordinates.
12. Use the definitions of reflexive, symmetric, anti-symmetric and transitive.
13. Recall the different types of relations.
14. Use the definitions of reflexive, symmetric, anti-symmetric and equivalence.
15. Recall the properties of relations.
16. Recall the properties of relations.
17. Use the definitions of reflexive, symmetric, anti-symmetric, transitive and equivalence.
18. Recall the definitions of reflexive, symmetric and anti-symmetric relations.
19. Recall the properties of relations.
20. Number of subsets of $A \times B$ is $2^{n(A).n(B)}$
21. Constant function contains only one element in the range.
22. If f(x) is an odd function, then $f(-x) = -f(x)$.
23. Recall the definition of function.
24. Recall the definition of function.
25. Substitute domain values in f(x).
26. Use the definition of a function.
27. Find $\dfrac{f(x)+f(-x)}{2}$ and verify even or odd.
28. Function is defined when denominator is not equal to zero.
29. Substitute the domain values in f(x) and find range.
30. Recall the definition of one-one function.
31. Divide each term by 'n' and find the positive factors of 6.

32. The greatest possible value of $n(A \cap B \cap C)$ is the least value amongst the values of $n(A \cap B)$, $(B \cap C)$ and $n(A \cap C)$.

33. A and B are disjoint $\Rightarrow A \cap B = \phi$ and $A \cap \phi = \phi$.

34. (i) A is proper subset of B, $A - B = \phi$ i.e., $n(A - B) = 0$.
(ii) Given $n(A) = 5$, the minimum number of elements in B is 6.
(iii) The minimum possible value of $n(A \Delta B)$ is $n(B) - n(A)$.

35. Number of elements in $(A \cap B \cap C)$ becomes the minimum number of elements in $B \cap C$.

36. Number of reflexive relations in a set containing 'n' elements is 2^{n^2-n}.

37. (i) Write the elements in R.
(ii) Apply definition of symmetric relation.

38. Recall the properties of relations.

39. $A \times A$ has maximum number of elements.

40. Use the definitions of reflexive, symmetric, anti-symmetric and equivalence.

41. (i) Substitute $f(xy)$, $f(x/y)$ in the given.
(ii) $\log ab = \log a + \log b$.
(iii) $\log\left(\frac{a}{b}\right) = \log a - \log b$.
(iv) Replace x by xy and also by $\frac{x}{y}$ in f(x) and simplify.

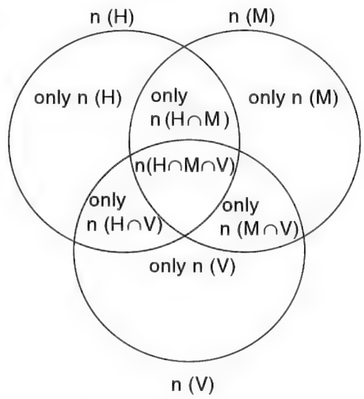
42. (i) Replace x by $\frac{x-1}{x+1}$ in f(x).
(ii) Then simplify

43. (i) Find $f^{-1}(x)$.
(ii) Put $x = -1, -2, 1, 2$ in $f^{-1}(x)$ and find their values.

44. Put $\alpha = 1$ in the given equation and simplify.

45. $\frac{1}{x}$ is not defined for $x = 0$ and logarithmic function takes only positive values.

46. (i) Let the number of students who like all the three be 'x.
(ii)



(iii) Substitute the given values in the above diagram.

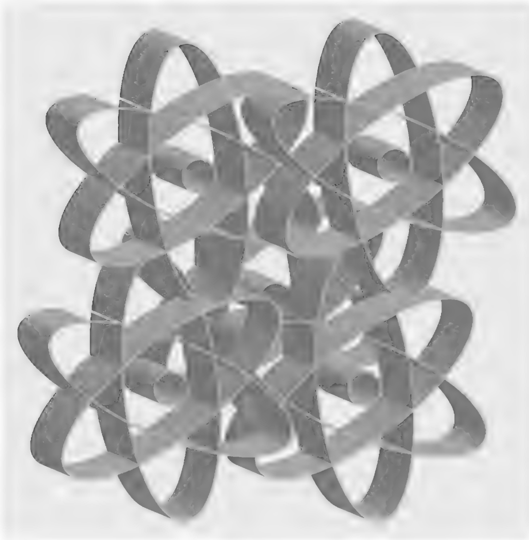
47. (i) Find gof(x).
(ii) We know that, $(gof)^{-1} = f^{-1}og^{-1}$.

48. (i) Let $f(x) = y$
(ii) Write the value of x in terms of y.

49. Recall the properties of equivalence relation.

50. (i) Put $2x + 3 = t$.
(ii) Replace x by t.
(iii) Again is put $t = 3x + 2$.

CHAPTER 7



Progressions

INTRODUCTION

Let us observe the following pattern of numbers.

(i) 5, 11, 17, 23, ...

(ii) 6, 12, 24, 48, ...

(iii) 4, 2, 0, –2, –4, ...

(iv) $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots\ldots\ldots$

In example (i), every number (except 5) is formed by adding 6 to the previous numbers. Hence a specific pattern is followed in the arrangement of these numbers. Similarly, in example (ii), every number is obtained by multiplying the previous number by 2. Similar cases are followed in examples (iii) and (iv).

Sequence

A systematic arrangement of numbers according to a given rule is called a sequence.

The numbers in a sequence are called its terms. We refer the first term of a sequence as T_1, second term as T_2 and so on. The nth term of a sequence is denoted by T_n, which may also be referred to as the general term of the sequence.

Finite and infinite sequences

1. A sequence which consists of a finite number of terms is called a finite sequence.

Example

(a) 2, 5, 8, 11, 14, 17, 20, 23 is the finite sequence of 8 terms.

2. A sequence which consists of an infinite number of terms is called an infinite sequence.

Example

3, 10, 17, 24, 31, ... is an infinite sequence.

NOTE: If a sequence is given, then we can find its nth term and if the nth term of a sequence is given we can find the terms of the sequence.

Example

Find the first four terms of the sequences whose nth terms are given as follows.

(i) $T_n = 3n + 1$

Substituting n = 1,

$T_1 = 3(1) + 1 = 4$

Similarly, $T_2 = 3(2) + 1 = 7$

$T_3 = 3(3) + 1 = 10$

$T_4 = 3(4) + 1 = 13$

∴ The first four terms of the sequence are 4, 7, 10, 13.

(ii) $T_n = 2n^2 - 3$

Substituting n = 1.

$T_1 = 2(1)^2 - 3 = -1$

Similarly, $T_2 = 2(2)^2 - 3 = 5$

$T_3 = 2(3)^2 - 3 = 15$

$T_4 = 2(4)^2 - 3 = 29$

∴ The first four terms of the sequence are –1, 5, 15, 29.

Series

The sum of the terms of a sequence is called the series of the corresponding sequence.

Example

1 + 2 + 3 + ... + n is a finite series of first n natural numbers.

The sum of first n terms of series is denoted by S_n.

Here, $S_n = T_1 + T_2 + \dots\dots\dots + T_n$.

Here, $S_1 = T_1$

$S_2 = T_1 + T_2$

$S_3 = T_1 + T_2 + T_3$

....

....

$S_n = T_1 + T_2 + T_3 + \dots\dots\dots + T_n$

We have,

$S_2 - S_1 = T_2$

$S_3 - S_2 = T_3$

Similarly,

$S_n - S_{n-1} = T_n$

Example

In the series, $T_n = 2n + 5$, find S_4.

Solution

$T_n = 2n + 5$

$T_1 = 2(1) + 5 = 7$

$T_2 = 2(2) + 5 = 9$

$T_3 = 2(3) + 5 = 11$

$T_4 = 2(4) + 5 = 13$

$S_4 = T_1 + T_2 + T_3 + T_4 = 7 + 9 + 11 + 13 = 40.$

Sequences of numbers which follow specific patterns are called progression. Depending on the pattern, the progressions are classified as follows.

(i) Arithmetic Progression

(ii) Geometric Progression

(iii) Harmonic Progression

Arithmetic progression (A.P.)

Numbers (or terms) are said to be in arithmetic progression when each one, except the first, is obtained by adding a constant to the previous number (or term).

An arithmetic progression can be represented by a, a + d, a + 2d, ..., [a + (n – 1)d]. Here, d is added to any term to get the next term of the progression. The term a is the first term of the progression, n is the number of terms in the progression and d is the common difference.

1. The nth term (general term) of an arithmetic progression is $T_n = a + (n - 1)d$

2. Sum to n terms of an A.P. = $S_n = \frac{n}{2}[2a + (n - 1)d]$

The sum to n terms of an A.P. can also be written in a different manner. That is,

sum of n terms = $\frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + \{a + (n - 1)d\}]$

But, when there are n terms in an A.P., a is the first term and $\{a + (n - 1)d\}$ is the last term. Hence, $S_n = \left(\frac{n}{2}\right)$ [first term + last term]

Arithmetic mean (A.M.)

The average of all the terms in an A.P. is called the arithmetic mean (A.M.) of the A.P.

The average of a certain numbers $= \dfrac{\text{Sum of all the numbers}}{\text{number of numbers}}$

$\therefore$ A.M. of n terms in an A.P. $= \dfrac{S_n}{n} = \dfrac{1}{n} \times \dfrac{n}{2}[\text{first term} + \text{last term}] = \dfrac{(\text{first term} + \text{last term})}{2}$

i.e., The A.M. of an A.P. is the average of the first and the last terms of the A.P.

The A.M. of an A.P. can also be obtained by considering any two terms which are EQUIDISTANT from the two ends of the A.P. and taking their average, i.e.,

(a) the average of the second term from the beginning and the second term from the end is equal to the A.M. of the A.P.

(b) the average of the third term from the beginning and the third term from the end is also equal to the A.M. of the A.P. and so on.

In general, the average of the kth term from the beginning and the kth term from the end is equal to the A.M. of the A.P.

If the A.M. of an A.P. is known, the sum to n terms of the series (S_n) can be expressed as S_n = n (A.M.)

In particular, if three numbers are in arithmetic progression, then the middle number is the A.M. i.e., if a, b and c are in A.P., then b is the A.M. of the three terms and $b = \dfrac{a+c}{2}$.

If a and b are any two numbers, then their A.M. $= \dfrac{a+b}{2}$.

Inserting arithmetic mean between two numbers:

When n arithmetic means $a_1, a_2, \ldots\ldots, a_n$ are inserted between a and b, then $a, a_1, a_2, \ldots\ldots, a_n, b$ are in A.P.

$\Rightarrow t_1 = a$ and $t_{n+2} = b$ of A.P.

The common difference of the A.P. can be obtained as follows:

Given that, n arithmetic means are there between a and b.

$\therefore\ a = t_1$ and $b = t_{n+2}$

Let d be the common difference.

$\Rightarrow b = t_1 + (n + 1)\,d$

$\Rightarrow b = a + (n + 1)d$

$\Rightarrow d = \dfrac{(b-a)}{(n+1)}$

Note:

(i) If three numbers are in A.P., we can take the three terms to be $(a - d)$, a and $(a + d)$.

(ii) If four numbers are in A.P., we can take the four terms to be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. The common difference in this case is 2d and not d.

(iii) If five numbers are in A.P., we can take the five terms to be $(a - 2d)$, $(a - d)$, a, $(a + d)$ and $(a + 2d)$.

Some important results

The sum to n terms of the following series are quite useful and hence should be remembered by students.

(i) Sum of first n natural numbers $= \sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

(ii) Sum of the squares of first n natural numbers $= \sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) Sum of the cubes of first n natural numbers $= \sum_{i=1}^{n} i^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \dfrac{n^2(n+1)^2}{4} = \left[\sum_{1}^{n} i\right]^2$

☛ *Example*

Find the 14th term of an A.P. whose first term is 3 and the common difference is 2.

Solution

The nth term of an A.P. is given by $t_n = a + (n - 1)d$, where a is the first term and d is the common difference.

$\therefore\ t_{14} = 3 + (14 - 1)\ 2 = 29$

☛ *Example*

Find the first term and the common difference of an A.P. if the 3rd term is 6 and the 17th term is 34.

Solution

If a is the first term and d is the common difference, then we have

$a + 2d = 6$ ------- (1)

$a + 16d = 34$ ----- (2)

On subtracting equation (1) from equation (2), we get

$14d = 28 \Rightarrow d = 2$

Substituting the value of d in equation (1), we get $a = 2$

$\therefore\ a = 2$ and $d = 2$

☛ *Example*

Find the sum of the first 22 terms of an A.P. whose first term is 4 and the common difference is 4/3.

Solution

Given that, $a = 4$ and $d = \dfrac{4}{3}$.

We have $S_n = \dfrac{n}{2}[2a + (n - 1)d]$

$$S_{22} = \left(\frac{22}{2}\right)\left[(2)(4)+(22-1)\left(\frac{4}{3}\right)\right] = (11)\ (8 + 28) = 396$$

Example

Divide 124 into four parts in such a way that they are in A.P. and the product of the first and the 4th part is 128 less than the product of the 2nd and the 3rd parts.

Solution

Let the four parts be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. The sum of these four parts is 124,

i.e., $4a = 124 \Rightarrow a = 31$

$(a - 3d)(a + 3d) = (a - d)(a + d) - 128$

$\Rightarrow a^2 - 9d^2 = a^2 - d^2 - 128$

$\Rightarrow 8d^2 = 128 \Rightarrow d = \pm 4$

As $a = 31$, taking $d = 4$, the four parts are 19, 27, 35 and 43.

NOTE: If d is taken as -4, then the same four numbers are obtained, but in decreasing order.

Example

Find the three terms in A.P., whose sum is 36 and product is 960.

Solution

Let the three terms of an A.P. be $(a - d)$, a and $(a + d)$.

Sum of these terms is $3a$.

$3a = 36 \Rightarrow a = 12$

Product of these three terms is

$(a + d)\, a\, (a - d) = 960$

$\Rightarrow (12 + d)(12 - d) = 80$

$\Rightarrow 144 - d^2 = 80 \Rightarrow d = \pm 8$

Taking $d = 8$, we get the terms as 4, 12 and 20.

NOTE: If d is taken as -8, then the same numbers are obtained, but in decreasing order.

Geometric progression (G.P.)

Numbers are said to be in geometric progression when the ratio of any quantity to the number that follows it is the same. In other words, any term of a G.P. (except the first one) can be obtained by multiplying the previous term by the same constant.

The constant is called the common ratio and is normally represented by r. The first term of a G.P. is generally denoted by a.

A geometric progression can be represented by $a, ar, ar^2, \ldots..$ where a is the first term and r is the common ratio of the G.P. nth term of the G.P. is ar^{n-1} i.e., $t_n = ar^{n-1}$

Sum to n terms $= S_n = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{r(ar^{n-1}) - a}{r - 1}$

$\therefore$ The sum to n terms of a geometric progression can also be written as

$$S_n = \frac{r\,(\text{Last term}) - \text{First term}}{r - 1}$$

NOTE:

1. If n terms viz., $a_1, a_2, a_3, \ldots\ldots\ldots a_n$ are in G.P., then the geometric mean (G.M.) of these n terms is given by = $\sqrt[n]{a_1 a_2 a_3 \ldots\ldots\ldots a_n}$
2. If three terms are in geometric progression, then the middle term is the geometric mean of the G.P., i.e., if a, b and c are in G.P., then b is the geometric mean of the three terms.
3. If there are two terms say a and b, then their geometric mean is given by G.M. = $\sqrt{ab}$.
4. When n geometric means are there between a and b, the common ratio of the G.P. can be derived as follows.
   Given that, n geometric means are there between a and b.
   $\therefore$ $a = t_1$ and $b = t_{n+2}$
   Let 'r' be the common ratio
   $\Rightarrow b = (t_1)(r^{n+1}) \qquad \Rightarrow b = a\, r^{n+1}$
   $\Rightarrow r^{n+1} = \dfrac{b}{a} \qquad \Rightarrow r = \sqrt[(n+1)]{\dfrac{b}{a}}$
5. For any two positive numbers a and b, their arithmetic mean is always greater than or equal to their geometric mean, i.e., for any two positive numbers a and b, $\dfrac{a+b}{2} \geq \sqrt{ab}$. The equality holds if and only if a = b.
6. When there are three terms in geometric progression, we can take the three terms to be a/r, a and ar.

Infinite geometric progression

If $-1 < r < 1$ (or $|r| < 1$), then the sum of a geometric progression does not increase infinitely but "converges" to a particular value, no matter how many terms of the G.P. we take. The sum of an infinite geometric progression is represented by S_∞ and is given by the formula,

$$S_\infty = \frac{a}{1-r}, \text{ if } |r| < 1.$$

☛ *Example*

Find the 7th term of the G.P. whose first term is 6 and common ratio is 2/3.

Solution

Given that, $t_1 = 6$ and $r = \dfrac{2}{3}$

We have $t_n = a \,.\, r^{n-1}$

$$t_7 = (6)\left(\frac{2}{3}\right)^6 = \frac{(6)(64)}{729} = \frac{128}{243}$$

☛ *Example*

Find the common ratio of the G.P. whose first and last terms are 25 and 1/625 respectively and the sum of the G.P. is 19531/625.

Solution

We know that the sum of a G.P is $\dfrac{\text{first term} - r(\text{last term})}{1-r}$

$$\Rightarrow \frac{19531}{625} = \frac{25-(r/625)}{1-r}$$

$\Rightarrow r = 1/5$

Example

Find three numbers of a G.P. whose sum is 26 and product is 216.

Solution

Let the three numbers be a/r, a and ar.

Given that,

$a/r \,.\, a \,.\, ar = 216;$

$\Rightarrow a^3 = 216; a = 6$

$a/r + a + ar = 26$

$\Rightarrow 6 + 6r + 6r^2 = 26r$

$\Rightarrow 6r^2 - 20r + 6 = 0$

$\Rightarrow 6r^2 - 18r - 2r + 6 = 0$

$\Rightarrow 6r\,(r - 3) - 2(r - 3) = 0$

$\Rightarrow r = 1/3$ (or) $r = 3$

Hence the three numbers are 2, 6 and 18 (or) 18, 6 and 2

Example

If $|\,x\,| < 1$, then find the sum of the series $2 + 4x + 6x^2 + 8x^3 + \ldots$

Solution

Let $S = 2 + 4x + 6x^2 + 8x^3 + \text{-------}$ (1)

$xS = 2x + 4x^2 + 6x^3 + \ldots..(2)$

(1) – (2) gives

$S\,(1 - x) = 2 + 2x + 2x^2 + 2x^3 + \ldots$

$= 2\,(1 + x + x^2 + \ldots)$

$1 + x + x^2 + \ldots$ is an infinite G.P with $a = 1, r = x$ and $|r| = |x| < 1$

$\therefore$ Sum of the series $= 1/1-x$

$\therefore\ S\,(1 - x) = 2/(1 - x)$

$\therefore\ S = 2/(1 - x)^2$

Example

Find the sum of the series 1, 2/5, 4/25, 8/125, ... ∞.

Solution

Given that, $a = 1, r = \frac{2}{5}$ and $|r| = \left|\frac{2}{5}\right| < 1$

$$\therefore\ S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{2}{5}} = 5/3$$

Harmonic progression (H.P.)

A progression is said to be a harmonic progression if the reciprocal of the terms in the progression form an arithmetic progression.

For example, consider the series $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$

The progression formed by taking reciprocals of terms of the above series is 2, 5, 8, 11,. . . . Clearly, these terms form an A.P. whose common difference is 3.

Hence, the given progression is a harmonic progression.

nth term of an H.P:

We know that if a, a + d, a + 2d,. . . .are in A.P., then the nth term of this A.P. is a + (n – 1) d. Its reciprocal is $\frac{1}{a + (n-1)d}$

So, nth term of an H.P. whose first two terms are $\frac{1}{a}$ and $\frac{1}{a+d}$ is $\frac{1}{a+(n-1)d}$.

Note: There is no concise general formula for the sum to n terms of an H.P.

Example

Find the 10th term of the H.P. $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \ldots\ldots$

Solution

The given H.P. is $\frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \ldots\ldots$

The corresponding A.P. is $\frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots\ldots$

Here $a = \frac{2}{3}$; $d = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ T_{10} of the corresponding A.P. is $a + (10-1)d = \frac{2}{3} + (9)\frac{1}{3} = \frac{11}{3}$

Hence required term in H.P. is $\frac{3}{11}$

Harmonic mean (H.M.)

If three terms are in H.P., then the middle term is the H.M. of other two terms.

The harmonic mean of two terms a and b is given by $\text{H.M.} = \frac{2ab}{a+b}$

Inserting n harmonic means between two numbers

To insert n harmonic means between two numbers, we first take the corresponding arithmetic series and insert n arithmetic means, and next, we find the corresponding harmonic series.

This is illustrated by the example below:

Example

Insert three harmonic means between $\frac{1}{12}$ and $\frac{1}{20}$

Solution

After inserting the harmonic means

let the harmonic progression be

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \frac{1}{a+4d}$

Given $\frac{1}{a} = \frac{1}{12}$ and $\frac{1}{a+4d} = \frac{1}{20} \Rightarrow a = 12$ and $d = 2$

$\therefore$ The required harmonic means are $\frac{1}{14}, \frac{1}{16}$ and $\frac{1}{18}$.

Relation between A.M., H.M. and G.M. of two numbers

Let x and y be two numbers.

$\therefore$ A.M. $= \frac{x+y}{2}$, G.M. $= \sqrt{xy}$ and H.M. $= \frac{2xy}{x+y}$

$\Rightarrow$ (A.M.) (H.M.) = (G.M.)^2

test your concepts

Very short answer type questions

1. Third term of the sequence whose nth term is $2n + 5$ is __________.
2. If a is the first term and d is the common difference of an A.P., then the $(n + 1)$th term of the A.P is ______.
3. If the sum of three consecutive terms of an A.P. is 9, then the middle term is ______.
4. General term of the sequence 5, 25, 125, 625, ……… is __________.
5. The arithmetic mean of 7 and 8 is ______.
6. The arrangement of numbers $\frac{1}{2}, \frac{-3}{4}, \frac{-5}{6}, \frac{-7}{8}, \ldots\ldots\ldots$ is an example of sequence. [True/False]
7. If $\frac{a}{2}$ is the first term and d is the common difference of an A.P., then the sum of n terms of the A.P. is ______.
8. In a sequence, if S_n is the sum of n terms and S_{n-1} is the sum of $(n - 1)$ terms, then the nth term is ______.
9. If $T_n = 3n + 8$, then $T_{n-1} =$ __________.

10. The sum of the first (n + 1) natural numbers is ______.

11. For a series in geometric progression, the first term is a and the second term is 3a. The common ratio of the series is ______.

12. In a series, starting from the second term, if each term is twice its previous term, then the series is in ______ progression.

13. All the multiples of 3 form a geometric progression. [True/False]

14. If a, b and c are in geometric progression then, a^2, b^2 and c^2 are in ______ progression.

15. If every term of a series in geometric progression is multiplied by a real number, then the resulting series also will be in geometric progression. [True/False]

16. Geometric mean of 5, 10 and 20 is ______.

17. Sum of the infinite terms of the G.P., –3, –6, –12, ... is 3. [True/False]

18. The reciprocals of all the terms of a series in geometric progression form a ______ progression.

19. The nth term of the sequence $\frac{1}{100}, \frac{1}{10000}, \frac{1}{1000000}$, ... is __________.

20. In a series, $T_n = x^{2n-2}$ $(x \neq 0)$, then write the infinite series.

21. The harmonic mean of 1, 2 and 3 is $\frac{3}{2}$. [True/False]

22. If a, b, c and d are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ are in ______ progression.

23. If the A.M. of two numbers is 9 and their H.M. is 4, then their G.M. is 6. [True/False]

24. If a, b and c are the arithmetic mean, geometric mean and harmonic mean of two distinct terms respectively, then b^2 is equal to ______.

25. If the sum of n terms which are in G.P. is a(r + 1), then the number of terms is _________. (Where a is the first term and r is the common ratio)

26. Write the first three terms of the sequence whose nth term is $T_n = 8 - 5n$.

27. Write the first three terms of the sequence whose nth term is $T_n = 5^{n+1}$.

28. If three arithmetic means are inserted between 4 and 5, then the common difference is ________.

29. If the 7th and the 9th terms of a G.P. are x and y respectively, then the common ratio of the G.P. is _________.

30. In a series, $T_n = 3 - n$, then S_5 = __________.

Short answer type questions

31. If the 5th term and the 14th term of an A.P. are 35 and 8 respectively, then find the 20th term of the A.P.

32. Which term of the series 21, 15, 9, ... is –39?

33. If the seventh term of an A.P. is 25 and the common difference is 4, then find the 15th term of A.P.

34. Find the general term of A.P. whose sum of n terms is given by $4n^2 + 3n$.

35. Find the sum of all three-digit numbers which leave a remainder 2, when divided by 6.

36. If the ratio of the sum of first three terms of a G.P. to the sum of first six terms is 448 : 455, then find the common ratio.

37. If in a G.P., 5th term and the 12th term are 9 and $\frac{1}{243}$ respectively, find the 9th term of G.P.

38. A person opens an account with Rs 50 and starts depositing every day double the amount he has deposited on the previous day. Then find the amount he has deposited on the 10th day from the beginning.

39. Find the sum of 5 geometric means between $\frac{1}{3}$ and 243, by taking common ratio positive.

40. Using progressions express the recurring decimal $2\cdot\overline{123}$ in the form of p/q, where p and q are integers.

41. A ball is dropped from a height of 64 m and it rebounces $\frac{3}{4}$ of the distance every time it touches the ground. Find the total distance it travels before it comes to rest.

42. Find the sum to n terms of the series 5 + 55 + 555 + ...

43. In an H.P., if the 3rd term and the 12th term are 12 and 3 respectively, then find the 15th term of the H.P.

44. If ℓth, mth and nth terms of an H.P. are x, y and z respectively, then find the value of $yz(m - n) + xz(n - \ell) + xy(\ell - m)$.

45. The A.M of two numbers is 40 more than G.M. and 64 more than H.M. Find the numbers.

Essay type questions

46. Find the sum to n terms of the series $1\cdot2\cdot3 + 2\cdot4\cdot6 + 3\cdot6\cdot9 + \ldots\ldots.$

47. One side of an equilateral triangle is 36 cm. The mid-points of its sides are joined to form another triangle. Again another triangle is formed by joining the mid-points of the sides of this triangle and the process is continued indefinitely. Determine the sum of areas of all such triangles including the given triangle.

48. Three numbers form a G.P. If the third term is decreased by 128 then, the three numbers, thus obtained, will form an A.P. If the second term of this A.P. is decreased by 16, a G.P. will be formed again. Determine the numbers.

49. If A, G and H are A.M., G.M. and H.M. of any two positive numbers, then prove that $A \geq G \geq H$.

50. The product of three numbers of a G.P. is $\frac{64}{27}$. If the sum of their products when taken in pairs is $\frac{148}{27}$, then find the numbers.

CONCEPT APPLICATION

Concept Application Level—1

1. Find t_5 and t_6 of the arithmetic progression 0, 1/4, 1/2, 3/4,....

(1) 1, 5/4 (2) 5/4, 1 (3) 1, 7/4 (4) 7/4, 1

2. If $t_n = 6n + 5$, then $t_{n+1} =$

(1) 6 n – 1 (2) 6n + 11 (3) 6n + 6 (4) 6n – 5

3. Which term of the arithmetic progression 21, 42, 63, 84, ... is 420?

(1) 19 (2) 20 (3) 21 (4) 22

4. Find the 15th term of the arithmetic progression 10, 4, –2, ...

(1) –72 (2) –74 (3) –76 (4) –78

5. If the kth term of the arithmetic progression 25, 50, 75, 100, ... is 1000, then k is ______.

(1) 20 (2) 30 (3) 40 (4) 50

6. The sum of the first 20 terms of an arithmetic progression whose first term is 5 and common difference is 4, is

(1) 820 (2) 830 (3) 850 (4) 860

7. Two arithmetic progressions have equal common differences. The first term of one of these is 3 and that of the other is 8, then the difference between their 100th terms is

(1) 4 (2) 5 (3) 6 (4) 3

8. If a, b and c are in arithmetic progression, then b + c, c + a and a + b are in

(1) Arithmetic Progression (2) Geometric Progression
(3) Harmonic Progression (4) None of these

9. The sum of the first 51 terms of the arithmetic progression whose 2nd term is 2 and 4th term is 8, is

(1) 3774 (2) 3477 (3) 7548 (4) 7458

10. Three alternate terms of an arithmetic progression are x + y, x – y and 2x + 3y, then x =

(1) –y (2) –2y (3) –4y (4) –6y

11. Find the 15th term of the series 243, 81, 27, ...

(1) $\frac{1}{3^{14}}$ (2) $\frac{1}{3^{8}}$ (3) $\left(\frac{1}{3}\right)^9$ (4) $\left(\frac{1}{3}\right)^{10}$

12. If t_8 and t_3 of a geometric progression are $\frac{4}{9}$ and $\frac{27}{8}$ respectively, then find t_{12} of the geometric progression.

(1) $\frac{64}{729}$ (2) $\frac{32}{243}$ (3) $\frac{729}{64}$ (4) $\frac{243}{32}$

13. If $t_n = 3^{n-1}$, then $S_6 - S_5 =$ ________.

(1) 243 (2) 81 (3) 77 (4) 27

14. Find the sum of the first 10 terms of geometric progression 18, 9, 4.5,......... .

(1) $9\dfrac{(2^{10}-1)}{2^8}$ (2) $9\dfrac{(2^{10}-1)}{2^{10}}$

(3) $36\left(\dfrac{2^{10}-1}{2^8}\right)$ (4) $8\dfrac{(2^{10}-1)}{2^8}$

15. If the 3rd, 7th and 11th terms of a geometric progression are p, q and r respectively, then the relation among p, q and r is

(1) $p^2 = qr$ (2) $r^2 = qp$ (3) $q^2 = p^2r^2$ (4) $q^2 = pr$

16. Evaluate $\Sigma(3 + 2^r)$, where r = 1, 2, 3,....10.

(1) 2051 (2) 2049 (3) 2076 (4) 1052

17. Find the sum of the series $\dfrac{27}{8}+\dfrac{9}{4}+\dfrac{3}{2}+.....\infty$.

(1) $\dfrac{81}{8}$ (2) $\dfrac{27}{8}$ (3) $\dfrac{81}{16}$ (4) $\dfrac{9}{8}$

18. If 3x – 4, x + 4 and 5x + 8 are the three positive consecutive terms of a geometric progression, then find the terms.

(1) 2, 8, 32 (2) 2, 10, 50 (3) 2, 6, 18 (4) 12, 6, 3

19. Find the geometric mean of the first twenty five powers of twenty five.

(1) 5^{13} (2) 5^{19} (3) 5^{24} (4) 5^{26}

20. Find the sum of 3 geometric means between $\dfrac{1}{3}$ and $\dfrac{1}{48}$ (r > 0).

(1) $\dfrac{1}{4}$ (2) $\dfrac{5}{24}$ (3) $\dfrac{7}{24}$ (4) $\dfrac{1}{3}$

21. If the second and the seventh terms of a Harmonic Progression are $\dfrac{1}{5}$ and $\dfrac{1}{25}$, then find the series.

(1) $1,\dfrac{1}{5},\dfrac{1}{9},$.......... (2) $\dfrac{1}{2},\dfrac{1}{5},\dfrac{1}{8},$...... (3) $\dfrac{1}{7},\dfrac{1}{5},\dfrac{1}{3},$........ (4) $\dfrac{1}{3},\dfrac{1}{5},\dfrac{1}{7},$.........

22. The 10th term of harmonic progression $\dfrac{1}{5},\dfrac{4}{19},\dfrac{2}{9},\dfrac{4}{17},$.......... is

(1) $\dfrac{11}{4}$ (2) $\dfrac{13}{4}$ (3) $\dfrac{4}{13}$ (4) $\dfrac{4}{11}$

23. If the ratio of the arithmetic mean and the geometric mean of two positive numbers is 3 : 2, then find the ratio of the geometric mean and the harmonic mean of the numbers.

(1) 2 : 3 (2) 9 : 4 (3) 3 : 2 (4) 4 : 9

24. If A, G and H are A.M, G.M and H.M. of any two given positive numbers, then find the relation between A, G and H.

(1) $A^2 = GH$ (2) $G^2 = AH$

(3) $H^2 = AG$ (4) $G^3 = A^2H$

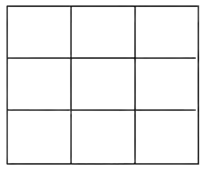
25. Find the least value of n for which the sum $1 + 2 + 2^2 + \ldots\ldots$ to n terms is greater than 3000.

(1) 8 (2) 10 (3) 12 (4) 15

26. Find the H.M. of $\frac{1}{7}$ and $\frac{1}{12}$.

(1) $\frac{1}{19}$ (2) $\frac{2}{19}$ (3) $\frac{3}{19}$ (4) $\frac{4}{19}$

27. Number of rectangles in the following figure is ___________ .



(1) 9 (2) 10

(3) 24 (4) 36

28. In a series, if $t_n = \frac{n^2 - 1}{n + 1}$, then $S_6 - S_3 =$ ______.

(1) 3 (2) 12 (3) 22 (4) 25

29. Find the number of terms to be added in the series 27, 9, 3, ……. so that the sum is $\frac{1093}{27}$.

(1) 6 (2) 7 (3) 8 (4) 9

30. Find the value of p (p > 0) if $\frac{15}{4} + p$, $\frac{5}{2} + 2p$ and $2 + p$ are the three consecutive terms of a geometric progression.

(1) $\frac{3}{4}$ (2) $\frac{1}{4}$ (3) $\frac{5}{3}$ (4) $\frac{1}{2}$

Concept Application Level—2

31. If $\frac{1}{b+c}, \frac{1}{c+a}$ and $\frac{1}{a+b}$ are in A.P, then a^2, b^2 and c^2 are in

(1) Geometric Progression (2) Arithmetic Progression

(3) Harmonic Progression (4) None of these

32. Among the following, which term belongs to the arithmetic progression –5, 2, 9, ...?

(1) 342 (2) 343 (3) 344 (4) 345

33. Five distinct positive integers are in arithmetic progression with a positive common difference. If their sum is 10020, then find the smallest possible value of the last term.

(1) 2002 (2) 2004 (3) 2006 (4) 2008

34. In a right triangle, the lengths of the sides are in arithmetic progression. If the lengths of the sides of the triangle are integers, which of the following could be the length of the shortest side?

(1) 2125 (2) 1700 (3) 1275 (4) 1150

35. If $S_1 = 3, 7, 11, 15, \ldots\ldots$ upto 125 terms and $S_2 = 4, 7, 10, 13, 16, \ldots\ldots$ upto 125 terms, then how many terms are there in S_1 that are there in S_2?

(1) 29 (2) 30 (3) 31 (4) 32

36. The first term and the mth term of a geometric progression are a and n respectively and its nth term is m. Then its (m + 1 – n)th term is ______.

(1) $\frac{ma}{n}$ (2) $\frac{na}{m}$ (3) mna (4) $\frac{mn}{a}$

37. The sum of the terms of an infinite geometric progression is 3 and the sum of the squares of the terms is 81. Find the first term of the series.

(1) 5 (2) $\frac{27}{5}$ (3) $\frac{31}{6}$ (4) $\frac{19}{3}$

38. If $\log_{\sqrt{2}} x + \log_{\sqrt{\sqrt{2}}} x + \log_{\sqrt{\sqrt{\sqrt{2}}}} x + \ldots\ldots$ upto 7 terms = 1016, the find the value of x.

(1) 4 (2) 16 (3) 64 (4) 2

39. For which of the following values of x is $8^{1+\sin x+\sin^2 x+\sin^3 x+\ldots\infty} = 64$?

(1) 60° (2) 135° (3) 45° (4) 30°

40. Find the sum of all the multiples of 6 between 200 and 1100.

(1) 96750 (2) 95760 (3) 97560 (4) 97650

41. If the kth term of a H.P. is ℓp and the ℓth term is kp and $k \neq \ell$, then the pth term is

(1) $k^2\ell$ (2) k^2p (3) p^2k (4) ℓk

42. If six harmonic means are inserted between 3 and $\frac{6}{23}$, then the fourth harmonic mean is

(1) $\frac{6}{11}$ (2) $\frac{6}{17}$ (3) $\frac{3}{7}$ (4) $\frac{3}{10}$

43. If a, b and c are positive numbers in arithmetic progression. and a^2, b^2 and c^2 are in geometric progression then a^3, b^3 and c^3 are in

(a) Arithmetic Progression

(b) Geometric Progression

(c) Harmonic Progression

(1) (a) and (b) only (2) only (c)

(3) (a), (b) and (c) (4) only (b)

44. The arithmetic mean A of two positive numbers is 8. The harmonic mean H and the geometric mean G of the numbers satisfy the relation $4H + G^2 = 90$. Then one of the two numbers is ________.

(1) 6 (2) 8 (3) 12 (4) 14

45. The infinite sum $\sum_{n=1}^{\infty}\left(\frac{5^n+3^n}{5^n}\right)$ is equal to

(1) $\frac{3}{2}$ (2) $\frac{3}{5}$

(3) $\frac{2}{3}$ (4) None of these

Concept Application Level—3

46. The numbers $h_1, h_2, h_3, h_4, \ldots\ldots, h_{10}$ are in harmonic progression and $a_1, a_2, \ldots\ldots, a_{10}$ are in arithmetic progression. If $a_1 = h_1 = 3$ and $a_7 = h_7 = 39$, then the value of $a_4 \times h_4$ is

(1) $\frac{13}{49}$ (2) $\frac{182}{3}$ (3) $\frac{7}{13}$ (4) 117

47. The difference between two hundred-digit numbers consisting of all 1's and a hundred-digit number consisting of all 2's is equal to

(1) $\underbrace{99\ldots\ldots9}_{100 \text{ times}}$ (2) $\left(\frac{333\ldots\ldots3}{80 \text{ times}}\right)^2$ (3) $\left(\frac{333\ldots\ldots3}{100 \text{ times}}\right)^2$ (4) $\underbrace{99\ldots\ldots9}_{200 \text{ times}}$

48. Find the value of $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{16}\right)\left(1+\frac{1}{256}\right)\ldots\ldots\infty$.

(1) 1 (2) 2 (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

49. The ratio of the sum of n terms of two arithmetic progressions is given by $(2n + 3) : (5n - 7)$. Find the ratio of their nth terms.

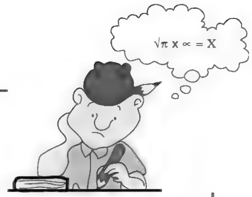
(1) $(4n + 5) : (10n + 2)$ (2) $(4n + 1) : (10n - 12)$

(3) $(4n - 1) : (10n + 8)$ (4) $(4n - 5) : (10n - 2)$

50. There are n arithmetic means (were $n \in N$) between 11 and 53 such that each of them is an integer. How many distinct arithmetic progressions are possible from the above data?

(1) 7 (2) 8 (3) 14 (4) 16

KEY



Very short answer type questions

1. 11
2. $a + nd$
3. 3
4. $5n$
5. 7.5
6. False
7. $\frac{n}{2}[a + (n - 1)d]$
8. $S_n - S_{n-1}$
9. $3n + 5$
10. $\frac{(n+1)\,(n+2)}{2}$
11. 3
12. geometric
13. False
14. geometric
15. True
16. 10
17. False
18. geometric
19. $\frac{1}{100^n}$
20. $1 + x^2 + x^4 + x^6 + \ldots\ldots\ldots\ldots$
21. False
22. arithmetic
23. True
24. ac
25. 2
26. 3, –2, –7.
27. 25, 125, 625.
28. $\frac{1}{4}$
29. $\pm\sqrt{\frac{y}{x}}$
30. 0

Short answer type questions

31. –10
32. –39
33. 57
34. $8n - 1$
35. 82650
36. $r = \frac{1}{4}$
37. $\frac{1}{9}$
38. Rs 25600
39. 121.
40. $\frac{707}{333}$
41. 448 m
42. $\frac{50}{81}(10^n - 1) - \frac{5n}{9}$
43. $\frac{12}{5}$
44. 0
45. 180 and 20

Essay type questions

46. $\frac{3}{2}\ n^2\,(n + 1)^2$
47. $432\sqrt{3}\ cm^2$
48. $\frac{8}{9}, \frac{104}{9}$ and $\frac{1352}{9}$
50. $1, \frac{4}{3}, \frac{16}{9}$

key points for selected questions

Very Short answer type questions

26. Substitute $n = 1, 2$ and 3 in T_n to get the first three terms of the sequence.

27. Substitute $n = 1, 2$ and 3 in T_n to get the first three terms of the sequence.

Short answer type questions

31. (i) In an A.P., $T_n = a + (n - 1)d$
(ii) Given $T_5 = 35$ and $T_{14} = 8$
(iii) Put $n = 5$ and 14, in T_7, get two equation in a and d.

(iv) Then solve the two equations for a and d

(v) Then substitute n = 20, a, d in T_n to get 20th term.

32. (i) By observation write the values of a and d.

(ii) Given $T_n = a + (n - 1) = - 39$.

(iii) Substitute a and d to get n.

33. (i) Given, d = 4 and $T_7 = 25$

(ii) From the above get a.

(iii) Then substitute n = 15, d = 4 and a in T_n to get 15th term.

34. (i) First of all find S_{n-1} from $S_n = 4n^2 + 3n$.

(ii) $T_n = S_n - S_{n-1}$

35. (i) First of all find the least and the greatest possible three digit number satisfies given condition.

(ii) $S_n = \dfrac{n(a+\ell)}{2}$

36. (i) Consider the sequences,

1, 2, 3 - - - - -

2, 4, 6 - - - - -

3, 6, 9 - - - - -

(ii) Find the T_n of the each sequences.

(iii) The required sum in the summation of product of nth terms of the sequences.

37. (i) T_n of G.P. is ar^{n-1}

(ii) Given $T_5 = 9$ and $T_{12} = \dfrac{1}{243}$

(iii) Then find a and r

(iv) Then find T_9 i.e., $a\,r^8$

38. (i) Given a = 50 and r = 2.

(ii) Find T_{10} by using $T_n = ar^{n-1}$.

39. (i) Given $a = \dfrac{1}{3}$ and $T_7 = ar^6 = 243$.

(ii) Find r, by substituting $a = \dfrac{1}{3}$ in $ar^6 = 243$.

(iii) Then find the sum of required terms by using $S_n = \dfrac{a(r^n - 1)}{(r-1)}$.

40. (i) $2.\overline{123} = 2 + 0.123 + 0.000123 + 0.000000123 +$ - - - - - - - -

(ii) Consider a = 0.123 and r = 0.0001.

(iii) Apply the formula $S_\infty = \dfrac{a}{1-r}$.

41. (i) Given a = 64, $r = \dfrac{3}{4}$.

(ii) Substitute a and r in $S_\infty = \dfrac{a}{1-r}$.

42. (i) Take out $\dfrac{5}{9}$,as common from the series.

Then we arrive with $\dfrac{5}{9}$ [9 + 99 + - - - -].

(ii) $= \dfrac{5}{9}\ [(10 - 1) + (10^2 - 1) + - - - - -)]$

(iii) Use the formula, $S_n = \dfrac{a(r^n - 1)}{(r-1)}$.

43. (i) Given $\dfrac{1}{a+2d} = 12$ and $\dfrac{1}{a+11d} = 3$.

(ii) Find a and d, then substitute a and d in $\dfrac{1}{a+14d}$.

44. (i) T_n of H.P. is $\dfrac{1}{a+(n-1)d}$

(ii) Given $T_\ell = x, T_m = y$ and $T_n = z$

(iii) Find $T_\ell - T_m, T_m - T_n, T_n - T_\ell$ and find the values of yz (m – n), ...in terms of y and x.

45. (i) Consider the two numbers as a and d

(ii) Then find A.M., G.M. and H.M.

(iii) Given that A.M. = G.M. + 40, A.M. = H.M. + 64

Essay type questions

46. (i) $S_n = \dfrac{a(r^n - 1)}{(r-1)}$

(ii) Given, $\dfrac{S_3}{S_6} = \dfrac{448}{455}$

47. (i) First of all find area of the triangle by using $\frac{\sqrt{3}a^2}{4}$.

(ii) If the area of first triangle is A then area of next triangle is $\frac{1}{4}$(A).

(iii) Substitute $a = A, r = \frac{1}{4}$ in $S_\infty = \frac{a}{1-r}$.

48. (i) Let the three terns of G.P. be $\frac{a}{r}$, a, ar.

(ii) Proceed with the directions given in the problem and find the numbers.

49. (i) Take two possible numbers as a and b.

(ii) Write A.M., G.M. and H.M.

(iii) Then consider $(a - b)^2 > 0$.

(iv) Add 4ab on both the sides.

(v) Get the relation between $\frac{a+b}{2}$ and $\sqrt{ab}$.

50. (i) Consider the three terms of G.P. as $\frac{a}{r}$, a and ar.

(ii) From the product of three numbers, find a.

(iii) Also given $\frac{a^2}{r} + a^2r + a^2 = \frac{148}{27}$.

(iv) Substitute a to get r.

KEY

Concept Application Level—1,2,3

1. 1	**22.** 4	**43.** 3	**47.** 3
2. 2	**23.** 3	**44.** 1	**48.** 2
3. 2	**24.** 2	**45.** 4	**49.** 2
4. 2	**25.** 3	**46.** 4	**50.** 3
5. 3	**26.** 2		
6. 4	**27.** 4		
7. 2	**28.** 2		
8. 1	**29.** 2		
9. 1	**30.** 2		
10. 4	**31.** 2		
11. 3	**32.** 4		
12. 1	**33.** 3		
13. 1	**34.** 3		
14. 1	**35.** 3		
15. 4	**36.** 2		
16. 3	**37.** 2		
17. 1	**38.** 2		
18. 3	**39.** 4		
19. 4	**40.** 4		
20. 3	**41.** 4		
21. 1	**42.** 3		

Concept Application Level—1,2,3

Key points for select questions

1. The nth term in A.P. = tn = a + (n – 1) d.
2. Substitute n = n + 1 in t_n.
3. Use formula to find nth term of an A.P.
4. Use the formula to find the nth term of an A.P.
5. Use the formula of nth term of an A.P.
6. Use the formula to find S_n of an A.P.
7. The difference of nth term of two A.P.'s having same common difference is the difference of their first terms.
8. If a, b and c are in A.P, then 2b = a + c.
9. Find a and d using the given data and use the formula to find S_n of A.P.

10. The difference between t_3 and t_1 is same as t_5 and t_3.

11. Use the formula to find the nth term of a G.P.

12. Find a and r using the given data.

13. $S_6 - S_5 = t_6$

14. Use the formula of S_n of a G.P.

15. $p = ar_1^{\,2}$, $q = ar_1^{\,6}$ and $r = ar_1^{\,10}$.

16. $\Sigma (3 + 2r) = 3\Sigma 1 + \sum_{r=1}^{10} 2^r$

17. Use the formula to find S_∞ of a G.P.

18. If a, b and c are in G.P., then $b^2 = ac$.

19. Form the series and use the formula to find S_n.

20. The 3 geometric means are t_2, t_3 and t_4 terms.

21. Use the formula of nth term of a H.P.

22. Use the formula of the nth term of a H.P.

23. Use the relation between A.M., G.M. and H.M.

24. (i) Consider two numbers as a and b.
    (ii) Then find A.M., G.M. and H.M.

25. (i) $S_n = \dfrac{a(r^n - 1)}{(r-1)}$
    (ii) $a = 1$ and $r = 2$
    (iii) Given that $S_n > 3000$
    (iv) Then find least possible value of n

26. H.M. of a and b is $\dfrac{2ab}{a+b}$.

28. $S_6 - S_3 = T_4 + T_5 + T_6$

29. Use the formula of S_n of a G.P.

30. If a, b and c are in G.P., then $b^2 = ac$.

31. $t_2 - t_1 = t_3 - t_2$

32. Each term is in the form of $7n + 2$, where $n = -1, 0, 1, 2, 3, \ldots\ldots\ldots$

33. Let the five integers be $a - 2d$, $a - d$, a, $a + d$ and $a + 2d$.

34. (i) The sides must be in the ratio 3 : 4 : 5.
    (ii) Shortest side should be a multiple of 3.

35. (i) Common terms in S_1 and S_2 are 7, 19, 31, ...
    (ii) The first term of S_1 is 3. The 125th term is 499.
    The first term of S_2 is 4. The 125th term is 376.
    (iii) The last common term is 367 (and not 499).
    (iv) $7 = 12(1) - 5$, $19 = 12(2) - 5, \ldots 367 = 12(31) - 5$.

36. (i) Use the formula to find nth term of a G.P.
    (ii) $ar^{m-1} = n$, $ar^{n-1} = m$.
    (iii) $(m + 1 - n)$th term $= ar^{m-n}$.

37. (i) Given $S_\infty = \dfrac{a}{1 - r} = 3$.
    (ii) $a^2 + a^2r^2 + a^2r^4 + \ldots\ldots\ldots = 81$

38. (i) Use $\log_{b^n} a = \dfrac{1}{n}\log_b a$.
    (ii) $S_n = \dfrac{a\,(r^n - 1)}{(r - 1)}$
    (iii) If $\log_a x = b$, then $a^b = x$

39. (i) Use the formula to find S_∞ of a G.P.
    (ii) Equate the powers on either sides by making equal bases.

40. Form the series and find the value of n and use the formula $S_n = \dfrac{n}{2}(2a + (n - 1)d)$.

41. $t_n = \dfrac{1}{a + (n - 1)d}$ in H.P.

42. (i) $\dfrac{1}{a} = 3$, $\dfrac{1}{a+7d} = \dfrac{6}{23}$.
    (ii) Find a and d by using above relation find t_5 using $\dfrac{1}{a+4d}$.

43. Use $b = \dfrac{a+c}{2}$ and $(b^2)^2 = a^2c^2$

44. Use $G^2 = A.H.$

45. (i) $\sum_{n=1}^{\infty}\left(1 + \left(\dfrac{3}{5}\right)^n\right) = 1 + \dfrac{3}{5} + \left(\dfrac{3}{5}\right)^2 + \ldots\ldots\ldots\infty$

(ii) $S_\infty = \frac{a}{1-r}$, where a is the first term and r is the common ratio.

46. (i) In A.P., $t_n = a + (n-1)d$ and in H.P., $t_n = \frac{1}{a+(n-1)d}$.

(ii) Find the values of a and d by using the data

47. Apply the principle $11 = 1 + 10^{2-1}$, 111 = and proceed.

48. (i) Let $p = (1 + \frac{1}{2})(1 + \frac{1}{4})(1 + \frac{1}{4})(1 + \frac{1}{4})$....

(ii) Multiply with $(1 - \frac{1}{2})$ on both sides.

(iii) $(a + b)(a - b) = a^2 - b^2$ a, $n - \infty \frac{1}{2^n} = 0$.

49. (i) Let a_1 and d_1 be the first term and common difference of the first A.P. and a_2, d_2 be the corresponding values for the second A.P.

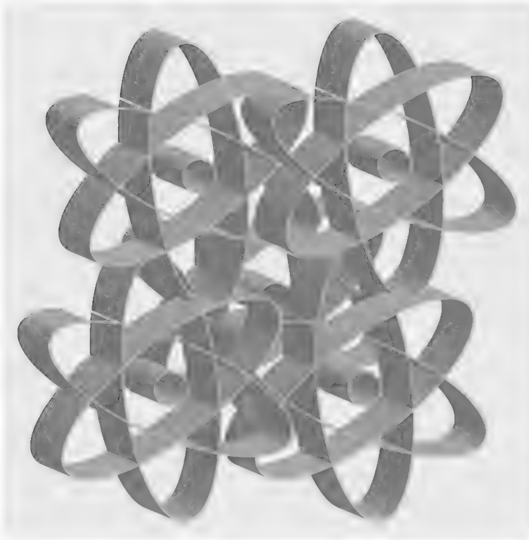
(ii) $S_n : S'_n$ has to be converted in the form of $t_n : t'_n$.

50. (i) n arithmetic means are in between a and b; $d = \frac{b-a}{n+1}$

(ii) evaluate for how many values of (n + 1), d is an integer.

CHAPTER 8

Trigonometry



INTRODUCTION

The word trigonometry is originated from the Greek word "tri" means three, "gonia" means angle and metron means measure. Hence the word trigonometry means three angle measure i.e., it is the study of geometrical figures which have three angles i.e., triangles.

The great Greek mathematician Hipapachus of 140 B.C. gave relation between the angles and sides of a triangle. Further trigonometry is developed by Indian (Hindu) mathematicians. This was migrated to Europe via Arabs.

Trigonometry plays an important role in the study of Astronomy, surveying, Navigation and Engineering. Now a days it is used to predict stock market trends.

Angle

A measure formed between two rays having a common initial point is called an angle. The two rays are called the arms or sides of the angle and the common initial point is called the vertex of the angle.

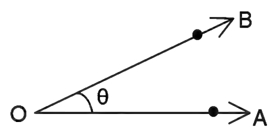


Figure 8.1

In the above figure OA is said to be the initial side and the other ray OB is said to be the terminal side of the angle.

The angle is taken positive when measured in anti-clockwise direction and is taken negative when measured in clockwise direction.

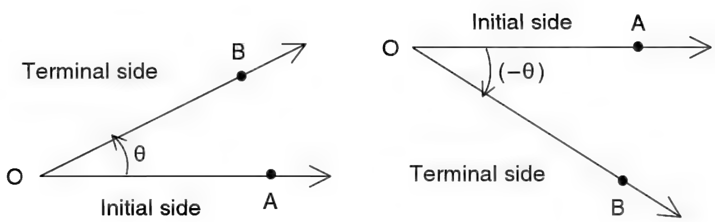


Figure 8.2

Systems of measurement of angle

We have the following systems of the measurement of angle.

1. **Sexagesimal system**
   In this system, the angle is measured in degrees(°).

   Degree: When the initial ray is rotated through $\left(\frac{1}{360}\right)^{\text{th}}$ of one revolution, we say that an angle of one degree (1°) is formed at the initial point. A degree is divided into 60 equal parts and each part is called one minute($1'$).
   Further, a minute is divided into 60 equal parts called seconds(").
   So, 1 right angle = 90°
   $1^\circ = 60'$ (minutes) and
   $1' = 60''$ (seconds)

NOTE: This system is also called as the British system.

2. **Centesimal system**
   In this system, the angle is measured in grades.
   Grade: When the initial ray is rotated through $\left(\frac{1}{400}\right)^{\text{th}}$ of one revolution, an angle of one grade is said to be formed at the initial point. It is written as 1^g.
   Further, one grade is divided into 100 equal parts called minutes and one minute is further divided into 100 equal parts called seconds.
   So, 1 right angle = 100^g
   $1^g = 100'$ (minutes) and
   $1' = 100''$ (seconds)

NOTE: This system is also called as the French system.

3. **Circular system**
   In this system, the angle is measured in radians.
   Radian: The angle subtended by an arc of length equal to the radius of a circle at its centre is said to have a measure of one radian. It is written as 1^c.

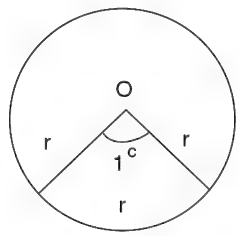


Figure 8.3

NOTE: This measure is also known as radian measure.

Relation between the units of the three systems

When a rotating ray completes one revolution, the measure of angle formed about the vertex is 360° or 400^g or $2\pi^c$
so,
$360^\circ = 400^g = 2\pi^c$
(or)
$90^\circ = 100^g = \frac{\pi^c}{2}$

For convenience, the above relation can be written as, $\frac{D}{90} = \frac{G}{100} = \frac{R}{\frac{\pi}{2}}$

where D denotes degrees, G grades and R radians.

Remember

1. $1° = \frac{\pi}{180}$ radians = 0.0175 radians (approximately)
2. $1^c = \frac{180}{\pi}$ degrees = 57°17' 44" (approximately)

Note:

1. The measure of an angle is a real number.
2. If no unit of measurement is indicated for any angle, it is considered as radian measure.

Example

1. Convert 45° into circular measure.

Solution

Given, D = 45°

We have, $\frac{D}{90} = \frac{R}{\pi/2}$

So, $\frac{45}{90} = \frac{R}{\pi/2}$

$\Rightarrow \frac{1}{2} \times \frac{\pi}{2} = R$ i.e., $R = \frac{\pi}{4}$

Hence, circular measure of 45° is $\frac{\pi^c}{4}$.

2. Convert 150^g into sexagesimal measure.

Solution

Given, $G = 150^g$

We have, $\frac{D}{90} = \frac{G}{100}$

So, $\frac{D}{90} = \frac{150}{100}$

$\Rightarrow D = \frac{3}{2} \times 90 = 135$

Hence, sexagesimal measure of 150^g is 135°.

3. What is the sexagesimal measure of angle measuring $\frac{\pi^c}{3}$?

Solution

Given, $R = \frac{\pi^c}{3}$

We have, $\dfrac{D}{90} = \dfrac{R}{\pi/2}$

So, $\dfrac{D}{90} = \dfrac{\pi/3}{\pi/2}$

$\Rightarrow D = \dfrac{2}{3} \times 90 = 60°$

Hence, the sexagesimal measure of $\dfrac{\pi^c}{3}$ is 60°.

Trigonometric ratios

Let AOB be a right triangle with $\angle AOB$ as 90°. Let $\angle OAB$ be θ. Notice that $0° < \theta < 90°$, i.e., θ is an acute angle.

We can define six possible ratios among the three sides of the triangle AOB, known as trigonometric ratios. They are defined as follows.

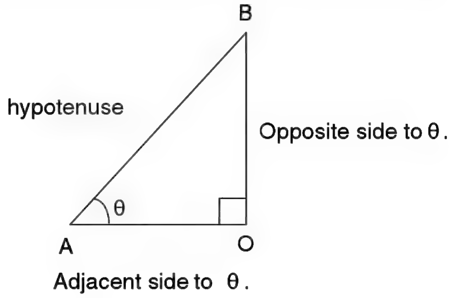


Figure 8.4

1. Sine of the angle θ (or) simply $\sin\theta$:

$$\sin\theta = \frac{\text{side opposite to angle } \theta}{\text{hypotenuse}} = \frac{OB}{AB}$$

2. Cosine of the angle θ or simply $\cos\theta$:

$$\cos\theta = \frac{\text{side adjacent to angle } \theta}{\text{hypotenuse}} = \frac{OA}{AB}$$

3. Tangent of the angle θ or simply $\tan\theta$:

$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{OB}{OA}$$

4. Cotangent of the angle θ or simply $\cot\theta$:

$$\cot\theta = \frac{\text{side adjacent to } \theta}{\text{side opposite to } \theta} = \frac{OA}{OB}$$

5. Cosecant of the angle θ or simply $\text{cosec}\theta$:

$$\text{cosec}\theta = \frac{\text{hypotenuse}}{\text{side opposite to } \theta} = \frac{AB}{OB}$$

6. Secant of the angle θ or simply $\sec\theta$:

$$\sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} = \frac{AB}{OA}$$

Observe that,

(i) $\text{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

(ii) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Example

If $\sin\theta = \frac{3}{5}$, then find the values of $\tan\theta$ and $\sec\theta$.

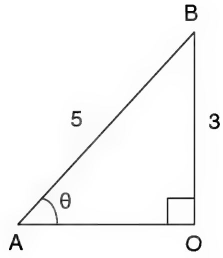


Figure 8.5

Solution

Given, $\sin\theta = \frac{3}{5}$

Let AOB be the right triangle such that $\angle OAB = \theta$

Assume that OB = 3 and AB = 5

Then, $OA = \sqrt{AB^2 - OB^2} = \sqrt{25-9} = 4$

So $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{OB}{OA} = \frac{3}{4}$ and

$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{AB}{OA} = \frac{5}{4}$

Some pythagorean triplets

(i) 3, 4, 5
(ii) 5, 12, 13
(iii) 8, 15, 17
(iv) 7, 24, 25
(v) 9, 40, 41

Trigonometric identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\sec^2\theta - \tan^2\theta = 1$
3. $\mathrm{cosec}^2\theta - \cot^2\theta = 1$

Table of values of trigonometric ratios for specific angles

$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\mathrm{cosec}\theta$	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0

From the above table, we observe that

1. $\sin\theta = \cos\theta$, $\tan\theta = \cot\theta$ and $\sec\theta = \mathrm{cosec}\theta$ if $\theta = 45°$
2. $\sin\theta$ and $\tan\theta$ are increasing functions when $0° \le \theta \le 90°$
3. $\cos\theta$ is a decreasing function when $0° \le \theta \le 90°$.

Worked out examples

Example

1. Find the value of $\tan 45° + 2\cos 60° - \sec 60°$

Solution

$\tan 45° + 2\cos 60° - \sec 60° = 1 + 2\left(\frac{1}{2}\right) - 2 = 1 + 1 - 2 = 0$

$\therefore \tan 45° + 2\cos 60° - \sec 60° = 0$

2. Using the trigonometric table, evaluate.
   (i) $\sin^2 30° + \cos^2 30°$
   (ii) $\sec^2 60° - \tan^2 60°$

Solution

(i) $\sin^2 30° + \cos^2 30° = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$

Hence, $\sin^2 30° + \cos^2 30° = 1$

(ii) $\sec^2 60° - \tan^2 60° = (2)^2 - \left(\sqrt{3}\right)^2 = 4 - 3 = 1$

Hence, $\sec^2 60° - \tan^2 60° = 1$

3. Find the values of $\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$ and $\tan 30°$. What do you observe?

Solution

$$\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3}.\frac{1}{\sqrt{3}}} = \frac{\frac{(3-1)}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}} \text{ and } \tan 30° = \frac{1}{\sqrt{3}}$$

Hence, $\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} = \tan 30°$

Trignometric ratios of compound angles

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and
   $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$ and
   $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
3. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Also, by taking A = B in the above relations, we get,

1. $\sin 2A = 2\sin A \cos A$
2. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.
3. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Examples

1. Find the value of sin75°.

Solution

We have, $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30° = \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}}.\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

2. Find the value of tan15°.

Solution

We have, $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$= \dfrac{(\sqrt{3}-1)}{(\sqrt{3}-1)} \times \dfrac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$

$= \dfrac{\left(\sqrt{3}-1\right)^2}{3-1} = \dfrac{3 + 1 - 2\sqrt{3}}{2} = \dfrac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

$\therefore \tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ or $2 - \sqrt{3}$

3. Eliminate θ from the equations $x = p \sin\theta$ and $y = q \cos\theta$.

Solution

We know that trigonometric ratios are meaningful when they are associated with some θ, i.e., we cannot imagine any trigonometric ratio with out θ. Eliminate θ means, eliminating the trigonometric ratio itself by suitable identities.

Given, $x = p \sin\theta$ and $y = q \cos\theta$

$\Rightarrow \frac{x}{p} = \sin\theta$ and $\frac{y}{q} = \cos\theta$

We know that, $\sin^2\theta + \cos^2\theta = 1$

So, $\left(\frac{x}{p}\right)^2 + \left(\frac{y}{q}\right)^2 = 1$

$\Rightarrow \frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$

Hence, the required equation is $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$

4. Find the relation obtained by eliminating θ from the equations $x = r \cos\theta + s \sin\theta$ and $y = r \sin\theta - s \cos\theta$.

Solution

Given, $x = r \cos\theta + s$ siwnθ

$\Rightarrow x^2 = (r\cos\theta + s\sin\theta)^2$

$= r^2 \cos^2\theta + 2rs \cos\theta.\sin\theta + s^2 \sin^2\theta$

Also $y = r \sin\theta - s \cos\theta$

$\Rightarrow y^2 = r^2 \sin^2\theta + s^2 \cos^2\theta - 2rs \sin\theta.\cos\theta$

$\Rightarrow x^2 + y^2 = r^2 (\cos^2\theta + \sin^2\theta) + s^2 (\sin^2\theta + \cos^2\theta)$

$= r^2 (1) + s^2 (1)$ [$\because \sin^2\theta + \cos^2\theta = 1$] $= r^2 + s^2$

Hence, the required relation is $x^2 + y^2 = r^2 + s^2$

5. Eliminate θ from the equations

$x = \operatorname{cosec}\theta + \cot\theta$

$y = \operatorname{cosec}\theta - \cot\theta$

Solution

Given, $x = \operatorname{cosec}\theta + \cot\theta$

and $y = \operatorname{cosec}\theta - \cot\theta$

Multiplying these equations, we get

$xy = (\operatorname{cosec}\theta + \cot\theta)(\operatorname{cosec}\theta - \cot\theta)$

$= \operatorname{cosec}^2\theta - \cot^2\theta = 1$

Hence, the required relation is $xy = 1$

6. Eliminate θ from the equations $m = \tan\theta + \cot\theta$ and $n = \tan\theta - \cot\theta$.

Solution

Given,

$m = \tan\theta + \cot\theta$ ---------- (1)

$n = \tan\theta - \cot\theta$ ---------- (2)

Adding (1) and (2), we get

$m + n = 2\tan\theta$

$\Rightarrow \tan\theta = \dfrac{m+n}{2}$

Subtracting (2) from (1), we get

$m - n = 2\cot\theta$

$\Rightarrow \cot\theta = \dfrac{m-n}{2}$

$\therefore \tan\theta.\cot\theta = \left(\dfrac{m+n}{2}\right).\left(\dfrac{m-n}{2}\right)$

$\Rightarrow \tan\theta.\dfrac{1}{\tan\theta} = \dfrac{m^2-n^2}{4}$

$\Rightarrow 1 = \dfrac{m^2-n^2}{4}$ (or) $m^2 - n^2 = 4$

Hence, by eliminating 'θ', we obtain the relation $m^2 - n^2 = 4$

7. If $\cos(A + B) = \dfrac{1}{2}$ and $\sec B = \sqrt{2}$, then find A and B.

Solution

Given,

$\text{Cos}(A + B) = \dfrac{1}{2}$

$\text{Cos}(A + B) = \cos 60°$

$A + B = 60°$--------- (1)

$\text{Sec } B = \sqrt{2} = \sec 45°$

$B = 45°$--------- (2)

From (1) & (2), we have

$A = 15°$ and $B = 45°$

8. Find the length of the chord which subtends an angle of 120° at the centre 'O' and which is at a distance of 5 cm from the centre.

Solution

Let the chord be AB and OD be the distance of chord from the centre of circle.

Given $\angle AOB = 120°$ and $OD = 5$ cm clearly, $\Delta OAD \cong \Delta OBD$, by S.S.S axiom

$\angle AOD = \angle BOD = \dfrac{1}{2}(\angle AOB) = 60°$ in ΔAOD.

$\tan 60° = \dfrac{AD}{OD}$

$\Rightarrow \sqrt{3} = \dfrac{AD}{5} \Rightarrow AD = 5\sqrt{3}$

the length of the chord $AB = 2AD = 10\sqrt{3}$ cm

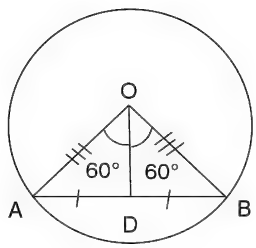


Figure 8.6

9. Evaluate $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

Solution

Given $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

Rationalize the denominator, i.e., $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}\cdot\dfrac{1-\sin\theta}{1-\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)^2}{(1)^2-\sin^2\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} (\because\ \sin^2\theta + \cos^2\theta = 1) = \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta \left(\because \sec\theta = \frac{1}{\cos\theta}, \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$$

Standard position of the angle

The angle is said to be in its standard position if its initial side coincides with the positive X-axis.

Note:

1. Rotation of terminal side in anti clock wise direction we consider the angle formed is positive and rotation in clock wise direction the angle formed is negative.
2. Depending upon the position of terminal side we decide the angle in different quadrants.

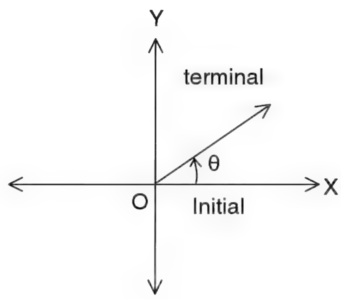


Figure 8.7

Coterminal angles

The angles that differ by either 360° or the integral multiples of 360° are called coterminal angles.

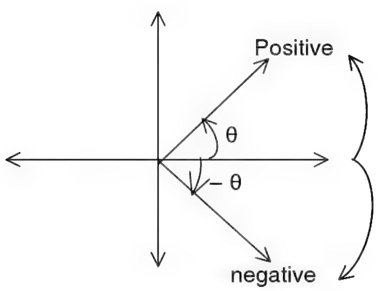


Figure 8.8

☛ *Example*

$60°, 360° + 60° = 420°, 2.360° + 60° = 780°$ are coterminal angles.

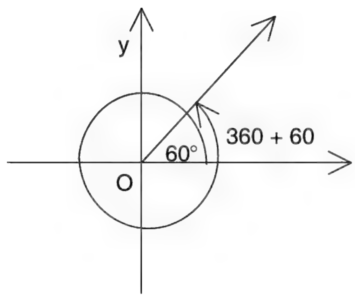


Figure 8.9

NOTE:

1. If θ is an angle then its coterminal angle is in the form of $(n.360 + \theta)$
2. The terminal side of coterminal angles in their standard position coincides.

Signs of trigonometric ratios

(i) If θ lies in the first quadrant, i.e., $0 < \theta < \frac{\pi}{2}$, then all the trigonometric ratios are taken positive.

(ii) If θ lies in the second quadrant, i.e., $\frac{\pi}{2} < \theta < \pi$, then only $\sin\theta$ and $\text{cosec}\theta$ are taken positive and all the other trigonometric ratios are taken negative.

(iii) If θ lies in the third quadrant, i.e., $\pi < \theta < \frac{3\pi}{2}$, then only $\tan\theta$ and $\cot\theta$ are taken positive and all the other trigonometric ratios are taken negative.

(iv) If θ lies in the fourth quadrant, i.e., $\frac{3\pi}{2} < \theta < 2\pi$, then only $\cos\theta$ and $\sec\theta$ are taken positive and all the other trigonometric ratios are taken negative.

Trigonometric ratios of (90° – θ)

$\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$; $\cot(90° - \theta) = \tan\theta$

$\text{cosec}(90° - \theta) = \sec\theta$; $\sec(90° - \theta) = \text{cosec}\theta$

Trigonometric ratios of (90° + θ)

$\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$

$\tan(90° + \theta) = -\cot\theta$; $\cot(90° + \theta) = -\tan\theta$

$\text{cosec}(90° + \theta) = \sec\theta$; $\sec(90° + \theta) = -\text{cosec}\theta$

Trigonometric ratios of (180° – θ)

$\sin(180° - \theta) = \sin\theta$; $\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$; $\cot(180° - \theta) = -\cot\theta$

$\text{cosec}(180° - \theta) = \text{cosec}\theta$; $\sec(180° - \theta) = -\sec\theta$

Trigonometric ratios of (180° + θ)

$\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$; $\cot(180° + \theta) = \cot\theta$
$\text{cosec}(180° + \theta) = -\text{cosec}\theta$; $\sec(180° + \theta) = -\sec\theta$

Similarly, the trigonometric ratios of $270° \pm \theta$ and $360° \pm \theta$ can be written.

NOTE: The trigonometric ratios of $(-\theta)$ are the same as the trigonometric ratios of $(360° - \theta)$.

So, $\sin(-\theta) = \sin(360° - \theta) = -\sin\theta$ and so on.

Examples

1. What is the value of tan 315°?

Solution

$\tan 315° = \tan(360° - 45°) = -\tan 45° = -1$

$\therefore \tan 315° = -1$

2. Find the value of $\sin^2 135° + \sec^2 135°$.

Solution

$$\sin 135° = \sin(180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}$$

$$\sec 135° = \sec(180° - 45°) = -\sec 45° = -\sqrt{2}$$

$$\therefore \sin^2 135° + \sec^2 135° = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\sqrt{2}\right)^2 = \frac{1}{2} + 2 = \frac{5}{2}$$

3. If $\cos A = \frac{5}{13}$ and A is not in first quadrant, then find the value of $\frac{\sin A - \cos A}{\tan A + 1}$

Solution

Given that $\cos A = \frac{5}{13}$ and A is not in first quadrant

A is in fourth quadrant

$\text{Sin } A = \frac{-12}{13}$ and $\text{Tan } A = \frac{12}{13}$

$$\text{Now, } \frac{\sin A - \cos A}{\tan A + 1} = \frac{\frac{-12}{13} - \frac{5}{13}}{\frac{12}{13} + 1} = \frac{-17}{13} \times \frac{13}{25}$$

$$= \frac{-17}{25} = \frac{\sin A - \cos A}{\tan A + 1} = \frac{-17}{25}$$

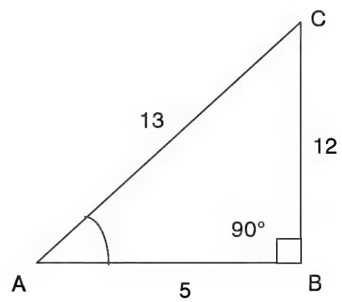


Figure 8.10

4. If ABCD is a cyclic quadrilateral, then find the value of cosAcosB – cosCcosD.

Solution

Given, ABCD is a cyclic quadrilateral

$A + C = 180°$ and $B + D = 180°$ ------ (1)

Now, cosAcosB – cosCcosD

$= \cos A \cos B - \cos(180° - A)\cos(180° - B) = \cos A - \cos B - \cos A(-\cos A)(-\cos B)$

$= \cos A \cos B - \cos A \cos B = 0$

5. If $\cot 15° = m$, then find $\dfrac{\cot 195° + \cot 345°}{\tan 15° - \cot 105°}$

Solution

Given, $\cot 15° = m = \tan 15° = \dfrac{1}{m}$ and $\tan 75° = m$ ($\because \tan(90° - 0) = \cot\theta$)

$$\frac{\cot 195° + \cot 345°}{\tan 15° - \cot 105°} = \frac{\cot(180° + 15°) + \cot(360° - 15°)}{\tan 15° - \cot(90° + 15°)} = \frac{-\cot 15° - \cot 15°}{\tan 15° - (-\tan 15°)}$$

$$= \frac{-2\cot 15°}{2\tan 15°} = -\cot^2 15° = -m^2 = \frac{\cot 195° + \cot 345°}{\tan 15° - \cot 105°} = -m^2$$

6. If $\sin\theta$ and $\cos\theta$ are the roots of the equation $mx^2 + nx + 1 = 0$, then find the relation between m and n.

Solution

The given equation is $mx^2 + nx + 1 = 0$

Here, $a = m$, $b = n$ and $c = 1$

$$\text{Sin }\theta + \cos\theta = \frac{-b}{a} = \frac{-n}{m}$$

$$\text{Sin }\theta\cos\theta = \frac{c}{a} = \frac{1}{m}$$

Consider

$$\text{Sin }\theta + \cos\theta = \frac{-n}{m}$$

$$\Rightarrow (\sin\theta + \cos\theta)^2 = \left(\frac{-n}{m}\right)^2$$

$$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \frac{n^2}{m^2}$$

$$= 1 + 2\sin\theta\cos\theta = \frac{n^2}{m^2} \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$1 + 2\left(\frac{1}{m}\right) = \frac{n^2}{m^2}\left(\because \sin\theta\cos\theta = \frac{1}{m}\right)$$

$$1 + \frac{2}{m} = \frac{n^2}{m^2}$$

$$\Rightarrow n^2 - m^2 = 2m$$

7. If $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, then find $\sin(\alpha + \beta)$.

Solution

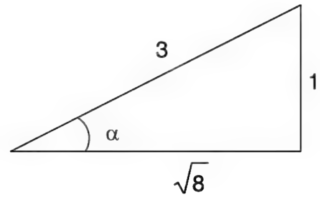


Figure 8.11

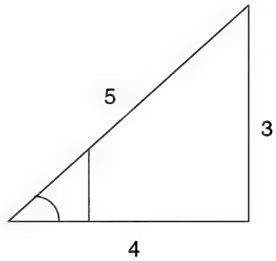


Figure 8.12

Given, $\sin\alpha = \frac{1}{3}$

$\text{Cos}\,\alpha = \frac{\sqrt{8}}{3}$

$\text{Cos}\,\beta = \frac{4}{5}$

$\text{Sin}\,\beta = \frac{3}{5}$

$\text{Sin}\,(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{1}{3}.\frac{4}{5} + \frac{\sqrt{8}}{3}.\frac{3}{5} = \frac{4+3\sqrt{8}}{15}$$

$$\sin(\alpha+\beta) = \frac{4+3\sqrt{8}}{15}$$

8. Express the following as a single trigonometric ratio

(i) $\sqrt{3}\,\cos\theta - \sin\theta$

(ii) $\sin\theta - \cos\theta$

Solution

(i) Given, $\sqrt{3}\,\cos\theta - \sin\theta = 2\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$

$$= 2\,(\cos\theta.\cos 30° - \sin\theta.\sin 30°) = 2\,(\cos(\theta+30°))$$

$$\Rightarrow \sqrt{3}\,\cos\theta - \sin\theta = 2\cos(\theta + 30°)$$

(ii) $\sin\theta - \cos\theta = \sqrt{2}\left(\frac{\sin\theta - \cos\theta}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta\right) = \sqrt{2}\left[\sin\theta\cos\left(\frac{\pi}{4}\right) - \cos\theta\sin\left(\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\,\sin\left(\theta - \frac{\pi}{4}\right) \because \sin(A-B) = \sin A\cos B - \cos A\sin B)$$

9. If A + B = 90°, then prove that

(i) $\sin^2 A + \sin^2 B = 1$

(ii) $\tan^2 A - \cot^2 B = 0$

Solution

Given, $A + B = 90°$

$\Rightarrow B = 90° - A$

(i) $\sin^2 A + \sin^2 B = \sin^2 A + \sin^2 (90° - A) = \sin^2 A + \cos^2 A$ ($\therefore \sin (90° - \theta 1 = \cos \theta) = 1$ $\sin^2 A + \sin^2 B = 1$

(ii) $\tan^2 A - \cot^2 B = \tan^2 A - \cot^2 (90° - A) = \tan^2 A - \tan^2 A$ ($\because \tan \theta = \cot (90° - \theta 1) = 0$ $\tan^2 A - \cot^2 B = 0$

10. Simplify the following.

(i) $\begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix}$

(ii) log (cot1°) + log cot2° + log (cot3°)+……+log(cot89°)

(iii) sin1°.sin2°…..sin181°

Solution

(i) $\begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix} = \cos A\,(-\cos A) - \sin A \sin A = -\cos^2 A - \sin^2 A$

$$= -(\sin^2 A + \cos^2 A) = -1$$

$$\begin{vmatrix} \cos A & \sin A \\ \sin A & -\cos A \end{vmatrix} = -1$$

(ii) log (cot1°) + log cot 2° + …. + log cot 89°

1. = log (cot 1° . cot 2° …..cot89°) [$\because$ log a + log b + ….. log n = log (abc ….n)]

= log (cot 1°. cot 2° …. cot 89°)

we know that,

cot 1° = tan 89°, cot 2° = tan 88° and so on

cot 1°. cot 89° = cot 1°.tan1° = 1 and so on

cot 1°.cot2° …cot 89° = 1

log (cot 1° cot 2° ….cot 89°) = log (1) = 0

(iii) sin 1°. sin 2°. sin 3° …… sin 181°

we know that, sin 180° = 0

sin 1°. sin 2° …. sin 180°.sin 181° = 0

Trigonometric tables

The values of trigonometric ratios of different angles can be found by using Natural sines, natural cosines, natural tangents etc. these tablular values are approximate and tally upto three decimal places.

The following example give an idea how to find any value of trigonometric ratios.

Example

1. Find t he value of sin $65° 28^1$.

Solution

Step (1): We have to refer the table of naturals ines in order to find the value of sin $65°28^1$.

Step (2): We have to slide our view from left to right in the row containing 65° until we reach the intersection with 28^1 and directly minutes table, so as to locate the nearest approximation in the minutes i.e., in this problem $24^1$and note down the value.

Step (3): We locate the difference of mintues (i.e., here 4^1) in the mean difference and add it to the value in step (2).

To get the value of sin $65° 28^1$, consider the following procedure.

0^1	6^1	12^1	18^1	24^1	.....	54^1	MEAN DIFFERENCES				
							1	2	3	4	5
0.9063	9070	9078	9085	9092	.....	9128	1	2	4	5	6

Now

Sin $65° 28^1$ = sin $65°.24^1+4^1$

= 0.9092 + 0.0005

= 0.9097

hence, sin $65° 28^1$ = 0.9097

2. Find the area of the right angle triangle with one of the acute angles being 65° and hypotenuse 6 cm.

Solution

Let the right triangle be ABC, $\angle B = 90°$, $\angle A = 65°$ and AC = 6 cm

From ΔABC

$$\text{Cos } 65° = \frac{AB}{AC}$$

$$0.4226 = \frac{AB}{6}$$

AB = 2.5356

$$\text{Sin } 65° = \frac{BC}{AC} = 0.9063 = \frac{BC}{6}$$

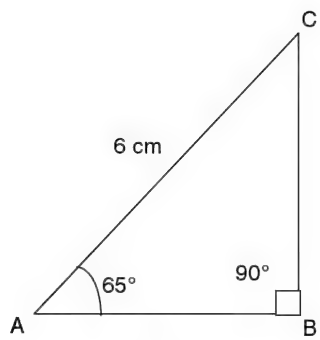


Figure 8.13

BC = 5.4378

$$\text{Area of } \Delta ABC = \frac{1}{2} AB \times BC = \frac{1}{2} (2.5356)\ (5.4378) = 6.8940 \text{ SQ. CM. (approximately)}$$

hence the area of the ΔABC is 6.8940 sq. cm.

3. Find the length of the chord which subtends an angle of 110° at the centre of the circle of radius 7 cm.

Solution

Let the chord be AB, O. be the centre of the circle and OD is shortest distance of the chord from the centre of circle.

Given OA = OB = 7 cm, $\angle AOB = 55°$

Clearly, $\Delta AOB \cong \Delta\ BOD$ by S.S.S. axiom

$$\angle AOD = \angle BOD = \frac{1}{2}\ \angle AOB = 55°$$

In ΔAOD,

$$\text{Sin } 55° = \frac{AD}{AO}$$

$$0.8192 = \frac{AD}{7}$$

$$AD = 5.7344$$

The length of the chord = 2 AD

= 2 (5.7344)

= 11.4688 cm

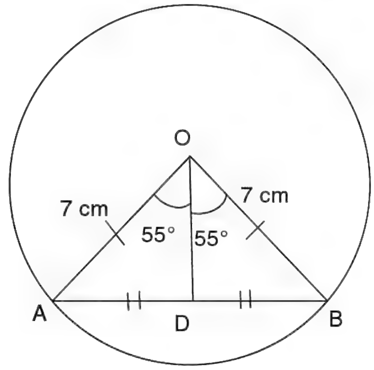


Figure 8.14

Heights and distances

(i) Let AB be a vertical line and PA and P^1B be two horizontal lines as shown in the figure above.

(ii) Let $\angle APB = \alpha$ and $\angle PBP^1 = \beta$. Then, (i) α is called the angle of elevation of the point B as seen from the point P and (ii) β is called the angle of depression of the point P as seen from the point B.

Note: Angle of elevation is always equal to the angle of depression.

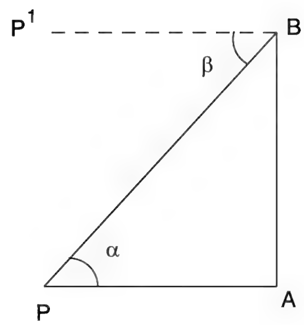


Figure 8.15

Applications

Examples

1. From a point on the ground which is at a distance of 50 m from the foot of the tower, the angle of elevation to the top of the tower is observed to be 30°. Find the height of the tower.

Solution

Let the height of the tower be h metres.

From ΔPAB, $\tan 30° = \dfrac{AB}{PA}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{50} \Rightarrow h = \frac{50}{\sqrt{3}} \text{ (or) } \frac{50\sqrt{3}}{3}$$

Hence, the height of the tower is $\dfrac{50\sqrt{3}}{3}$ m

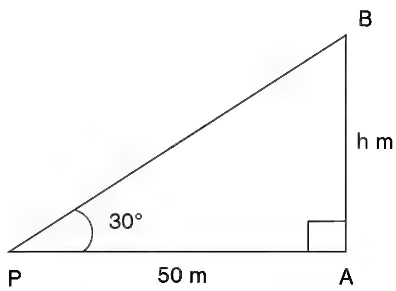


Figure 8.16

2. The angle of elevation of the top of a tower is 45°. On walking 20 m towards the tower along the line joining the foot of the observer and foot of the tower, the angle of elevation is found to be 60°. Find the height of the tower.

Solution

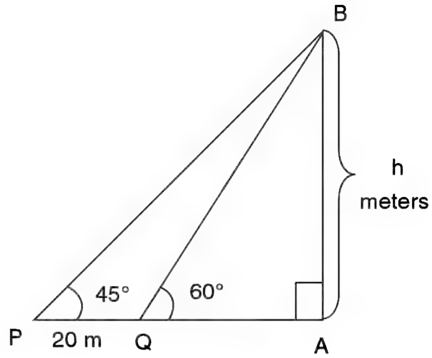


Figure 8.17

Let the height of the tower be h metres.

Let QA = x metres.

In Δ PAB,

$$\tan 45° = \frac{AB}{PA} \Rightarrow 1 = \frac{h}{20 + x} \Rightarrow 20 + x = h$$

$$\Rightarrow x = h - 20 -- \quad (1)$$

From Δ QAB,

$$\tan 60° = \frac{AB}{QA}$$

$$\Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow h = \sqrt{3}\,x \Rightarrow h = \sqrt{3}(h - 20), \text{(using (1))}$$

$$\Rightarrow (\sqrt{3} - 1)\, h = 20\sqrt{3} \Rightarrow h = \frac{20\sqrt{3}}{\sqrt{3}-1} = \frac{20\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{20\,(3+\sqrt{3})}{3-1} = 10\,(3+\sqrt{3})$$

Hence, the height of the tower is $10\,(3 + \sqrt{3})$ m.

3. From the top of a building 100 m high, the angles of depression of the bottom and the top of an another building just opposite to it are observed to be 60° and 45° respectively. Find the height of the building.

Solution

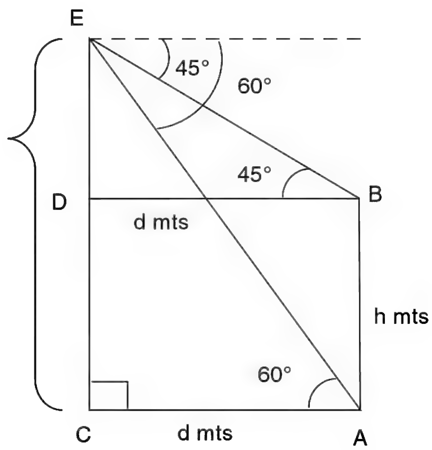


Figure 8.18

Let the height of the building be h metres.

Let AC = BD = d metres

From ΔBDE,

$$\tan 45° = \frac{ED}{BD} \Rightarrow 1 = \frac{100-h}{d}$$

$$\Rightarrow d = 100 - h -- \quad (1)$$

From Δ ACE,

$$\tan 60° = \frac{CE}{AC}$$

$\Rightarrow \sqrt{3} = \dfrac{100}{d} \qquad \Rightarrow \sqrt{3}\ d = 100$

$\Rightarrow \sqrt{3}\ (100 - h) = 100$ (using (1))

$\Rightarrow 100 - h = \dfrac{100}{\sqrt{3}} \qquad \Rightarrow h = 100 - \dfrac{100}{\sqrt{3}}$

$= 100\left(\dfrac{\sqrt{3} - 1}{\sqrt{3}}\right) = \dfrac{100(3 - \sqrt{3})}{3}$

Hence, the height of the tower is $\dfrac{100(3-\sqrt{3})}{3}$ m

test your concepts

Very short answer type questions

1. If $\sin\theta = \dfrac{1}{2}$ where $0° \le \theta \le 180°$, then the possible values of θ are __________.

2. $\cot\theta$ interms of $\sin\theta$ = _________ $(0 \le \theta \le 90°)$.

3. If A and B are two complementary angles, then $\sin A . \cos B + \cos A . \sin B$ = _________.

4. If the angle of a sector is 45° and the radius of the sector is 28 cm then the length of the arc is ______.

5. If ABCD is a cyclic quadrilateral, then $\tan A + \tan C$ = ___________.

6. $\dfrac{1 - \cos 2\theta}{2}$ = _________(in terms of $\sin\theta$).

7. cos 1°.cos2°.cos3°-------- cos 120° = _____________.

8. The angle $\dfrac{3\pi}{2}$ is equivalent to __________ in centesimal system.

9. If $A + B = 360°$, then $\dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ = _________.

10. If $\tan\theta + \cot\theta = 2$, then $\tan^{10}\theta + \cot^{10}\theta$ = _________ (where $0 < \theta < 90°$).

11. Write an equation eliminating θ from the equations $a = d\sin\theta$ and $c = d\cos\theta$.

12. Convert 250^g into other two measures.

13. $\sin(180 + \theta) + \cos(270 + \theta) + \cos(90 + \theta) + \sin(360 + \theta)$ = ______________.

14. If $\sin\theta + \cos\theta = 1$ and $0° \le \theta \le 90°$, then the possible values of θ are ________.

15. Evaluate $\sin^2 45° + \cos^2 60° + \text{cosec}^2 30°$

16. If ABCD is a cyclic quadrilateral, then find the value of $\cos A + \cos B + \cos C + \cos D$

17. $\text{cosec}(7\pi + \theta).\sin(8\pi + \theta)$ = ____________.

18. If $\sin\theta_1 = \frac{7}{25}$ and $\cos\theta_2 = \frac{24}{25}$, then find the relation between θ_1 and θ_2.

19. Find the value of tan 1140°.

20. If $\sin(A + B) = \cos(A - B) = \frac{\sqrt{3}}{2}$ then cot 2A = ______.

21. If Δ ABC is an isosceles triangle and right angled at B, then $\frac{\tan A + \tan C}{\cot A + \cot C}$ = ______________.

22. $[\sin(x - \pi) + \cos(x - \pi/2)].\cos(x - 2\pi)$ = _________.

23. $\tan(A + B)\tan(A - B)$ = ___________.

24. The angles of a quadrilateral are in the ratio 1 : 2 : 3 : 4. then the smallest angle in the centesimal system is ___________.

25. If $\alpha + \beta = 90°$ and $\alpha = \frac{\beta}{2}$ then $\tan\alpha.\tan\beta$ = __________.

26. $[\sin\alpha + \sin(180 - \alpha) + \sin(180 + \alpha)]\operatorname{cosec}\alpha$ = ____________.

27. Express $\frac{\tan\theta + 1}{\tan\theta - 1}$ as a single trigonometric ratio.

28. If $\operatorname{cosec}\theta + \cot\theta = 3$, then find $\cos\theta$.

29. The top of a building from a fixed point is observed at an angle of elevation 60° and the distance from the foot of the building to the point is 100 m. then the height of the building is __________.

30. If $\cot\theta = \frac{4}{3}$ where $180° < \theta < 270°$, then $\sin\theta + \cos\theta$ = _________.

Short answer type questions

31. If the tip of the pendulum of a clock travels 13.2 cm in one oscillation and the length of the pendulum is 6.3 cm, then the angle made by the pendulum in half oscillation in radian system is ___________.

32. If $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ are in H.P., then $\frac{\sin\theta + \tan\theta}{\cos\theta}$ = __________.

33. $\cot\frac{\pi}{18}.\cot\frac{\pi}{9}.\cot\frac{\pi}{4}.\cot\frac{4\pi}{9}.\cot\frac{7\pi}{18}$ = ____________.

34. If $\cot\theta = \frac{3}{4}$ and θ is acute, then find the value of $\frac{\tan\theta + \cot\theta}{\sec\theta + \operatorname{cosec}\theta}$.

35. Simplify $\sin(A + 45°)\sin(A - 45°)$

36. Eliminate θ from the following equations : $x = a\sin\theta$, $y = b\cos\theta$ and $z = a\sin^2\theta + b\cos^2\theta$

37. If $\sin A = \frac{3}{5}$ and A is not in the first quadrant, then find $\frac{\cos A + \sin 2A}{\tan A + \sec A}$.

38. If $\cos(A - B) = \frac{5}{13}$ and $\sin(A + B) = \frac{4}{5}$, then find sin 2B.

39. If $\operatorname{cosec}\theta - \cot\theta = 2$, then find the value of $\operatorname{cosec}^2\theta + \cot^2\theta$.

40. Prove that $\frac{1+\cos A}{1-\cos A} = (\operatorname{cosec}A + \cot A)^2$

41. If $\tan 28° = n$, then find the value of $\frac{\tan 152° + \tan 62°}{\tan 242° + \tan 28°}$.

42. If $3\sin A + 4\cos A = 4$, then find $4\sin A - 3\cos A$.

43. A ladder of length 50 m rests against a vertical wall, at height of 30 m from the grand. Find the inclination of the ladder with the horizontal. Also find the distance between the foot of the ladder and the wall.

44. Eliminate θ from the following equations: $x\sin\alpha + y\cos\alpha = p$ and $x\cos\alpha - y\sin\alpha = q$

45. Prove that $\left(\frac{1-\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{1+\sin\alpha}\right)\left(\sec\alpha + \frac{1}{\cot\alpha}\right) = 2$

Essay type questions

46. Show that $6(\sin x + \cos x)^4 + 12(\sin x - \cos x)^2 + 8(\sin^6 x + \cos^6 x) = 26$.

47. Prove that $\frac{\cot A + \operatorname{cosec}A - 1}{\cot A - \operatorname{cosec}A + 1} = \frac{1+\cos A}{\sin A}$

48. The angle of depression of the top of the tower from the top of a building is 30° and angle of elevation of the top of the tower from the bottom of the building is 45° and if the height of the tower is 20m then find the height of the building.

49. A vertical pole is 60 m high. The angles of depression of two points P and Q on the ground are 30° and 45° respectively. If the points P and Q lie on either side of the pole, then find the distance PQ.

50. Prove that $\sin^8\theta - \cos^8\theta = \cos 2\theta\,(2\sin^2\theta\cos^2\theta - 1)$

CONCEPT APPLICATION

Concept Application Level—1

1. If $\sin x° = \sin \alpha x$, then α is

(1) $\frac{180}{\pi}$ (2) $\frac{\pi}{270}$ (3) $\frac{270}{\pi}$ (4) $\frac{\pi}{180}$

2. If in a triangle ABC, A and B are complementary, then tanC is

(1) ∞ (2) 0 (3) 1 (4) $\sqrt{3}$

3. If $\sin\alpha = \frac{4}{5}$ and $\cos\beta = \frac{4}{5}$, then which of the following is true?

(1) $\alpha < \beta$ (2) $\alpha > \beta$ (3) $\alpha = \beta$ (4) None of these

4. $\sin^2 20 + \sin^2 70$ is equal to ______ .

(1) 1 (2) -1 (3) 0 (4) 2

5. $\cos 50° 50^1 \cos 9° 10^1 - \sin 50° 50^1 \sin 9° 10^1 =$

(1) 0 (2) $\frac{1}{2}$ (3) 1 (4) $\frac{\sqrt{3}}{2}$

6. $\sin\theta \cos(90° - \theta) + \cos\theta \sin(90° - \theta)$ ______ .

(1) -1 (2) 2 (3) 0 (4) 1

7. A wheel makes 20 revolutions per hour. The radians it turns through 25 minutes is

(1) $\frac{50\pi^c}{7}$ (2) $\frac{250\pi^c}{3}$ (3) $\frac{150\pi^c}{7}$ (4) $\frac{50\pi^c}{3}$

8. $\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta} =$

(1) -1 (2) 2 (3) 0 (4) 1

9. Simplified expression of $(\sec\theta + \tan\theta)(1 - \sin\theta)$ is

(1) $\sin^2\theta$ (2) $\cos^2\theta$ (3) $\tan^2\theta$ (4) $\cos\theta$

10. If $a = \sec\theta - \tan\theta$ and $b = \sec\theta + \tan\theta$, then

(1) $a = b$ (2) $\frac{1}{a} = \frac{-1}{b}$ (3) $a = \frac{1}{b}$ (4) $a - b = 1$

11. If $\sec\alpha + \tan\alpha = m$, then $\sec^4\alpha - \tan^4\alpha - 2\sec\alpha\tan\alpha$ is

(1) m^2 (2) $-m^2$ (3) $\frac{1}{m^2}$ (4) $\frac{-1}{m^2}$

12. If $\sin^4 A - \cos^4 A = 1$, then $(A/2)$ is $(0 < A \le 90°)$

(1) 45° (2) 60° (3) 30° (4) 40°

13. The value of tan 15° tan 20° tan 70° tan 75° is

(1) -1 (2) 2 (3) 0 (4) 1

14. In a ΔABC, $\tan\left(\frac{A + C}{2}\right) =$

(1) $\tan\frac{B}{2}$ (2) $\cot\frac{B}{2}$ (3) $-\tan B$ (4) $\cot B$

15. If tan (A – 30°) = 2 – $\sqrt{3}$, then find A.

(1) $\frac{\pi^c}{2}$ (2) $\frac{\pi^c}{4}$ (3) $\frac{\pi^c}{6}$ (4) $\frac{\pi^c}{3}$

16. If $\sin^4\theta - \cos^4\theta = K^4$ then $\sin^2\theta - \cos^2\theta$ is

(1) K^4 (2) K^3 (3) K^2 (4) K

17. $\frac{\tan^3\theta - 1}{\tan\theta - 1} =$

(1) $\sec^2\theta + \tan\theta$ (2) $\sec^2\theta - \tan\theta$ (3) 0 (4) $\tan\theta - \sec^2\theta$

18. For all values of θ, 1 + cosθ can be ______ .

(1) positive (2) negative (3) non-positive (4) non-negative

19 If sin 3θ = cos (θ – 6°), where 3θ and (θ – 6°) are acute angles then the value of θ is

(1) 42° (2) 24° (3) 12° (4) 26°

20. (cosecA – sinA) (secA – cosA) (tanA + cotA) =

(1) –1 (2) 2 (3) 0 (4) 1

21. If x = a (cosec θ + cot θ) and y = b (cot θ – cosec θ), then

(1) xy – ab = 0 (2) xy + ab = 0 (3) $\frac{x}{a} + \frac{y}{b} = 1$ (4) $x^2 y^2 = ab$

22. The value of $\frac{\cos^4 x + \cos^2 x \sin^2 x + \sin^2 x}{\cos^2 x + \sin^2 x \cos^2 x + \sin^4 x}$ is

(1) 2 (2) 1 (3) 3 (4) 0

23. $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$ is equal to

(1) $2\sec^2\theta$ (2) $2\cos^2\theta$ (3) 0 (4) 1

24. If tan (α + β) = $\frac{1}{2}$ and tan α = $\frac{1}{3}$, then tan β =

(1) $\frac{1}{6}$ (2) $\frac{1}{7}$ (3) 1 (4) $\frac{7}{6}$

25. The value of log sin 0° + log sin 1° + log sin 2° + …….+ log sin 90° is

(1) 0 (2) 1 (3) –1 (4) Undefined

26. Which of the following is not possible?

(1) sin θ = $\frac{3}{5}$ (2) sec θ = 100 (3) cosec θ = 0.14 (4) None of these

27. $\sin^2 20° + \cos^2 160° - \tan^2 45° =$

(1) 2 (2) 0 (3) 1 (4) -2

28. $\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}+\dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}=$

(1) $\dfrac{2}{1-2\cos^2\theta}$ (2) $\dfrac{2}{2\sin^2\theta-1}$ (3) Both (1) and (2) (4) None of these

29. The length of the side (in cm) of an equilateral triangle inscribed in a circle of radius 8 cm is

(1) $16\sqrt{3}$ (2) $12\sqrt{3}$ (3) $8\sqrt{3}$ (4) $10\sqrt{3}$

30. Which among the following is true?

(1) $\sin 1° > \sin 1^c$ (2) $\sin 1° < \sin 1^c$ (3) $\sin 1° = \sin 1^c$ (4) None of these

Concept Application Level—2

31. If $2\sin\alpha + 3\cos\alpha = 2$, then $3\sin\alpha - 2\cos\alpha =$

(1) ± 3 (2) ± 1 (3) 0 (4) ± 2

32. If $\dfrac{\sin^2\alpha - 3\sin\alpha + 2}{\cos^2\alpha} = 1$, then α can be

(1) 60° (2) 45° (3) 0° (4) 30°

33. If $\cot A = \dfrac{5}{12}$ and A is not in the first quadrant, then $\dfrac{\sin A - \cos A}{1 + \cot A}$ is

(1) $\dfrac{-74}{25}$ (2) $\dfrac{-84}{221}$ (3) $\dfrac{-87}{223}$ (4) None of these

34. If $\dfrac{1+\sin\alpha}{1-\sin\alpha} = \dfrac{m^2}{n^2}$, then $\sin\alpha$ is

(1) $\dfrac{m^2+n^2}{m^2-n^2}$ (2) $\dfrac{m^2-n^2}{m^2+n^2}$ (3) $\dfrac{m^2+n^2}{n^2-m^2}$ (4) $\dfrac{n^2-m^2}{m^2+n^2}$

35. If $\sin\theta - \cos\theta = \dfrac{3}{5}$, then $\sin\theta\cos\theta =$

(1) $\dfrac{16}{25}$ (2) $\dfrac{9}{16}$ (3) $\dfrac{9}{25}$ (4) $\dfrac{8}{25}$

36. If ABCD is a cyclic quadrilateral, then the value of $\cos^2 A - \cos^2 B - \cos^2 C + \cos^2 D$ is

(1) 0 (2) 1 (3) -1 (4) 2

37. The length of minute hand of a wall clock is 12cm. Find the distance covered by the tip of the minute hand in 25 minutes.

(1) $\dfrac{220}{7}$ cm (2) $\dfrac{110}{7}$ cm (3) $\dfrac{120}{7}$ cm (4) $\dfrac{240}{7}$ cm

38. $\sin^2 2° + \sin^2 4° + \sin^2 6° + \ldots\ldots\ldots\ldots + \sin^2 90° =$ ________

(1) 22 (2) 23 (3) 44 (4) 45

39. A straight highway leads to the foot of a tower of height 50 m. From the top of the tower, the angles of depression of two cars standing on the highway are 30° and 60°. What is the distance between the two cars? (in metres)

(1) $\frac{100}{\sqrt{3}}$ (2) $50\sqrt{3}$ (3) $\frac{50}{\sqrt{3}}$ (4) $100\sqrt{3}$

40. The angle of elevation of the top of a hill at the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30°. If the tower is 50 m high, what is the height of the hill?

(1) 180 m (2) 150 m (3) 100 m (4) 120 m

41. $\tan 38° - \cot 22° =$

(1) $\frac{1}{2}$ cosec 38° sec 22° (2) 2 sin 22° cos 38°

(3) $-\frac{1}{2}$ cosec 22° sec 38° (4) None of these

42. $\frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{1-\cos\theta} =$

(1) $2\sin\theta$ (2) $2\cos\theta$ (3) $2\operatorname{cosec}\theta$ (4) $2\sec\theta$

43. $\sqrt{-4+\sqrt{8+16\operatorname{cosec}^4\alpha+\sin^4\alpha}} =$

(1) $\operatorname{cosec}\alpha - \sin\alpha$ (2) $2\operatorname{cosec}\alpha + \sin\alpha$
(3) $2\operatorname{cosec}\alpha - \sin\alpha$ (4) $\operatorname{cosec}\alpha - 2\sin\alpha$

44. The angles of depression of the top and the bottom of a 7 m tall building from the top of a tower are 45° and 60° respectively. Find the height of the tower in metres.

(1) $7(3+\sqrt{3})$ (2) $\frac{7}{2}(3-\sqrt{3})$

(3) $\frac{7}{2}(3+\sqrt{3})$ (4) $7(3-\sqrt{3})$

45. If tan 86° = m, then $\frac{\tan 176° + \cot 4°}{m + \tan 4°}$ is

(1) $\frac{m^2-1}{m^2+1}$ (2) $\frac{m^2+1}{1-m^2}$ (3) $\frac{1-m^2}{1+m^2}$ (4) $\frac{m^2+1}{m^2-1}$

Concept Application Level—3

46. There is a small island in the river which is 100 m wide and a tall tree stands on the island. P and Q are points directly opposite each other on the two banks and in line with the tree. If the angles of

elevation of the top of the tree from P and Q are respectively are 30° and 45°, find the height of the tree (in metres).

(1) $50(\sqrt{3}-1)$ (2) $50(\sqrt{3}+1)$

(3) $100(\sqrt{3}+1)$ (4) $100(\sqrt{3}-1)$

47. A balloon is connected to a meteorological ground station by a cable of length 215 m inclined at 60° to the horizontal. Determine the height of the balloon from the ground. Assume that there is no slack in the cable.

(1) $107.5\sqrt{3}$ m (2) $100\sqrt{3}$ m (3) $215\sqrt{3}$ m (4) $215/\sqrt{3}$ m

48. If sin2A = 2sinAcosA and sin20° = K, then the value of cos20° cos40° cos80° cos160° =

(1) K (2) $-\sqrt{1-K^2}$ (3) $\dfrac{\sqrt{1-K^2}}{8}$ (4) $-\dfrac{\sqrt{1-K^2}}{8}$

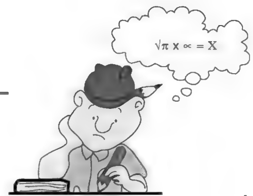
49. If $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta = 2\sqrt{2}$, then the value of θ can be

(1) 0° (2) –45° (3) 30° (4) –60°

50. A circus artist climbs from the ground along a rope which is stretched from the top of a vertical pole and tied at the ground at a certain distance from the foot of the pole. The height of the pole is 12 m and the angle made by the rope with the ground is 30°. Calculate the distance covered by the artist in reaching the top of the pole.

(1) 24 m (2) 6 m (3) 12 m (4) None of these

KEY



Very short answer type questions

1. 30° or 150°
2. $\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$
3. 1
4. 22 cm
5. 0°.
6. $\sin^2\theta$
7. 0
8. 300^g
9. 0
10. 2
11. $a^2 + c^2 = d^2$
12. $\dfrac{5\pi^c}{4}$
13. 0
14. 0° and 90°
15. $\dfrac{19}{4}$
16. 0
17. – 1
18. $\theta_1 = \theta_2$
19. $\sqrt{3}$
20. 0
21. 1
22. 0
23. $\dfrac{\tan^2 A - \tan^2 B}{1-\tan^2 A\tan^2 B}$.
24. 40^g
25. 1
26. 1

27. $-\tan\left(\theta+\frac{\pi}{4}\right)$

28. $\frac{4}{5}$

29. $100\sqrt{3}$ m

30. $\frac{-7}{5}$

37. $\frac{22}{25}$

38. $\frac{-16}{65}$

39. $\frac{17}{8}$

41. $\frac{1-n^2}{1+n^2}$

Short answer type questions

31. $\frac{\pi^c}{3}$

32. 2

33. 1

34. $\frac{5}{7}$

35. $-\frac{1}{2}\cos 2A$

36. $bx^2 + ay^2 = abz$

42. ± 3

43. 40 m.

44. $x^2 + y^2 = p^2 + q^2$

Essay type questions

48. $\frac{20\,(1+\sqrt{3})}{\sqrt{3}}$m

49. $60\ (\sqrt{3}+1)$m

key points for selected questions

Very short answer type questions

12. Use the relation $\frac{D}{90}=\frac{G}{100}=\frac{R}{\frac{\pi}{2}}$.

15. Substitute the values and simplify.

16. As ABCD is a cyclic quadrilateral, A + C = 180° and B + D = 180°.

18. (i) Find $\cos\theta_1$ and $\sin\theta_2$.
    (ii) Then substitute the values in $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$.

19. $\tan(n \times 360° + \theta) = \tan\theta$

20. (i) Find the angles of A+B and A–C
    (ii) Solve A+B, A–B for A.

21. A+C = 90°

24. Sum of angles of a quadrilateral = 360°

25. Obtain the values of α and β using the given conditions.

27. Use tan(A+B) formulae.

28. (i) If $\operatorname{cosec}\theta + \cot\theta = x$, then $\operatorname{cosec}\theta - \cot\theta = \frac{1}{x}$.
    (ii) Then find $\operatorname{cosec}\theta$ and $\sin\theta$.
    (iii) $\cos\theta = \sqrt{1-\sin^2\theta}$

30. (i) Consider the adjacent side and opposite side as 4K, 3K
    (ii) Find hypotenuse, then find $\sin\theta$, $\cos\theta$.

Short answer type questions

31. Use formula $\because = r\theta$.

32. If a, b, c are in HP then $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are A.P.

33. (i) Convert $\cot\frac{7\pi}{18}, \cot\frac{4\pi}{9}$ angles in $\tan\frac{\pi}{18}.\tan\frac{\pi}{9}$.
    (ii) $\tan\theta.\cot\theta = 1$

34. (i) Consider the adjacent side and the opposite sides as 3k and 4k respectively.
(ii) Then find hypotenuse.
(iii) Then find $\tan\theta$, $\sec\theta$ and $\text{cosec}\theta$.

35. Recall the formula of, $\sin(A \pm B)$.

36. Find $\sin^2\theta$ and $\cos^2\theta$ and then substitute in $z = a\sin^2\theta + b\cos^2\theta$

37. (i) sinA is positive in Q_1 and Q_2
(ii) Then find cosA, tanA and secA according to the second quadrant.
(iii) Substitute the values and simplify.

38. (i) First of all find $\sin(A - B)$ and $\cos(A + B)$
(ii) $\sin 2B = \sin[(A + B) - (A - B)]$

39. (i) If $\text{cosec}\theta + \cot\theta = x$, then $\text{cosec}\theta - \cot\theta = \frac{1}{x}$
(ii) Find $\text{cosec}\theta$ and $\cot\theta$

40. (i) Consider LHS expression.
(ii) Multiply both the numerator and the denominator with $(1 + \cos A)$.
(iii) Use $\cos^2 A + \sin^2 A = 1$ and simplify.

41. (i) $152° = 180° - 28°$
$62° = 90° - 28°$
$242° = 270° - 28°$
(ii) If $\tan\theta = x$, then $\cot\theta = \frac{1}{x}$.

42. (i) Let $4\sin A - 3\cos A = K$.
(ii) Square on both the sides of the given equation and assumed equation.
(iii) And then add the equations and find k.

43. (i) Draw the diagram according to the given data.
(ii) use $\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}}$ to find θ.
(iii) Then find the required distance by using $\tan\theta$.

44. Square and add the given equations.

Essay type questions

46. (i) Take $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x\cos^2 x$.
(ii) Expand the other two terms and simplify.

47. (i) Replace $\text{cosec}^2A - \cot^2A$ by 1 in the numerator.
(ii) Then take out $(\cot A + \text{cosec}\,A)$ as common.
(iii) Then cancel the common factor.
(iv) Then write cot A in terms of cosA and sin A.

48. (i) Draw the diagram according to the given data.
(ii) Let height of the building be h m.
(iii) Write tan 30° and tan 45° from the diagram.

49. (i) Draw the diagram according to the given data.
(ii) Write $\tan 30^0$ and $\tan 45^0$ from the diagram.

50. $\sin^2\theta + \cos^2\theta = 1$.

KEY

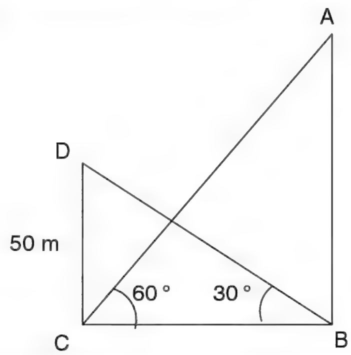
Concept Application Level—1,2,3

1. 4	7. 4	13. 4	19. 2
2. 1	8. 4	14. 2	20. 4
3. 2	9. 4	15. 2	21. 2
4. 1	10. 3	16. 1	22. 2
5. 2	11. 3	17. 1	23. 1
6. 4	12. 1	18. 4	24. 2
			25. 4

Concept Application Level—1,2,3

Key points for select questions

1. Use the relation between degrees and radians.
2. $A + B = 90°$ and $A + B + C = 180°$.
3. Find $\sin\beta$ and compare $\sin\alpha$ and $\sin\beta$.
4. $\sin\theta = \cos(90 - \theta)$
5. Use $\cos A \cos B - \sin A \sin B = \cos(A + B)$.
6. $\cos(90 - \theta) = \sin\theta$ and $\sin(90 - \theta) = \cos\theta$.
7. Find the angle covered in one hour.
8. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
9. Write $\sec\theta$ and $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.
10. Use the identity $\sec^2\theta - \tan^2\theta = 1$.
11. Simplify the given expression and find $\sec\alpha - \tan\alpha$.
12. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
13. $\tan\theta = \cot(90 - \theta)$ and $\tan\theta.\cot\theta = 1$.
14. $A + B + C = 180°$.
15. Use the identity $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
16. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$.
17. Use $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
18. Recall the range of $\cos\theta$.
19. Use complementary angles.
20. Replace cosecA by $\dfrac{1}{\sin A}$ and secA by $\dfrac{1}{\cos A}$.
21. Use $\operatorname{cosec}^2\theta - \cot^2\theta = 1$
22. Simplify the numerator and denominator by taking common terms appropriately
23. Take LCM and simplify.
24. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.
25. $\log a + \log b + \log c + \ldots\ldots = \log(a.b.c\ldots.)$.
26. Recall the ranges of $\sin\theta$ and $\cos\theta$.
27. $\sin(180 - \theta) = \sin\theta$.
28. Simplify the expression.
29. Equal chords subtend equal angles at the centre. ($2\sin 60°$).
30. (i) $1°$ is always less than 1^c.
    (ii) The value $\sin\theta$ increases from $0°$ to $90°$.
31. Put $3\sin\alpha - 2\cos\alpha = k$, square the two equations and add.
32. Use the identity $\cos^2\alpha + \sin^2\alpha = 1$ and convert the equation into quadratic form in terms of $\sin\alpha$ and the solve.
33. (i) cot A is positive in the first and the third quadrants. As cot A not in first quadrant, cot A in third quadrant.
    (ii) Using the right angle triangle find the values of sinA, cosA and cotA in third quadrant.
34. Apply componendo – dividendo rule i.e., if $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.
35. Square the given equation, then use the identity $\cos^2\theta + \sin^2\theta = 1$ to evaluate the value of $\sin\theta\cos\theta$.
36. (i) In a cyclic quadrilateral, sum of the opposite angles is $180°$.
    (ii) Use the result, $\cos(180 - \theta) = -\cos\theta$ and simplify.
37. $\ell = r\theta$, where ℓ = length, r = radians and θ = angle in radians.
38. Use the results $\sin^2 88 = \cos^2 2$ and $\sin^2\theta + \cos^2\theta = 1$ to simplify the given expression.
39. Use $\tan\theta$ = opposite side/adjacent side in the triangles so formed.
40.



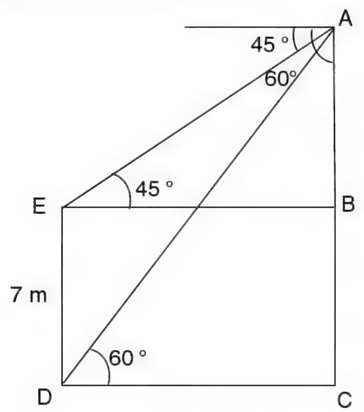
Use the diagram, by taking the values of tan30° from ΔBCD and tan60° from ΔABC, find CD.

41. (i) Convert the given trigonometric values in terms of $\sin\theta$, $\cos\theta$, then simplify to obtain the required value.
(ii) Use the formula, cost (A+B) = cosAcosB – sinA.sinB.

42. Take L.C.M. and use the identity $\sin^2\theta + \cos^2\theta = 1$ to simplify the given equation.

43. Express the square root function as $(a + b)^2$ and remove the first square root. Again express the obtained function as $(a - b)^2$ and then simplify.

44. The following figure is drawn as per the description in the problem.



Use the diagram, to find DC from ΔADC and AB from ΔAEB as DC = BE.

45. Express tan176° = – cot 86°, cot 4° = tan 86° and tan 4° = cot 86°. Then substitute these values to simplify the given expression.

46. Draw the figure as per the description given in the problem. Two right triangles are formed. Use appropriate trigonometric ratios by considering each of the triangle and find the height of the tree.

47. Draw a right angle triangle according to the data and take the ratio of $\sin\theta$ to evaluate the height of the balloon from the ground.

48. (i) Multiply and divide the given expression with sin 20°.
(ii) Use sin2A = 2sinAcosA
(iii) sin(180–A) = sinA.

49. (i) Divide the equation with $2\sqrt{2}$.

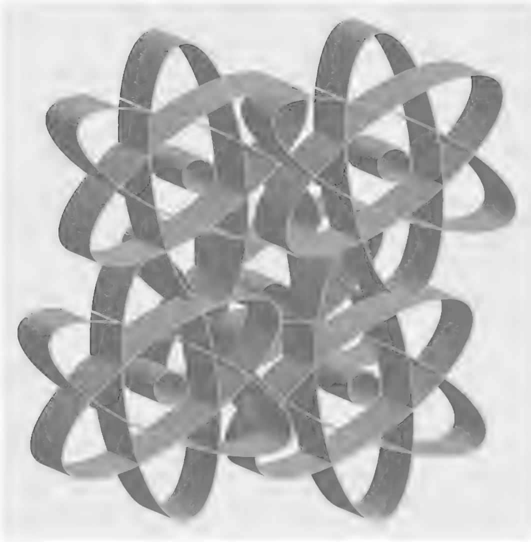
Then substitute $\frac{1}{2}$ = cos60° and $\frac{\sqrt{3}}{2}$ = sin60°

(ii) Now apply the formula, cos(A + B) = cosAcosB – sinAsinB then obtain the value of θ.

50. (i) Draw a figure as per the situation described then use the concept, $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$
(ii) Given θ and opposite side find hypotenuse which is the required length of the rope.

CHAPTER 9

Limits



INTRODUCTION

Limit of a function

Let y = f(x) be a function of x and let 'a' be any real number.

We first understand what a 'limit' is. A limit is the value, a function approaches, as the independent variable of the function (usually 'x') gets nearer and nearer to a particular value. In other words, when x is very close to a certain number, say a, what is f(x) very close to? It may be equal to f(a) but may be different. It may exist even when f(a) is not defined.

Meaning of '$x \to a$'

Let x be a variable and 'a' be a constant. If x assumes values nearer and nearer to 'a', then we say that 'x tends to a' or 'x approaches a' and is written as '$x \to a$'. By $x \to a$, we mean that $x \neq a$ and x may approach 'a' from left or right, which is explained in the example given below.

What is the limit of the function $f(x) = x^2$ as x approaches 3? The expression "the limit as x approaches to 3" is written as: $\lim\limits_{x \to 3}$. Let us check out some values of $\lim\limits_{x \to 3}$, as x increases and gets closer to 3, without ever exactly getting three.

when x = 2.9, f(x) = 8.4100

when x = 2.99, f(x) = 8.9401

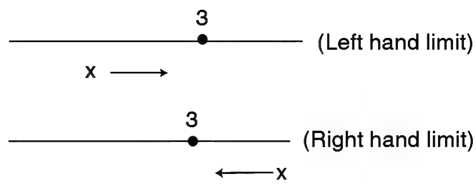
when x = 2.999, f(x) = 8.9940

when x = 2.9999, f(x) = 8.9994

As x increases and approaches 3, f(x) gets closer and closer to 9 and since x tends to 3 from the left this is called the 'left-hand limit' and is written as $\lim\limits_{x \to 3}$.

Now, let us see what happens when x is greater than 3.

when x = 3.1, f(x) = 9.6100



when $x = 3.01, f(x) = 9.0601$

when $x = 3.001, f(x) = 9.0060$

when $x = 3.0001, f(x) = 9.0006$

As x decreases and approaches 3, f(x) still approaches 9. This is called the 'right-hand limit' and is written as $\lim\limits_{x\to 3^+}$.

As $\mathop{\mathrm{Lt}}\limits_{x\to 3^-} f(x) = \mathop{\mathrm{Lt}}\limits_{x\to 3^+} f(x) = 9$, we write that $\lim\limits_{x\to 3} x^2 = 9$.

Meaning of the symbol: $\lim\limits_{x\to a} f(x) = \ell$

Let f(x) be a function of x where x takes values closer and closer to 'a' ($\neq$ a), and let f(x) assume values nearer and nearer to ℓ.

We say, f(x) tends to the limit 'ℓ' as x tends to 'a'.

The following are some simple algebraic rules of limits.

(i) $\lim\limits_{x\to a} k\, f(x) = k \lim\limits_{x\to a} f(x)$

(ii) $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(iii) $\lim\limits_{x\to a} [f(x) \,.\, g(x)] = \lim\limits_{x\to a} f(x) \,.\, \lim\limits_{x\to a} g(x)$

(iv) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ (where $\mathop{\mathrm{Lt}}\limits_{x\to a} g(x) \neq 0$)

NOTE:

1. If the left hand limit of a function is not equal to the right hand limit of the function, then the limit does not exist.
2. A limit equal to infinity does not imply that the limit does not exist.

Example

The limit of a chord of a circle passing through a fixed point Q and a variable point P as P approaches Q is the tangent to the circle at P.

If PQ is chord of a circle, when P approaches Q along the circle, then the chord becomes the tangent to the circle at P.

Problems: While evaluating the limits we use the following methods.

(i) Method of substitution.

(ii) Method of factorization.

(iii) Method of rationalization.

(iv) Using the formula $\mathop{\mathrm{Lt}}\limits_{x\to a} \left[\dfrac{x^n - a^n}{x - a}\right] = na^{n-1}$.

(i) Method of substitution

In this method we directly substitute the value of x in the given function to obtain the limit value.

Examples 1. $\underset{x\to 2}{\text{Lt}}\left[x^2+5x+6\right]=(2)^2+5(2)+6=20$

2. $\underset{x\to 3}{\text{Lt}}\left[\dfrac{x^2+4x+1}{x+5}\right]=\dfrac{(3)^2+4(3)+1}{3+5}=\dfrac{22}{8}=\dfrac{11}{4}$

Indeterminate form

If $f(x) = \dfrac{x^2-9}{x-3}$, then $f(3) = \dfrac{3^2-9}{3-3}=\dfrac{0}{0}$, which is not defined. By substituting $x = 3$, the function assumes a form whose value can't be determined form is called an indeterminate form.

(ii) Method of factorization

If $\dfrac{f(x)}{g(x)}$ assumes an indeterminate form when $x = a$, then there exists a common factor for $f(x)$ and $g(x)$. We remove the common factors, and use the substitution method to find the limit.

Example

Evaluate $\underset{x\to 2}{\text{Lt}}\left[\dfrac{x^2-5x+6}{x^2-3x+2}\right]$

Solution

When $x = 2$, $\dfrac{4-10+6}{4-6+2}=\dfrac{0}{0}$ an indeterminate form.

Now, $x^2 - 5x + 6 = (x - 3)(x - 2)$

$x^2 - 3x + 2 = (x - 1)(x - 2)$

$$\underset{x\to 2}{\text{Lt}}\left[\frac{x^2-5x+6}{x^2-3x+2}\right] = \underset{x\to 2}{\text{Lt}}\left[\frac{(x-3)(x-2)}{(x-1)(x-2)}\right]$$

$$\underset{x\to 2}{\text{Lt}}\left[\frac{x-3}{x-1}\right]=\left[\frac{2-3}{2-1}\right]=-1$$

(iii) Method of rationalization

If $\dfrac{f(x)}{g(x)}$ assumes an indeterminate form when $x = a$ and $f(x)$ or $g(x)$ are irrational functions, then we rationalize $f(x)$, $g(x)$ and cancel the common factor, then use the substitution method to find the limit.

Example

Evaluate $\underset{x\to 0}{\text{Lt}}\left[\dfrac{x}{1-\sqrt{1-x}}\right]$.

Solution

When $x = 0$, the given function assumes the form $\dfrac{0}{1-\sqrt{1-0}}=\dfrac{0}{0}$, an indeterminate form.

Since g(x), the denominator is an irrational form, we multiply the numerator and denominator with the rationalizing factor of g(x).

$$\underset{x\to 0}{\text{Lt}}\left[\frac{x}{1-\sqrt{1-x}}\right]=\underset{x\to 0}{\text{Lt}}\frac{x(1+\sqrt{1-x})}{x} = \underset{x\to 0}{\text{Lt}}\,(1+\sqrt{1-x})= 1+ \sqrt{1} = 2.$$

Example

Evaluate $\underset{x\to 3}{\text{Lt}}\dfrac{(3-\sqrt{6+x})}{x-3}$.

Solution

When $x = 3$, $\dfrac{3-\sqrt{6+3}}{3-3} = \dfrac{0}{0}$, an indeterminate form. As f(x) is irrational, we multiply the numerator and denominator with the rationalizing factor of f(x).

$$\underset{x\to 3}{\text{Lt}}\frac{(3-\sqrt{6+x})}{x-3} = \underset{x\to 3}{\text{Lt}}\frac{(3-\sqrt{6+x})}{x-3}\times\frac{(3+\sqrt{6+x})}{(3+\sqrt{6+x})} = \underset{x\to 3}{\text{Lt}}\frac{9-(6+x)}{(x-3)(3+\sqrt{6+x})}$$

$$= \underset{x\to 3}{\text{Lt}}\frac{3-x}{(x-3)(3+\sqrt{6+x})} = \underset{x\to 3}{\text{Lt}}\frac{-1}{(3+\sqrt{6+x})} = \frac{-1}{(3+\sqrt{6+3})}$$

$$= \frac{-1}{(3+3)} = -\frac{1}{6}.$$

Example

Evaluate $\underset{x\to 0}{\text{Lt}}\dfrac{\sqrt{4+x}-\sqrt{4-x}}{\sqrt{9+x}-\sqrt{9-x}}$.

Solution

When $x = 0$, $\dfrac{\sqrt{4}-\sqrt{4}}{\sqrt{9}-\sqrt{9}} = \dfrac{0}{0}$, an indeterminate form.

Here f(x) and g(x) both are irrational functions. We multiply both the numerator and the denominator with their rationalizing factors.

$$\underset{x\to 0}{\text{Lt}}\frac{\sqrt{4+x}-\sqrt{4-x}}{\sqrt{9+x}-\sqrt{9-x}}\times\frac{\sqrt{4+x}+\sqrt{4-x}}{\sqrt{9+x}+\sqrt{9-x}}\times\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}$$

$$\underset{x\to 0}{\text{Lt}}\frac{(4+x)-(4-x)}{(9+x)-(9-x)}\times\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}} = \underset{x\to 0}{\text{Lt}}\frac{2x}{2x}\times\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{\sqrt{9+x}+\sqrt{9-x}}{\sqrt{4+x}+\sqrt{4-x}} = \frac{\sqrt{9}+\sqrt{9}}{\sqrt{4}+\sqrt{4}} = \frac{2\sqrt{9}}{2\sqrt{4}} = \frac{3}{2}$$

(iv) Using the formula

(i) $\underset{x\to a}{\text{Lt}}\left[\dfrac{x^n - a^n}{x-a}\right] = na^{n-1}$ (where 'n' is a any rational number)

Proof

Case 1: n is a positive integer.

$$\underset{x\to a}{\text{Lt}}\left[\frac{x^n - a^n}{x-a}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots\ldots+a^{n-1})}{x-a}\right]$$

$$= \underset{x\to a}{\text{Lt}}\left[x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots\ldots\ldots\ldots+a^{n-1}\right]$$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots\ldots\ldots + a^{n-1} \text{ (n terms)}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots\ldots\ldots + a^{n-1} \text{ (n terms)} = n.a^{n-1}.$$

Case 2: n is negative integer.

Let n = –m (m is positive integer)

$$\underset{x\to a}{\text{Lt}}\left[\frac{x^n - a^n}{x-a}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{x^{-m} - a^{-m}}{x-a}\right]$$

$$= \underset{x\to a}{\text{Lt}}\frac{\frac{1}{x^m}-\frac{1}{a^m}}{x-a} = \underset{x\to a}{\text{Lt}}\frac{\frac{a^m - x^m}{x^m a^m}}{x-a} = \underset{x\to a}{\text{Lt}}\frac{-(x^m - a^m)}{(x-a)x^m a^m} = -\underset{x\to a}{\text{Lt}}\frac{x^m - a^m}{x-a} \times \underset{x\to a}{\text{Lt}}\frac{1}{x^m a^m}$$

$$= -m\,a^{m-1}\frac{1}{a^m a^m} \text{ } (\because \text{m is positive integer using case(1))}$$

$$= -m\frac{a^{m-1}}{a^{2m}} = -m\,a^{m-1-2m} = (-m)a^{-m-1}$$

$$\Rightarrow \underset{x\to a}{\text{Lt}}\left[\frac{x^m - a^n}{x-a}\right] = na^{n-1} \text{ } (\because n = -m)$$

Case 3: n is a rational number.

Let $n = \dfrac{p}{q}$ where p, q are integers and $q \neq 0$.

$$= \underset{x\to a}{\text{Lt}}\left[\frac{x^n - a^n}{x-a}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x-a}\right] = \underset{x\to a}{\text{Lt}}\left[\frac{(x^{\frac{1}{q}})^p - (a^{\frac{1}{q}})^p}{x-a}\right]$$

Let $x^{1/q} = y$ then $x = y^q$

$a^{1/q} = b$ then $a = b^q$.

$x \to a \Rightarrow y^q \to b^q \Rightarrow y \to b$.

$$\underset{x\to a}{\text{Lt}}\left[\frac{(x^{\frac{1}{q}})^p - (a^{\frac{1}{q}})^p}{x-a}\right] = \underset{y\to b}{\text{Lt}}\left[\frac{y^p - b^p}{y^q - b^q}\right]$$

$$= \underset{y\to b}{\text{Lt}} \frac{\dfrac{y^p - b^p}{y-b}}{\dfrac{y^q - b^q}{y-b}} = \frac{\underset{y\to b}{\text{Lt}} \dfrac{y^p - b^p}{y-b}}{\underset{y\to b}{\text{Lt}} \dfrac{y^q - b^q}{y-b}} = \frac{p.b^{p-1}}{q.b^{q-1}} \ (\because \ p, q \text{ are integers}).$$

$$= \left(\frac{p}{q}\right) b^{p-1-(q-1)} = \left(\frac{p}{q}\right) b^{p-q} = \left(\frac{p}{q}\right) b^{q(\frac{p}{q}-1)} = \left(\frac{p}{q}\right) (b^q)^{(\frac{p}{q}-1)}$$

$$\underset{x\to a}{\text{Lt}} \left[\frac{x^n - a^n}{x-a}\right] = na^{n-1} \ (\because \ n = \frac{p}{q}; b^q = a)$$

$\therefore$ Hence when n is rational number,

$$\underset{x\to a}{\text{Lt}} \left[\frac{x^n - a^n}{x-a}\right] = na^{n-1}$$

Note: $\underset{x\to a}{\text{Lt}} \left[\dfrac{x^m - a^n}{x^n - a^n}\right] = \dfrac{m}{n} a^{m-n}$

Example

Evaluate $\underset{x\to 4}{\text{Lt}} \left[\dfrac{x^4 - 256}{x-4}\right]$.

Solution

$$\underset{x\to 4}{\text{Lt}} \left[\frac{x^4 - 256}{x-4}\right]$$

$$= \underset{x\to 4}{\text{Lt}} \left[\frac{x^4 - 4^4}{x-4}\right]$$

$= 4 \times 4^{4-1} = 4 \times 4^3 = 256.$

Example

Evaluate $\underset{x\to 3}{\text{Lt}} \left[\dfrac{x^5 - 243}{x^2 - 9}\right]$

Solution

$$\underset{x\to 3}{\text{Lt}} \left[\frac{x^5 - 243}{x^2 - 9}\right] = \underset{x\to 3}{\text{Lt}} \left[\frac{x^5 - 3^5}{x^2 - 3^2}\right]$$

$$= \frac{5}{2} 3^{5-2} \left(\because \ \underset{x\to a}{\text{Lt}} \left[\frac{x^m - a^n}{x^n - a^n}\right] = \frac{m}{n} a^{m-n}\right)$$

$$= \frac{5}{2}.3^3 = \frac{135}{2}.$$

Example

If $\underset{x\to 2}{\text{Lt}}\left[\frac{x^n - 2^n}{x-2}\right] = 32$, then find the value of n.

Solution

$$\underset{x\to 2}{\text{Lt}}\left[\frac{x^n - 2^n}{x-2}\right] = n.2^{n-1}$$

$32 = n.2^{n-1}$

$4 \times 8 = n.2^{n-1}$

$4 \times 2^3 = n.2^{n-1}$

$4 \times 2^{4-1} = n \times 2^{n-1}$

$\therefore\ n = 4.$

Limits as x tends to infinity

We know, when x tends to infinity $\frac{1}{x}$ tends to 0. While evaluating limits at infinity put $x = \frac{1}{y}$;

Example

Evaluate $\underset{x\to\infty}{\text{Lt}}\frac{2x+3}{x-5}$.

Solution

Put $x = \frac{1}{y}$; if $x \to 0, y \to 0$

$$\therefore\ \underset{x\to\infty}{\text{Lt}}\frac{2x+3}{x-5} \Rightarrow \underset{y\to 0}{\text{Lt}}\frac{\frac{2}{y}+3}{\frac{1}{y}-5} = \underset{y\to 0}{\text{Lt}}\frac{2+3y}{1-5y} = \frac{2+0}{1-0} = 2.$$

Example

Evaluate $\underset{x\to\infty}{\text{Lt}}\frac{3x^2+4x+5}{4x^2+7}$.

Solution

Put $x = \frac{1}{y}$, when $x \to \infty, y \to 0$

$$\underset{x\to\infty}{\text{Lt}}\frac{3x^2+4x+5}{4x^2+7} = \underset{y\to 0}{\text{Lt}}\frac{3.\frac{1}{y^2}+4.\frac{1}{y}+5}{4.\frac{1}{y^2}+7}$$

$$= \underset{y\to 0}{\text{Lt}}\frac{3+4y+5y^2}{4+7y^2} = \frac{3+0+0}{4+7(0)} = \frac{3}{4}.$$

test your concepts

Very short answer type questions

1. When the number of sides of a polygon tends to infinity, it approaches ________.
2. The scientists who put calculus in mathematical form are ________ and ________.
3. The radius of circle tends to zero then it approaches a ________.
4. ________ helps in finding the areas of the curves.
5. Evaluate $\underset{x \to 2}{\text{Lt}}(2x - 2)$.
6. The limiting position of a secant is ________.
7. $\underset{x \to a}{\text{Lt}}\ x^n + ax^{n-1} + a^2x^{n-2} + \ldots.. + a^n =$ ________.
8. $\underset{x \to 3}{\text{Lt}}\ \log(4x - 11) =$ ________.
9. A circle is inscribed in a polygon. As the number of sides increases, the difference in areas of circle and polygon ________.
10. $\underset{x \to 0}{\text{Lt}}\ \dfrac{x^2 + 8x}{x} =$ ________.
11. $\underset{n \to \infty}{\text{Lt}}\ \dfrac{n(n+1)}{n^2} =$ ________.
12. $\underset{n \to \infty}{\text{Lt}}\ \dfrac{1}{n} =$ ________.
13. $\underset{x \to 1}{\text{Lt}}\ \dfrac{\sqrt[5]{x} - 1}{\sqrt[4]{x} - 1} =$ ________.
14. $\underset{x \to -a}{\text{Lt}}\ \dfrac{x^n + a^n}{x + a}$ (where n is an odd natural number) = ________.
15. $\underset{x \to 0^-}{\text{Lt}}\ \dfrac{|x|}{x} =$ ________.
16. $\underset{x \to 3}{\text{Lt}}\ \sqrt{9 - x^2} =$ ________.
17. Evaluate: $\underset{x \to 3}{\text{Lt}}\ \dfrac{|x-3|}{x-3}$.
18. Evaluate $\underset{x \to a}{\text{Lt}}\left[\dfrac{x^n - a^n}{x^m - a^m}\right]$
19. $\underset{x \to 5}{\text{Lt}}\ \dfrac{\sqrt{x} - \sqrt{5}}{x - 5} =$ ________.
20. If $\underset{x \to 0}{\text{Lt}}\left[\dfrac{2x^2 + 3x + b}{x^2 + 4x + 3}\right] = 2$, then the value of b is ______.

21. What is the value of $\underset{x\to 1}{\text{Lt}}(x^3+1)(x^2-2x+4)$?

22. Evaluate: $\underset{x\to 3}{\text{Lt}}\left[\dfrac{x^3-27}{x-3}\right]$.

23. Evaluate: $\underset{x\to \infty}{\text{Lt}}\left[\dfrac{x^2+x+6}{x+1}\right]$.

24. Evaluate: $\underset{x\to \infty}{\text{Lt}}\left[\dfrac{4x^n+1}{5x^{n+1}+6}\right]$.

25. Evaluate: $\underset{x\to a}{\text{Lt}}\dfrac{x^{1/4}-a^{1/4}}{x^4-a^4}$.

26. Evaluate: $\underset{x\to -2}{\text{Lt}}\dfrac{x^7+128}{x+2}$.

27. Evaluate: $\underset{x\to 1}{\text{Lt}}\left[\dfrac{x^2+3x+2}{x^2-5x+3}\right]$.

28. Evaluate: $\underset{x\to 5}{\text{Lt}}\sqrt{25-x^2}$.

29. In finding $\underset{x\to a}{\text{Lt}}f(x)$, we replace x by 1/n then the limit becomes ________.

30. $\underset{x\to \infty}{\text{Lt}}\dfrac{7x-3}{8x-10}=$ ________.

Short answer type questions

31. Evaluate: $\underset{x\to \infty}{\text{Lt}}\dfrac{11|x|+7}{8|x|-9}$.

32. Evaluate: $\underset{x\to 2}{\text{Lt}}\left[\dfrac{2x^2-9x+10}{5x^2-5x-10}\right]$.

33. Evaluate: $\underset{x\to 3}{\text{Lt}}\dfrac{x^5-243}{x-3}$.

34. Evaluate: $\underset{x\to \infty}{\text{Lt}}\dfrac{x^5+3x^4-4x^3-3x^2+2x+1}{2x^5+4x^2-9x+16}$.

35. Evaluate: $\underset{x\to a}{\text{Lt}}\dfrac{x^{14}-a^{14}}{x^{-7}-a^{-7}}$.

36. Evaluate: $\underset{x\to 2}{\text{Lt}} \dfrac{x-2}{\sqrt{x+2}-2}$.

37. Evaluate: $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{5+x}-\sqrt{5-x}}{\sqrt{10+x}-\sqrt{10-x}}$.

38. Evaluate: $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x+x^2+x^3}-1}{x}$.

39. $\underset{x\to\infty}{\text{Lt}} \dfrac{x^n+a^n}{x^n-a^n}=$

40. Evaluate: $\underset{x\to 1}{\text{Lt}} \left[\dfrac{x^4-2x^3-x^2+2x}{x-1}\right]$.

41. Evaluate: $\underset{x\to a}{\text{Lt}} \dfrac{\sqrt{x+a}-\sqrt{2a}}{x-a}$.

42. Evaluate: $\underset{n\to\infty}{\text{Lt}} \left(\sum_{r=0}^{n} \dfrac{1}{2^r}\right)$.

43. If $\underset{x\to -3}{\text{Lt}} \dfrac{x^k+3^k}{x+3}=405$, where k is an odd natural number then, k = _______.

44. $\underset{x\to 4^-}{\text{Lt}} \dfrac{|x-4|}{x-4}=$

45. $\underset{x\to 3}{\text{Lt}} \dfrac{\log(2x-3)-\log(3x+2)}{\log(2x+1)}=$

Essay type questions

46. Evaluate: $\underset{x\to\infty}{\text{Lt}} \dfrac{n(1+4+9+16+\ldots\ldots+n^2)}{n^4+8n^3}$.

47. Evaluate: $\underset{n\to\infty}{\text{Lt}} \dfrac{n^2(1+2+3+4+\ldots\ldots+n)}{n^4+4n^2}$.

48. $\underset{x\to 1}{\text{Lt}} \dfrac{\sqrt{x+1}-\sqrt{5x-3}}{\sqrt{2x+3}-\sqrt{4x+1}}=$

49. $\underset{n\to\infty}{\text{Lt}} \dfrac{1+3+5+7+\ldots\ldots\text{n terms}}{2+4+6+8+\ldots\ldots\text{n terms}}=$

50. $\underset{x\to 3}{\text{Lt}} \left[\dfrac{2}{x-3}+\dfrac{2}{x^2-7x+12}\right]=$

CONCEPT APPLICATION

Concept Application Level—1

1. Evaluate $\underset{x\to 3}{\text{Lt}}\,(4x^2 + 3) =$

(1) 36 (2) 39 (3) 40 (4) None of these

2. $\underset{x\to\infty}{\text{Lt}}\,\dfrac{2x+4}{x-2} =$

(1) 1 (2) 0 (3) 2 (4) 6

3. $\underset{x\to 5^+}{\text{Lt}}\,\dfrac{|x-5|}{x-5} =$

(1) 1 (2) -1 (3) 0 (4) Cannot say

4. Evaluate $\underset{x\to 1}{\text{Lt}}\,\dfrac{2x^2+4x+4}{2x-1}$.

(1) 1 (2) 10 (3) 20 (4) 5

5. Evaluate $\underset{x\to 20}{\text{Lt}}\,\dfrac{\sqrt{x+5}+5}{\sqrt{x+5}-5}$.

(1) 1 (2) 2 (3) 4 (4) ∞

6. $\underset{x\to 0}{\text{Lt}}\,\dfrac{\sqrt{8-3x}+\sqrt{8+4x}}{\sqrt{2-3x}} =$

(1) 5 (2) 3 (3) 2 (4) 4

7. $\underset{x\to\infty}{\text{Lt}}\,\dfrac{(x-1)(2x-1)}{(x-4)(x-7)} =$

(1) 0 (2) 1 (3) 2 (4) -1

8. $\underset{x\to 2}{\text{Lt}}\,\dfrac{x^{-8}\ \frac{1}{256}}{x-2} =$

(1) $-\dfrac{1}{32}$ (2) $-\dfrac{1}{128}$ (3) $-\dfrac{1}{256}$ (4) $-\dfrac{1}{64}$

9. $\underset{x\to 2}{\text{Lt}}\,\dfrac{x^2+2x-8}{2x^2-3x-2} =$

(1) $\dfrac{1}{5}$ (2) $\dfrac{6}{5}$ (3) $\dfrac{3}{2}$ (4) $-\dfrac{1}{6}$

10. $\underset{x\to 4^-}{\text{Lt}}\,\sqrt{16-x^2}$ is a/an

(1) complex number (2) real number (3) natural number (4) integer

11. $\underset{x\to 0}{\mathrm{Lt}} \log\left(\frac{2x+1}{5x+4}\right) =$

(1) $-2\log 2$ (2) $\log 4$ (3) $-\log\left(\frac{1}{2}\right)$ (4) $-3\log 2$

12. $\underset{x\to 3}{\mathrm{Lt}} \frac{\sqrt{x+13}+\sqrt{x+6}}{\sqrt{x+1}-2} =$

(1) 1 (2) 7 (3) 14 (4) None of these

13. $\underset{x\to 5}{\mathrm{Lt}} \frac{\sqrt{x+20}-\sqrt{3x+10}}{5-x} =$

(1) $-\frac{2}{5}$ (2) $\frac{2}{5}$ (3) $\frac{1}{5}$ (4) $\frac{-1}{5}$

14. $\underset{x\to -3}{\mathrm{Lt}} \frac{x^2-2x-15}{x^2+2x-3} =$

(1) 1 (2) 2 (3) -3 (4) -4

15. $\underset{x\to\infty}{\mathrm{Lt}} \frac{(x-5)(x+7)}{(x+2)(5x+1)} =$

(1) $-\frac{1}{5}$ (2) 25 (3) 5 (4) $\frac{1}{5}$

16. Evaluate, $\underset{x\to 2}{\mathrm{Lt}} \frac{f(x)-f(2)}{x-2}$, where $f(x) = x^2 - 4x$.

(1) -1 (2) 2 (3) 0 (4) 4

17. Evaluate $\underset{x\to 9}{\mathrm{Lt}} \frac{2x-7\sqrt{x}+3}{3x-11\sqrt{x}+6}$.

(1) $\frac{3}{4}$ (2) $\frac{5}{3}$ (3) $\frac{5}{7}$ (4) $\frac{3}{7}$

18. $\underset{x\to b}{\mathrm{Lt}} \frac{\sqrt{x}-\sqrt{b}}{x-b} =$

(1) $\frac{1}{2b}$ (2) $\frac{\sqrt{2}}{b}$ (3) $\frac{1}{2\sqrt{b}}$ (4) None of these

19. $\underset{x\to 3}{\mathrm{Lt}} \frac{x-3}{\sqrt{x+6}-\sqrt{2x+3}} =$

(1) -5 (2) -6 (3) 9 (4) 6

20. $\underset{x\to -2}{\mathrm{Lt}} \frac{x+2}{\sqrt{2x+8}-\sqrt{2-x}} =$

(1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) 1 (4) $\frac{4}{3}$

21. Evaluate $\underset{x\to 1}{\text{Lt}}\left[\dfrac{\log x^3+\log\left(\dfrac{1}{x^2}\right)+\log 2}{\log(x^3+3)}\right]$.

(1) $\dfrac{3}{2}$ (2) 1 (3) $\dfrac{1}{2}$ (4) 2

22. $\underset{x\to 3}{\text{Lt}}\dfrac{x^{-5}-\dfrac{1}{243}}{x-3}=?$

(1) $-\dfrac{5}{729}$ (2) $-\dfrac{5}{243}$ (3) $\dfrac{5}{81}$ (4) None of these

23. $\underset{x\to\infty}{\text{Lt}}\dfrac{4x-3}{(2x+3)}=$

(1) 0 (2) 1 (3) $\dfrac{1}{2}$ (4) 2

24. Evaluate: $\underset{d\to 3}{\text{Lt}}\dfrac{d^3-27}{d-3}$

(1) 3 (2) 9 (3) 27 (4) None of these

25. $\underset{x\to 3^-}{\text{Lt}}\sqrt{9-x^2}$ is a/an

(1) natural number (2) real number (3) imaginary number (4) integer

26. Evaluate $\underset{x\to 4}{\text{Lt}}\dfrac{3x-8\sqrt{x}+4}{5x-9\sqrt{x}-2}$.

(1) $\dfrac{1}{5}$ (2) $\dfrac{4}{11}$ (3) $\dfrac{3}{10}$ (4) 5

27. For some real number k, the value of $\underset{x\to -k}{\text{Lt}}\dfrac{x^5+k^5}{x+k}$ can be

(1) 50 (2) 60 (3) 70 (4) 80

28. Evaluate $\underset{x\to 1}{\text{Lt}}\dfrac{f(x)-f(1)}{x-1}$, where $f(x) = x^2 - 2x$.

(1) -1 (2) 0 (3) 1 (4) 2

29. Evaluate $\underset{x\to a}{\text{Lt}}\dfrac{x^{1/5}-a^{1/5}}{x^{-4/5}-a^{-4/5}}$.

(1) $\dfrac{-a}{4}$ (2) $\dfrac{-1}{4a}$ (3) $\dfrac{1}{4a}$ (4) $\dfrac{a}{4}$

30. $\underset{x\to\infty}{\text{Lt}} \dfrac{6x^4+7x^3+2x+1}{x^4+1} =$

(1) 1 (2) 2 (3) 3 (4) 6

Concept Application Level—2

31. Evaluate $\underset{x\to -2}{\text{Lt}} \dfrac{2x^3-x^2-13x-6}{3x^2+x-10}$.

(1) $-\dfrac{5}{7}$ (2) $-\dfrac{17}{5}$ (3) $-\dfrac{15}{11}$ (4) $-\dfrac{10}{3}$

32. Evaluate $\underset{x\to -1}{\text{Lt}} \dfrac{\log x^2 - \log\left(\dfrac{1}{x^4}\right) + \log 3}{\log\left(\dfrac{x^3}{-3}\right)}$.

(1) 1 (2) 0 (3) -1 (4) Does not exist

33. $\lim\limits_{x\to\infty} \left\{\dfrac{3x}{\sqrt{x^2+5x-6}+2x}\right\} =$

(1) -1 (2) 1 (3) 0 (4) ∞

34. $\underset{x\to\infty}{\text{Lt}} \dfrac{(4x+5)(2x-1)}{(27x^2+1)} = ?$

(1) $\dfrac{4}{27}$ (2) $\dfrac{8}{27}$ (3) $\dfrac{2}{27}$ (4) $\dfrac{6}{27}$

35. $\underset{x\to 1}{\text{Lt}} \dfrac{x^3-6x^2+11x-6}{x^2-5x+4} =$

(1) $\dfrac{2}{3}$ (2) $-\dfrac{5}{3}$ (3) $\dfrac{5}{3}$ (4) $\dfrac{-2}{3}$

36. If $\underset{x\to a}{\text{Lt}} \dfrac{x^5-a^5}{x-a} = 5$, then find the sum of the possible real values of a.

(1) 0 (2) 2
(3) 3 (4) Cannot be determined

37. $\underset{x\to 0}{\text{Lt}} \dfrac{x}{\sqrt{3+x}-\sqrt{3-x}} =$

(1) $\sqrt{3}$ (2) $\sqrt{-3}$ (3) $2\sqrt{3}$ (4) $-2\sqrt{3}$

38. $\underset{x\to 1}{\text{Lt}} \dfrac{4-\sqrt{15+x}}{1-x} =$

(1) $\dfrac{1}{6}$ (2) $\dfrac{1}{8}$ (3) $\dfrac{1}{10}$ (4) $\dfrac{1}{12}$

39. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x+x^2+x^3}-1}{x} = ?$

(1) 1 (2) 2 (3) $\dfrac{1}{2}$ (4) $\dfrac{1}{4}$

40. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt[4]{x+16}-2}{x} =$

(1) $\dfrac{1}{16}$ (2) $\dfrac{1}{8}$ (3) $\dfrac{1}{32}$ (4) $\dfrac{1}{20}$

41. $\underset{x\to \infty}{\text{Lt}} \dfrac{(x+2)^{10}+(x+4)^{10}+.....+(x+20)^{10}}{x^{10}+1} =$

(1) 12 (2) 20 (3) 40 (4) 10

42. $\underset{n\to \infty}{\text{Lt}} \dfrac{2+6+10+14+.....2n\text{ terms}}{2+4+6+8+.....n\text{ terms}} =$

(1) 4 (2) 16 (3) 8 (4) 10

43. Evaluate: $\underset{x\to -a}{\text{Lt}} \dfrac{1}{x}\left[\dfrac{1}{x+a}+\dfrac{b+a}{x^2-bx+ax-ab}\right]$

(1) $\dfrac{1}{a}$ (2) $\dfrac{1}{a+b}$ (3) $\dfrac{1}{a(a+b)}$ (4) $\dfrac{1}{a(a-b)}$

44. $\underset{x\to 5}{\text{Lt}} \dfrac{\sqrt{2x-3}-\sqrt{3x-8}}{\sqrt{2x-1}-\sqrt{3x-6}} =$

(1) $\dfrac{2\sqrt{7}}{3}$ (2) $\dfrac{3}{2\sqrt{7}}$ (3) $\dfrac{4}{3}\sqrt{7}$ (4) $\dfrac{3}{\sqrt{7}}$

45. If $\underset{x\to a}{\text{Lt}} \dfrac{x^7-a^7}{x-a} = 7$, then find the number of possible real values of a.

(1) 0 (2) 1 (3) 2 (4) Cannot say

Concept Application Level—3

46. $\underset{n\to \infty}{\text{Lt}} \dfrac{1+4+9+16+.......+n^2}{4n^3+1} =$

(1) $\dfrac{1}{12}$ (2) $\dfrac{1}{24}$ (3) $\dfrac{1}{8}$ (4) $\dfrac{1}{16}$

47. $\underset{x\to 0}{Lt}\dfrac{\sqrt[5]{x+32}-2}{x}=$

(1) $\dfrac{1}{16}$ (2) $\dfrac{1}{45}$ (3) $\dfrac{1}{80}$ (4) $\dfrac{1}{100}$

48. $\underset{x\to 2}{Lt}\dfrac{\sqrt{x^3-2x^2+2x-3}-1}{x-2}=$

(1) 4 (2) 8 (3) 6 (4) 3

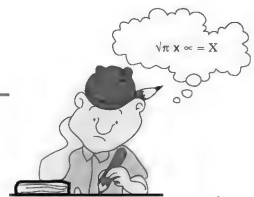
49. Evaluate $\underset{n\to\infty}{Lt}\dfrac{(1^2+2^2+3^2+.....+n^2)}{n^2(1+3+5+7+.....+n\,\text{terms})}$

(1) 0 (2) 1 (3) ∞ (4) Does not exist

50. Evaluate $\underset{x\to 0}{Lt}\dfrac{\sqrt{6+x}-\sqrt{6-x}}{\sqrt{8+x}-\sqrt{8-x}}$.

(1) $\sqrt{3}$ (2) $\dfrac{-5\sqrt{3}}{3}$ (3) $\dfrac{4\sqrt{3}}{3}$ (4) $\dfrac{2\sqrt{3}}{3}$.

KEY



Very short answer type questions

1. Circle
2. Newton and Leibnitz
3. Point
4. Integration
5. 2
6. tangent
7. na^n
8. 0
9. Decreases.
10. 8
11. 1
12. 0
13. $\dfrac{4}{5}$
14. $n(-a)^{n-1}$
15. –1
16. does not exist
17. limit does not exist
18. $\dfrac{n}{m}a^{n-m}$
19. $\dfrac{1}{2\sqrt{5}}$
20. 6
21. 6.
22. 27
23. ∞
24. 0
25. $\dfrac{1}{16}.a^{\frac{-15}{4}}$
26. 448
27. –6
28. limit does not exist
29. $\underset{\frac{1}{n}\to a}{Lt} f(1/n)$
30. $\dfrac{7}{8}$

Short answer type questions

31. $\dfrac{11}{8}$
32. $\dfrac{-1}{15}$
33. 405
34. $\dfrac{1}{2}$
35. $-2a^{21}$.
36. 4

37. $\sqrt{2}$

38. $\frac{1}{2}$

39. 1

40. -2

41. $\frac{1}{2\sqrt{2a}}$

42. 2

43. -1

44. cannot say

45. $\text{Log}_7\left(\frac{6}{11}\right)$

Essay type questions

46. $\frac{1}{3}$

47. $\frac{1}{2}$

48. $\sqrt{10}$

49. 1

50. 0

key points for selected questions

Very short answer type questions

17. (i) First of all evaluate LHL and RHL.
(ii) Then check whether limit exist or not.

20. (i) Put = 0.
(ii) Then solve the equation obtained.

21. Put $x = 1$.

22. (i) Factorize the numerator.
(ii) Then cancel the common factor.
(iii) Then put $x = 3$

23. If the degree of the numerator is greater than that of denominator and $x \to \infty$, then the limit is ∞.

25. Apply the formula $\underset{x\to a}{\text{Lt}} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

26. (i) Write $x^7 + 128 = x^7 - (-2)^7$ and $x + 2 = x - (-2)$
(ii) Apply the formula $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$

27. Put $x = 1$

28. Put $x = 5$.

Short answer type questions

31. (i) Divide each term with $|x|$.
(ii) If $|x| \to \infty$ then $\frac{1}{|x|} \to 0$

32. (i) Factorise both the numerator and the denominator.
(ii) Then cancel the common factor (s)
(iii) Then put $x = 2$.

33. (i) Express 243 as 3^5.
(ii) Apply the formula $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$

34. If the degrees of both numerator and the denominator are equal and $x \to \infty$, then the required limit is the ratio of the coefficients of the highest power of x.

35. Apply the formula $\underset{x\to a}{\text{Lt}} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

36. (i) Rationalize the denominator.
(ii) Then cancel the common factor.
(iii) Then put $x = 2$.

37. (i) Multiply both numerator and denominator. With the RF of the numerator and also with the RF of the denominator.

(ii) Then cancel the common term.
(iii) Then put x = 0.

38. (i) First of all rationalise the numerator.
(ii) Then cancel the common factor
(iii) Then put x = 0.

39. (i) Divide the numerator and the denominator by x^n.
(ii) Use, $x^n \to \infty$, $\frac{1}{x^n} \to 0$ and evaluate the limit.

40. (i) Factorise the numerator by using remainder theorem.
(ii) Then cancel the common factor.
(iii) Then put x = 1

41. (i) First of all rationalize the numerator.
(ii) Then cancel the common factor
(iii) Then put x = a.

42. (i) First of all write the sum of first n terms of the series.
(ii) As $\frac{1}{2} < 1$ and $n \to \infty$, then $\left(\frac{1}{2}\right)^n \to 0$

43. Use the formula, $\underset{x \to a}{\mathrm{Lt}} \frac{x^n - a^n}{x - a}$.

44. When n < 4, $|n - 4| = -(n - 4)$.

45. Substitute x = 3.

Essay type questions

46. (i) Find the sum of squares of first n natural numbers.
(ii) If the degree of the numerator is greater than that of denominator and $x \to \infty$, then limit is ∞.

47. (i) Find the sum of first n natural numbers.
(ii) If the degrees of both numerator and the denominator are equal and $n \to \infty$, then the required limit is the ratio of the coefficients of the highest powers of x.

48. Rationalize both the numerator and the denominator and cancel the common factor, then substitute x = 1.

49. (i) Sum of n terms of AP whose first term is 'a' and common difference is 'd', is $\frac{n}{2}2a + (n - d)d$.
(ii) The highest power of n is to be taken common from both numerator and denominator and put $\frac{1}{n} = 0$.

50. Add the two fractions and remove the common factor x − 3, then substitute x = 3.

KEY

Concept Application Level—1,2,3

1. 2	**2.** 3	**19.** 2	**20.** 4
3. 1	**4.** 2	**21.** 3	**22.** 1
5. 4	**6.** 4	**23.** 4	**24.** 3
7. 3	**8.** 4	**25.** 2	**26.** 2
9. 2	**10.** 2	**27.** 4	**28.** 2
11. 1	**12.** 4	**29.** 1	**30.** 4
13. 3	**14.** 2	**31.** 3	**32.** 3
15. 4	**6.** 3	**33.** 2	**34.** 2
17. 3	**18.** 3	**35.** 4	**36.** 1
		37. 1	**38.** 2

39. 3 **40.** 3
41. 4 **42.** 3
43. 3 **44.** 4
45. 3 **46.** 1
47. 3 **48.** 4
49. 1 **50.** 4

Concept Application Level—1,2,3

Key points for select questions

1. Substitute x = 3 in the given function.
2. Divide both numerator and denominator by x then use if $x \to \infty$ then $\frac{1}{x} \to 0$
3. When x > 5, $|x-5| = x-5$.
4. Substitute x = 1 in the given expression.
5. Substitute x = 20.
6. Substitute x = 0.
7. Divide both numerator and denominator by x^2. Now substitute, $\frac{1}{x} = 0$
8. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x-a} = na^{n-1}$.
9. Factorise the numerator and denominator and remove common factor. Now substitute x = 2.
10. Apply the concept of left hand limit.
11. Substitute x = 0.
12. Rationalize the denominator.
13. Rationalize the numerator and cancel common factor. Then substitute, x = 5.
14. Factorize the numerator and denominator and cancel the common factor and then substitute x = –3.
15. Divide both the numerator and the denominator by x^2. Then substitute, $\frac{1}{x} = 0$.
16. Substitute then factorize numerator f(x), then cancel the common factor in both the numerator and the denominator, then substitute x = 2.
17. Factorize numerator and denominator, eliminate the common factor then substitute x = 9.
18. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}$.
19. Rationalize the demnominator and remove the common factor. Now substitute x = 3.
20. Rationalize the denominator and cancel the common factor and then substitute x = –2.
21. Put x =1 in the given limit.
22. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x-a}$ by converting the given limit into the form.
23. Divide both the numerator and the denominator by x and apply the concept hat, as $x \to \infty$, $\frac{1}{x} \to 0$.
24. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x-a} = n \cdot a^{n-1}$.
25. Apply the concept of left hand limit.
26. Factorize the numerator and denominator and cancel the common factor which is in both numerator and denominator then substitute x = 4.
27. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x-a} = na^{n-1}$.
28. Calculate f(1) substitute f(x) and f(1), factorise numerator, then cancel the common factor and substitute x = 1.
29. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$.
30. (i) The highest power of x is to be taken common from numerator and denominator.

    (ii) Put $\frac{1}{x}, \frac{1}{x^2}$.....=0 as $x \to \infty$ $\frac{1}{x}, \frac{1}{x^2} \to 0$.

31. Factorize the numerator and the denominator and cancel the common factor then substitute $x = -2$.

32. Put $x = -1$ in the given limit.

33. Numerator and denominator are of the same degree.

$\therefore$ Limiting value of the given function

$$= \frac{\text{Coefficient of } x \text{ in numerator}}{\text{Coefficient of } x \text{ in denominator}}$$

34. Divide both the numerator and the denominator by x^2 and then use when $x \to \infty \Rightarrow \frac{1}{x} \to 0$.

35. Factorize the numerator and the denominator, and cancel the common factor, then substitute $x = 1$.

36. (i) $\underset{x\to a}{\text{Lt}} \frac{x^m - a^m}{x - a} = ma^{m-1}$.

(ii) Equating ma^{m-1} with 5 obtain the value of a.

(iii) Add all the possible real values of a.

37. Rationalize the denominator and cancel common factor and then substitute $x = 0$.

38. Rationalize the numerator and cancel the common factor and then substitute $x = 1$.

39. Rationalize the numerator and cancel the common factor x, then substitute $x = 0$.

40. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$.

41. Take common the highest power of x and cancel it, then substitute $\frac{1}{x} = 0$ as $x \to \infty$, $\frac{1}{x} \to 0$.

42. Find the sum of 2n terms in numerator and sum of n terms in denominator and take common the highest power of n in both the numerator and the denominator and then substitute $\frac{1}{n} = 0$ as $n \to \infty$, $\frac{1}{n} \to 0$.

43. Simplify and then factorise the numerator. Cancel the common factor and then substitute $x = -a$.

44. Rationalize the numerator and denominator and cancel the common factor and then substitute $x = 5$.

45. (i) Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^m - a^m}{n^n - a^n} = \frac{m}{n} a^{m-n}$.

(ii) $7a^6 = 7$

(iii) Now, find the number of possible real values of a.

46. Write formula for sum of the squares of first n natural numbers. i.e., $\frac{n(n+1)(2n+1)}{6}$.

Divide the numerator and the denominator by n^3. Then substitute $\frac{1}{n} = 0$ as $n \to \infty, \frac{1}{n} \to 0$

47. Use the formula, $\underset{x\to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$.

48. (i) Rationalize the numerator and find its factors

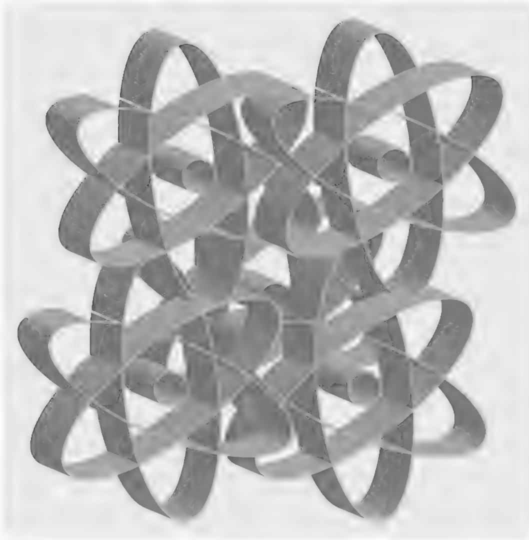
(ii) Cancel the common factor of numerator and denominator and substitute the value of limit.

49. Substitute the formulae for sum of the squares of n natural numbers and sum of n natural number and sum of n odd natural numbers and simplify.

50. Rationalize both numerator and denominator.

CHAPTER 10

Matrices



INTRODUCTION

A *matrix* is a rectangular arrangement of a set of elements in the form of horizontal and vertical lines. The elements can be numbers (real or complex) or variables. Matrices is the plural of matrix.

Horizontal line of elements is called a row and the vertical line of elements is called a column.

The rectangular array of elements in a matrix are enclosed by brackets [ ] or parenthesis ( ).

Generally we use capital letters to denote matrices.

Examples

1. $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \end{bmatrix}$ is a matrix having 2 rows and 3 columns.

   Here the elements of matrix are numbers.

2. $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix having 2 rows and 2 columns.

   Here the elements of matrix are variables.

Order of a matrix

If a matrix A. has "m" rows and "n" columns, then $m \times n$ is called the order (or type) of matrix, and is denoted as $A_{m \times n}$.

Examples

1. $A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 4 & 1 \end{bmatrix}$ is a matrix consisting of 2 rows and 3 columns. So its order is 2×3.

2. $B = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is a matrix consisting of 3 rows and 1 column. So its order is 3×1.

So in general, a set of mn elements can be arranged as a matrix having m rows and n columns as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & . & a_{2n} \\ a_{31} & . & . & . & . & . \\ . & . & . & a_{ij} & . & . \\ . & . & . & . & . & . \\ a_{m1} & . & . & . & . & a_{mn} \end{bmatrix}_{m \times n} \quad \text{or } A = [a_{ij}]_{m \times n}$$

In the above matrix a_{ij} represents an element of the matrix occurring in ith row and jth column. In general, a_{ij} is called (i,j)th element of the matrix.

Hence in a particular matrix, we can note that (3,4)th element is the element occurring in 3rd row and 4th column.

(1,3)rd element is the element occurring is 1st row and 3rd column.

Example

$$\text{Let } P = \begin{bmatrix} 2 & 3 & 51 \\ 4 & -2 & -3 \\ 5 & -31 & 1 \end{bmatrix}$$

In this matrix we have
(1, 1)th element = 2; (1, 2)th element = 3
(1, 3)th element = 51; (2, 1)th element = 4
(2, 2)th element = –2; (2, 3)th element = –3
(3, 1)th element = 5; (3, 2)th element = –31, (3, 3)th element = 1.

In compact form any matrix A can be represented as

$A = \left[a_{ij}\right]_{m \times n}$ where $1 \le i \le m,$
$1 \le j \le n.$

Various types of matrices

1. **Rectangular Matrix:** In a matrix if number of rows is not equal to number of columns, then the matrix is called a rectangular matrix.

Examples

1. $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}_{3\times2}$;A has 3 rows and 2 columns.

2. $B = \begin{bmatrix} -2 & 3 & 1 & -4 \\ 5 & -1 & 0 & 1 \end{bmatrix}_{2\times 4}$; B has 2 rows and 4 columns.

2. **Row matrix:** A matrix which has only one row is called a row matrix.

Examples

1. $[5 \quad -3 \quad 2 \quad 1]$ is a row matrix of order 1×4.
2. $[4 \quad 3 \quad 13 \quad -4 \quad -31]$ is a row matrix of order 1×5.
   In general, order of any row matrix is $1 \times n$, where n is number of columns and $n = 2, 3, 4 \ldots\ldots$

3. **Column Matrix:** A matrix which has only one column is called a column matrix.

Examples

1. $\begin{bmatrix} 5 \\ -3 \\ 2 \\ -1 \end{bmatrix}_{4\times 1}$ is a column matrix of order 4×1.
2. $\begin{bmatrix} -3 \\ 5 \\ 20 \\ -2006 \\ 2 \\ -1 \end{bmatrix}_{6\times 1}$ is a column matrix of order 6×1.

In general, order of any column matrix is $m \times 1$, where m is number of rows in the matrix and $m = 2, 3, 4, \ldots\ldots.$

4. **Null matrix or zero matrix:** If every element of a matrix is zero, then the matrix is called null matrix or zero matrix. A zero matrix of order $m \times n$ is denoted by $O_{m \times n}$ or in short by O.

Examples

1. $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{2\times 4}$ 2. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$ 3. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

5. **Square matrix:** In a matrix, if number of rows is equal to number of columns, then the matrix is called a square matrix. A matrix of order $n \times n$ is termed as a square matrix of order n.

Examples

1. $\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$ is a square matrix of order 2.

2. $\begin{bmatrix} a & b & -3 \\ 4 & c & -2 \\ y & x & z \end{bmatrix}$ is a square matrix of order 3.

Principal diagonal of a square matrix: In a square matrix A of order n, the elements a_{ii} (i.e., $a_{11}, a_{22}, \ldots\ldots, a_{nn}$) constitute principal diagonal. The elements a_{ii} are called elements of principal diagonal.

Examples

1. $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

The elements 2 and 5 constitute the principal diagonal of A.

2. $P = \begin{bmatrix} -3 & 4 & 5 \\ 6 & -2 & 1 \\ a & 3 & b \end{bmatrix}$

The elements –3, –2 and b constitute principal diagonal of P.

6. **Diagonal matrix:** In a square matrix, if all the non-diagonal elements are zeroes and at least one principal diagonal element is non-zero, then the matrix is called diagonal matrix.

Examples

1. $\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ is a diagonal matrix of order 2.

2. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a diagonal matrix of order 3

7. **Scalar matrix:** In a matrix, if all the diagonal elements are equal and rest of the elements are zeroes, then the matrix is called scalar matrix.

Examples

1. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is a scalar matrix of order 2.

2. $\begin{bmatrix} 31 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 31 \end{bmatrix}$ is a scalar matrix of order 3.

8. **Identity matrix (or) unit matrix:** In a square matrix, if all the principal diagonal elements are unity and rest of the elements are zeroes, then the square matrix is called identity matrix or unit matrix. It is denoted by I.

Examples

1. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2

2. $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 4.

Comparable matrices

Two matrices A and B can be compared, only when they are of same order.

Example

Consider two matrices A and B given by

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 21 \\ 10 & -4 \end{bmatrix}_{3\times 2} \quad \text{and } B = \begin{bmatrix} -3 & 10 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}_{3\times 2}$$

Matrices A and B can be compared as both of them are of order 3×2

Equality of two matrices

Two matrices are said to be equal only when (i) they are of same order and (ii) corresponding elements of both the matrices are equal.

Examples

If $\begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 6 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ a & 2 \\ 6 & b \end{bmatrix}$ are equal matrices, then a = 1 and b = –2.

Operations on matrices

Multiplication of a matrix by a scalar

If every element of a matrix A is multiplied by a number (real or complex) k, the matrix obtained is k times A and is denoted by kA and the operation is called scalar multiplication.

Example

1. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix}$, then find (i) $-A$ (ii) $3A$ (iii) $\frac{1}{4}A$.

(i) $$-A = -\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -1\times 2 & -1\times 3 & -1\times(-1) \\ -1\times 5 & -1\times 6 & -1\times 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -5 & -6 & -1 \end{bmatrix}$$

(ii) $3A = 3\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3\times2 & 3\times3 & 3(-1) \\ 3\times5 & 3\times6 & 3\times1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & -3 \\ 15 & 18 & 3 \end{bmatrix}$

(iii) $\frac{1}{4}A = \frac{1}{4}\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\times2 & \frac{1}{4}\times3 & \frac{1}{4}\times(-1) \\ \frac{1}{4}\times5 & \frac{1}{4}\times6 & \frac{1}{4}\times1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & \frac{-1}{4} \\ \frac{5}{4} & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$

NOTE:

1. If a and b are any two scalars and P is a matrix, then a (bP) = (ab)P
2. If m and n are any two scalars and A is a matrix, then (m + n) A = mA + nA

Addition of matrices

1. Two matrices A and B can be added only when they are of same order.

Example

Let $A = \begin{bmatrix} -3 & 2 & 1 \\ 5 & 6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -13 & 21 & 33 \\ -52 & 4 & 49 \end{bmatrix}$

Here both matrices A and B are of order 2 × 3. So, they can be added.

2. The **sum** matrix of two matrices A and B is obtained by adding the corresponding elements of A and B and the **sum** matrix of same order as that of A or B.

3. Here $A + B = \begin{bmatrix} -3+(-13) & 2+21 & 1+33 \\ 5+(-52) & 6+4 & -5+49 \end{bmatrix} = \begin{bmatrix} -16 & 23 & 34 \\ -47 & 10 & 44 \end{bmatrix}$

NOTE: If two matrices are of different orders, then their sum is not defined.

Properties of matrix addition

1. Matrix addition is commutative i.e., if A and B are two matrices of same order, then A + B = B + A
2. Matrix addition is associative, i.e., if A, B and C are three matrices of same order, then A + (B + C) = (A + B) + C.
3. Additive identity:

   If $O_{m \times n}$ is a null matrix of order m × n and A is any matrix of order m × n, then A + O = O + A = A
   So, O is called additive identity.
4. Additive inverse:

   If $A_{m \times n}$ is any matrix of order m × n, then A + (–A) = (–A) + A = O
   So,–A is called additive inverse of the matrix A.
5. If k is a scalar and A and B are two matrices of same order, then k(A+B) = kA + kB

Matrix subtraction

1. Matrix subtraction is possible only when both the matrices are of same order.
2. The difference of two matrices of same type (or order) A and B i.e., A – B, is obtained by subtracting corresponding element of B from that of A.
3. The difference matrix is of the same order as that of A or B.

☛ *Examples*

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix}$ then find A – B.

$$A - B = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 4 & -2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-(-3) & 3-1 \\ -1-4 & 4-(-2) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -5 & 6 \end{bmatrix}$$

Transpose of a matrix

For a given matrix A, the matrix obtained by interchanging its rows and columns is called transpose of the matrix A and is denoted by A^T.

☛ *Examples*

1. If $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & -6 & 11 & -1 \end{bmatrix}$, then

$$\text{Transpose of } A = A^T = \begin{bmatrix} 2 & 5 \\ -1 & -6 \\ 3 & 11 \\ 4 & -1 \end{bmatrix}$$

Here we can note that order of A is 2×4, while that of A^T is 4×2.

2. $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, then $A^T = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 \\ 2004 \\ 53 \\ -47 \end{bmatrix}$, then $A^T = [-1 \quad 2004 \quad 53 \quad -47]$

4. If $A = [5003]$, then $A^T = [5003]$

NOTE:

1. If the order of a matrix is $m \times n$, then order of transpose of the matrix is $n \times m$.
2. $\left(A^T\right)^T = A$

Examples

Let $A = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix}$$

$$\left(A^T\right)^T = \begin{bmatrix} -2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix}^T = \begin{bmatrix} -2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix} = A$$

3. If A and B are two matrices of same order, then $(A + B)^T = A^T + B^T$
4. If k is a scalar and A is any matrix, then $(kA)^T = kA^T$.

Symmetric matrix

A square matrix is said to be symmetric if the transpose of the given matrix is equal to the matrix itself. Hence a square matrix A is symmetric
$\Rightarrow A = A^T$

Examples

1. If $A = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}$

   Thus we can observe that $A^T = A$ so A is a symmetric matrix.

2. Similarly for $P = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & -1 \\ 43 & 4 & 2 \end{bmatrix}$,

   $$P^T = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & -1 \\ 43 & 4 & 2 \end{bmatrix}^T = \begin{bmatrix} -3 & 5 & 43 \\ 5 & 4 & 4 \\ 43 & -1 & 2 \end{bmatrix} = P$$

   So P is a symmetric matrix.

Skew-symmetric matrix

A square matrix A is said to be skew-symmetric if $A^T = -A$, i.e., transpose of the matrix is equal to its additive inverse.

Examples

1. If $A = \begin{bmatrix} 0 & 2006 \\ -2006 & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & -2006 \\ 2006 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2006 \\ -2006 & 0 \end{bmatrix} = -A$

   So A is a skew-symmetric matrix.

NOTE: (i) For a square matrix A; $\frac{1}{2}(A + A^T)$ is always a symmetric matrix.

(ii) For a square matrix A, $\frac{1}{2}(A - A^T)$ is always a skew symmetric matrix.

Multiplication of matrices

Two matrices A and B can be multiplied only if the number of columns in A is equal to the number of rows in B.

Suppose order of matrix A is $m \times q$. Then order of matrix B, such that AB exists, should be of the form $q \times n$. Further order of the product matrix AB will be $m \times n$.

Now consider a matrix A of order 2×3 and another matrix B of the order 3×4. As the number of columns in A(=3) is equal to number of rows in B(=3). So AB exists and it is of the order 2×4. We can obtain the product matrix AB as follows.

(1, 1)th element of AB

= sum of products of elements of first row of A with the corresponding elements of first column of B.

(1, 2)th element of AB

= sum of products of elements of first row of A with the corresponding elements of second column of B.

(2, 1)th element of AB

= sum of products of elements of second row of A with the corresponding elements of first column of B and so on.

In general (I, j)th element of AB

= sum of products of elements of ith row in A with the corresponding elements of jth column in B.

Following example will clearly illustrate the method.

$$\text{Let } A = \begin{bmatrix} 2 & -1 \\ 1 & 7 \\ 3 & 5 \end{bmatrix}_{3\times 2}, B = \begin{bmatrix} -5 & 6 & 4 \\ 9 & 11 & 8 \end{bmatrix}_{2\times 3}$$

As A is of order 3×2 and B is of order 2×3, AB will be of the order 3×3.

$$\therefore AB = \begin{bmatrix} 2\times(-5)+(-1)\times 9 & 2\times 6+(-1)\times 11 & 2\times 4+(-1)\times 8 \\ 1\times(-5)+7\times 9 & 1\times 6+7\times 11 & 1\times 4+7\times 8 \\ 3\times(-5)+5\times 9 & 3\times 6+5\times 11 & 3\times 4+5\times 8 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -10+(-9) & 12+(-11) & 8+(-8) \\ -5+63 & 6+77 & 4+56 \\ -15+45 & 18+55 & 12+40 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -19 & 1 & 0 \\ 58 & 83 & 60 \\ 30 & 73 & 52 \end{bmatrix}_{3\times 3}$$

In general if $A = [a_{ip}]$ is a matrix of order $m \times q$ and $B = [b_{pj}]$ is a matrix of order $q \times n$, then the product matrix $AB = Q = [x_{ij}]$ will be of the order $m \times n$ and is given by $x_{ij} = \sum_{p=1}^{q} a_{ip} b_{pj}$

This evaluation is made clear in the following.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & . & a_{1n} \\ x_{21} & . & . & . & . & . \\ a_{31} & . & . & . & . & . \\ . & . & . & x_{ij} & . & . \\ . & . & . & . & . & . \\ x_{m1} & .x_{m2} & .x_{m3} & . & . & x_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & . & a_{iq} & . & a_{1n} \\ a_{21} & & & . & . & . \\ a_{il} & . & . & a_{iq} & . & a_{in} \\ . & . & . & & . & . \\ . & . & . & . & . & . \\ a_{m1} & . & . & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & b_{iq} & . & b_{1n} \\ b_{21} & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{il} & . & . & b_{ij} & . & b_{in} \\ . & . & . & . & . & . \\ b_{m1} & . & . & b_{mq} & . & b_{mn} \end{bmatrix}$$

Properties of matrix multiplication

1. In general, matrix multiplication is not commutative i.e., AB ≠ BA

Examples

Let $A = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} -3\times2+1\times0 & -3\times(-1)+1\times1 \\ 0\times2+2\times0 & 0(-1)+2\times1 \end{bmatrix} =$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-3)+(-1)\times0 & 2\times1+(-1)\times2 \\ 0(-3)+1\times0 & 0\times1+1\times2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$$

So we can observe that AB ≠ BA

2. Matrix multiplication is associative
   i.e., A (BC) = (AB) C
3. Matrix multiplication is distributive over addition i.e.,
   (i) A(B + C) = AB + AC,
   (ii) (B + C)A = BA + CA
4. For any two matrices A and B if AB = O, then it is not necessarily imply that A = O or B = O or both A and B are zero.

Example

Let $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -12 \\ -4 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 8 & -12 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 2(8)+4(-4) & 2(-12)+4(6) \\ 4(8)+8(-4) & 4(-12)+8(6) \end{bmatrix}$$

$$= \begin{bmatrix} 16-16 & -24+24 \\ 32-32 & -48+48 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence we can observe that, though AB = O, A ≠ O and B ≠ O.

5. For any three matrices A, B and C if AB = AC, then it is not necessarily imply that B = C or A = O (But in case of any three real numbers a, b and c if ab = ac and a ≠ 0, then it is necessary that b = c)

Example

Let $A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$ $B = \begin{bmatrix} 10 & 5 \\ 15 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} -10 & 35 \\ 25 & -5 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} 10 & 5 \\ 15 & 10 \end{bmatrix} = \begin{bmatrix} 5\times10+10\times15 & 5\times5+10\times10 \\ 10\times10+20\times15 & 10\times5+20\times10 \end{bmatrix}$$

$$= \begin{bmatrix} 50+150 & 25+100 \\ 100+300 & 50+200 \end{bmatrix} = \begin{bmatrix} 200 & 125 \\ 400 & 250 \end{bmatrix}$$

$$AC = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}\begin{bmatrix} -10 & 35 \\ 25 & -5 \end{bmatrix} = \begin{bmatrix} 5(-10)+10(25) & 5(35)+10(-5) \\ 10(-10)+20(25) & 10(35)+20(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -50+250 & 175-50 \\ -100+500 & 350-100 \end{bmatrix} = \begin{bmatrix} 200 & 125 \\ 400 & 250 \end{bmatrix}$$

Here AB = AC but B ≠ C

6. If A is a square matrix of order n and I is the identity matrix of order n, then AI = IA = A. i.e., I is the multiplicative identity matrix.

Examples

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = A$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = A$$

∴ AI = IA = A; Here I is called Identity matrix.

7. If matrix A is multiplied by a null matrix, then the resultant matrix is null matrix, i.e., AO = OA = O.

8. If A and B are two matrices such that AB exists, then $(AB)^T = B^T A^T$.

Note: If A_1, A_2, A_3 ……., A_n are n matrices, then $(A_1A_2A_3 \ldots\ldots A_n)^T = A_n^T A^T_{n-1} \ldots\ldots A_1^T$

9. If A is any square matrix, then $(A^T)^n = (A^n)^T$.
10. If A and B are any two square matrices, then

    (i) $(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B)$
    $= A^2 + AB + BA + B^2$

    (ii) $(A - B)^2 = (A - B)(A - B) = A(A - B) - B(A - B)$
    $= A^2 - AB - BA + B^2$

    (iii) $(A + B)(A - B) = A(A - B) + B(A - B)$
    $= A^2 - AB + BA - B^2$

We have learnt about the order of matrix, different kinds of matrices, some operations like transpose, addition, subtraction, multiplication of matrices. We also learnt different properties of matrix multiplication.

In this chapter we will learn how to find determinant and inverse of a 2×2 square matrix. We also learn about how to apply the concept of matrices to solve a system of linear equations in two variables.

Determinant

For a given 2×2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the real number $(ad - bc)$ is defined as the determinant of A and is denoted by $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

☛ *Example* If $A = \begin{bmatrix} 2 & -5 \\ 6 & 3 \end{bmatrix}$, then determinant of $A = |A| = \begin{vmatrix} 2 & -5 \\ 6 & 3 \end{vmatrix} = 2(3) - (-5) \times 6 = 36$.

Singular matrix

If determinant of a square matrix is zero, then the matrix is called a singular matrix.

☛ *Example* For the square matrix $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 6 & 9 \\ 2 & 3 \end{vmatrix} = 6 \times 3 - 9 \times 2 = 18 - 18 = 0.$$

So A is a singular matrix.

Non-singular matrix

If determinant of a square matrix is not equal to zero, then the matrix is called non-singular matrix.

☛ *Example* For the square matrix $A = \begin{bmatrix} 2 & -4 \\ 5 & 3 \end{bmatrix}$,

$$|A| = \begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = 2(3) - (-4) \times 5 = 6 + 20 = 26 \neq 0.$$

So A is a non-singular matrix.

Multiplicative inverse of a square matrix

For every non-singular square matrix A of order n, there exists a non-singular square matrix B of same order, such that AB = BA = I. (Note that I is unit matrix of order n). Here B is called multiplicative inverse of A and is denoted as $A^{-1} \Rightarrow B = A^{-1}$

NOTE: If AB = KI, then $A^{-1} = \frac{1}{K}B$.

Multiplicative inverse of a 2 × 2 square matrix

For a 2 × 2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can show that $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

NOTE:

1. For a singular square matrix $|A| = 0$, and so its multiplicative inverse doesn't exist. Conversely if a matrix A doesn't have multiplicative inverse, then $|A| = 0$.
2. If A is a square matrix and K is any scalar, then $(KA)^{-1} = \frac{1}{K}A^{-1}$.
3. For any two square matrices A and B of same order $(AB)^{-1} = B^{-1}A^{-1}$.

Method for finding inverse of a 2 × 2 square matrix

We know that for a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

From this formula we can find A^{-1} using the following steps.

1. Find whether $|A| = 0$ or not. If $|A| = 0$, then the given matrix is singular, so A^{-1} doesn't exist. If $|A| \neq 0$, then the matrix has a multiplicative inverse and can be found by the following steps (2), (3) and (4).
2. Interchange the elements of principal diagonal.
3. Multiply the other two elements by –1.
4. Multiply each element of the matrix by $\frac{1}{|A|}$.

Example

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -4 \\ 3 & -5 \end{bmatrix}$.

Solution

$$|A| = \begin{vmatrix} 2 & -4 \\ 3 & -5 \end{vmatrix} = -10 + 12 = 2 \neq 0$$

$\therefore$ A is non singular and A^{-1} exists.

$$\therefore A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & 4 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{4}{2} \\ -\frac{3}{2} & \frac{2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & 2 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

Solution of simultaneous linear equations in two variables

The concept of matrices and determinants can be applied to solve a system of linear equations in two or more variables. Here we present two such methods. First one is Matrix Inversion method and the second one is Cramer's method.

1. **Matrix inversion method**

   Let us try to understand the method through an example.

Example

Solve the simultaneous linear equations

$2x - 5y = 1, 5x + 3y = 18.$

Solution

Given system of linear equations can be written in matrix form as shown below.

$$\begin{bmatrix} 2x-5y \\ 5x+3y \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

The L.H.S matrix can be further written as product of two matrices as shown below.

$$\begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix} \text{ or } AX = B \rightarrow \quad (1)$$

Here $A = \begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}$ is called coefficient matrix, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is called variable matrix and $B = \begin{bmatrix} 1 \\ 18 \end{bmatrix}$ is called constant matrix. Now we need to find values of x and y i.e., the matrix X

To find X premultiplying both the sides of (1) with A^{-1}.

$\Rightarrow A^{-1}(AX) = A^{-1}B.$

or $(A^{-1}A)X = A^{-1}B$ [since A (BC) = (AB) C]

or $IX = A^{-1}B$, [$\because A^{-1}A = I$] or $X = A^{-1}B$, [$IX = X$]

$X = A^{-1}B.$

So to find X we have to find inverse of coefficient matrix (i.e., A) and multiply it with B.

$$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 5 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

$$= \frac{1}{2\times 3-(-5)\times 5}\begin{bmatrix} 3 & 5 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

$$= \frac{1}{31}\begin{bmatrix} 3\times 1+5\times 18 \\ -5\times 1+2\times 18 \end{bmatrix}$$

$$= \frac{1}{31}\begin{bmatrix} 93 \\ 31 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\text{Thus } X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow x = 3, y = 1.$$

Thus in general any system of linear equations $px + qy = a$, and $rx + sy = b$ can be represented in matrix form (i.e., AX=B) as $\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

Here A is coefficient matrix $= \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, X is variables matrix $= \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \end{bmatrix}$ is constant matrix.

$$X = A^{-1} B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}^{-1}\begin{bmatrix} a \\ b \end{bmatrix}$$

Note:

1. Matrix inversion method is applicable only hen the coefficient matrix A is non-singular i.e., $|A| \neq 0$. If $|A| = 0$, then A^{-1} doesn't exist and so the method is not applicable.
2. This method can be extended for a system of linear equations in more than 2 variables.

2. **Cramer's rule or Cramer's method**

   This is another method of solving system of linear equations using concept of determinants. Unlike matrix inversion method, in this method we don't need to find the inverse of coefficient matrix.

Example

Solve the system of linear equations

$3x + 4y = 2$, $5x - 3y = 13$ by Cramer's method.

The system of equations can be written in matrix form (i.e., AX = B) as shown below.

$$\begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

Here $A = \begin{bmatrix} 3 & 4 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$

To find the solution by Cramer's method we define two matrices B_1 and B_2. The matrix B_1 is obtained by replacing first column matrix A by the column in B. similarly B_2 is obtained by replacing column 2 of matrix A by the column in B.

i.e., $B_1 = \begin{bmatrix} 2 & 4 \\ 13 & -3 \end{bmatrix}$, $B_2 = \begin{bmatrix} 3 & 2 \\ 5 & 13 \end{bmatrix}$

Now $x = \dfrac{|B_1|}{|A|}$

$$= - \frac{\begin{vmatrix} 2 & 4 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}} = \frac{2(-3) - 4(13)}{3(-3) - 4(5)}$$

$$= \frac{-6 - 52}{-9 - 20} = 58/29 = 2 \text{ and}$$

$$y = \frac{|B_2|}{|A|} = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 13 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & -3 \end{vmatrix}}$$

$$= \frac{3 \times 13 - 2 \times 5}{3(-3) - 4 \times 5} = 29/-29 = -1$$

Thus in general for a system of linear equations $px + qy = a$, $rx + sy = b$, solution by Cramer's method

is $x = \dfrac{\begin{vmatrix} a & q \\ b & s \end{vmatrix}}{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} p & a \\ r & b \end{vmatrix}}{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}$

Note:

1. If the coefficient matrix A is singular, then $|A| = 0$, and so the method is not applicable.
2. This method can be extended to system of equations in more than two variables.

test your concepts

Very short answer type questions

1. Who gave the name 'matrix' to a rectangular arrangement of certain numbers in some rows and columns?

2. If $a_{ij} = 0$ $(i \neq j)$ and $a_{ij} = 4$ $(i = j)$, then the matrix $A = [a_{ij}]_{n \times n}$ is a __________ matrix.

3. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & m \end{bmatrix}$ is a scalar matrix, then $x + m =$ _________.

4. The order of column matrix containing n rows is ________.

5. If $P = \begin{bmatrix} 3 & 0 \\ 0 & \lambda \end{bmatrix}$ is scalar matrix then $\lambda =$ _____.

6. If $\begin{bmatrix} 4 & -3 \\ 2 & 16 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 2^t \end{bmatrix}$ then t =_____.

7. If $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, then find $|A|$.

8. The product of two matrices i.e., AB = I, then B is called the _____ of A and written as _____.

9. If $(A + B^T)^T$ is a matrix of order 4×3, then the order of matrix B is _____________.

10. If $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} + \begin{bmatrix} -y & 4 \\ 7 & x \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 7 & 6 \end{bmatrix}$ then x = _________ and y = _________.

11. Is $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ singular?

12. If A is any square matrix, then $\frac{1}{2}(A - A^T)$ is a _________ matrix.

13. If the determinant of a square matrix is non-zero, then the matrix is called a _____ matrix.

14. $(AB)^{-1} =$ _____.

15. The inverse of matrix A, if $A^2 = I$, is ___________.

16. The additive inverse of $\begin{bmatrix} -1 & 3 & 4 \\ 5 & -7 & 8 \end{bmatrix}$ is _________.

17. If $A \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, then the order of A is _________.

18. If the order of matrices A, B and C are 3×4, 7×3 and 4×7 respectively, then the order of (AC)B is _________.

19. Express the equations $2x - y + 6 = 0$ and $6x + y + 8 = 0$, in the matrix equation form.

20. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} =$ _____.

21. If $\begin{vmatrix} 2 & -4 \\ 9 & d-3 \end{vmatrix} = 4$, then d = _____.

22. If AB = KI, where $K \in R$, then $A^{-1} =$ _____.

23. If p = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then P^{-1} = _____.

24. The value of $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ = _____.

25. If K is real number, then $(KA)^{-1}$ = _____.

26. The matrix A = $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ is singular then a = _____.

27. If A and B commute, then $(A + B)^2$ = ________.

28. If $|A| = 5$, $|B_1| = 5$ and $|B_2| = 25$, then find the values of x and y in Cramer's method.

29. If A = $[s \quad 2]$ and B = $\begin{bmatrix} x \\ y \end{bmatrix}$ then AB = _____.

30. The matrix obtained by multiplying each of the given matrix A with –1 is called the _________ of A and is denoted by _________.

Short answer type questions

31. If $A = [a_{ij}]_{2\times2}$ such that $a_{ij} = i - j + 3$, then find A.

32. If $A + B^T = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ and $A^T - B = \begin{bmatrix} 7 & 8 \\ -1 & 3 \end{bmatrix}$, then find matrices A and B.

33. If $\begin{pmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{4}{6} & -\frac{1}{7} \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then find a, b, c and d.

34. If B = $\begin{bmatrix} -1 & 0 \\ 2 & 4 \end{bmatrix}$ and $f(x) = x^2 - 4x + 5$, then find f(B).

35. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix}$ then find A (B + C).

36. If $A \times \begin{bmatrix} -3 & 4 \\ 5 & 10 \end{bmatrix} = [13 \ 6]$, then find A.

37. Two friends Jack and Jill attend IIT entrance test which has three sections; Maths, Physics and Chemistry. Each question in Maths, Physics and Chemistry carry 5 marks, 8 marks and 3 marks respectively. Jack attempted 10 questions in Maths, 12 in Physics and 6 in Chemistry while Jill attempted 18, 5 and 9 questions in Maths, Physics and Chemistry respectively. Assuming that all the questions attempted were correct, find the individual marks obtained by the boys by showing the above information as a matrix product.

38. If A and B are two matrices such that $A + B = \begin{bmatrix} 3 & 8 \\ 11 & 6 \end{bmatrix}$ and $A - B = \begin{bmatrix} 5 & 2 \\ -3 & -6 \end{bmatrix}$, then find the matrices A and B.

39. Compute the product $\begin{pmatrix} -5 & 1 \\ 6 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 4 & -5 & -1 \\ 5 & 6 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$.

40. If $A = \begin{pmatrix} 7 & 2 \\ 18 & 5 \end{pmatrix}$, then show that $A - A^{-1} = 12I$.

41. If $A = \begin{pmatrix} 9 & -7 \\ -4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & -7 \\ -4 & -9 \end{pmatrix}$, then find AB and hence find A^{-1}.

42. Given $A = \begin{pmatrix} 3 & p \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 5 & 6 \end{pmatrix}$. If AB = BA, then find p.

43. If $A = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find the matrix X such that $4A - 2X + I = O$.

44. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 9 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 6 \\ 2 & 1 \end{bmatrix}$, then find $2A + 3B - 4C$.

45. Find the possible orders possible of matrices A and B if they have 18 and 19 elements respectively.

Essay type questions

46. If $A = \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$, then find $A + 10\,A^{-1}$.

47. Solve the following simultaneous equations using Cramer's method.

$$\frac{3x - 5y}{18} = 1,\ 2y - 4x + 10 = 0.$$

48. If $A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$ find AB, $(AB)^{-1}$, A^{-1}, B^{-1} and $B^{-1}A^{-1}$. What do you notice?

49. Solve the following system of linear equations using matrix inversion method.

$5x - 3y = -13,\ 2x + 5y = 1$

50. If $A = \begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$, then show that $A + 23\,A^{-1} = 6I$.

CONCEPT APPLICATION

Concept Application Level—1

1. If $\begin{vmatrix} 2 & -3 \\ p-4 & 2p-1 \end{vmatrix} = -6$, then p =

(1) 8/7 (2) 7/8 (3) 5 (4) 0

2. If $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b+c \\ b-c & d \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 3 & 2 \end{pmatrix}$, then $(a-b)+(c-d) =$

(1) –2 (2) 9 (3) 2 (4) –1

3. If $\begin{vmatrix} 5 & -3 \\ 6 & -a \end{vmatrix} = 4$, then 5a – 4 =

(1) 0 (2) 10 (3) 14 (4) 14/5

4. If $\begin{bmatrix} 2 & -3 \\ 5x+4 & 4 \end{bmatrix}$ has a multiplicative inverse, then x cannot be

(1) $\frac{3}{4}$ (2) $\frac{4}{5}$ (3) $\frac{-3}{4}$ (4) $\frac{-4}{3}$

5. If $A = \begin{bmatrix} 8 & 7 \\ -9 & -8 \end{bmatrix}$, then $A^{-1} =$ ______________.

(1) A (2) –A (3) 2A (4) $\begin{bmatrix} 8 & 7 \\ -(-9) & -8 \end{bmatrix}$

6. Given $A = \begin{bmatrix} 4 & -2 \\ 2a-1 & 5a-3 \end{bmatrix}$ and if A doesn't have multiplicative inverse, then 12a – 13 =

(1) 6 (2) 7/12 (3) 12/7 (4) –6

7. If $A = \begin{pmatrix} 7 & 2 \\ -3 & 9 \end{pmatrix}$, $B = \begin{pmatrix} p & 2 \\ -3 & 5 \end{pmatrix}$ and AB = BA, then find p.

(1) –2 (2) 1
(3) 3 (4) p does not have a unique value

8. Given $A = \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$, then $A^{-1} =$ __________.

(1) $\begin{pmatrix} -1 & -3 \\ 2 & 5 \end{pmatrix}$ (2) $\begin{pmatrix} -1 & -3 \\ -2 & -5 \end{pmatrix}$ (3) $\begin{pmatrix} -1 & -3 \\ 2 & -5 \end{pmatrix}$ (4) $\begin{pmatrix} 1 & 3 \\ 2 & -5 \end{pmatrix}$

9. If $\begin{vmatrix} 7a-5b & 3c \\ -1 & 2 \end{vmatrix} = 0$, then which of the following is true?

(1) $14a + 3c = 5b$ (2) $14a - 3c = 5b$
(3) $14a + 3c = 10b$ (4) $14a + 10b = 3c$

10. If a square matrix A is skew symmetric, then which of the following is correct?

(1) A^T is skew symmetric (2) A^{-1} is skew symmetric
(3) A^{2007} is skew symmetric (4) All the above

11. If $|A| = 47$, then find $|A^T|$.

(1) -47 (2) 47
(3) 0 (4) Cannot be determined

12. There are 25 software engineers and 10 testers in infosys and 15 software engineers and 8 testers in Wipro. In both the companies, a software engineer is paid Rs 5000 per month and a tester is paid Rs 3000 per month. Find the total amount paid by each of the companies per month by representing the data in matrix form.

(1) $\begin{pmatrix} 155000 \\ 99000 \end{pmatrix}$ (2) $\begin{pmatrix} 23000 \\ 24000 \end{pmatrix}$ (3) $\begin{pmatrix} 50000 \\ 30000 \end{pmatrix}$ (4) $\begin{pmatrix} 155000 \\ 100000 \end{pmatrix}$

13. If $\det(A) = 5$, then find $\det(15A)$ where A is of order 2×2.

(1) 225 (2) 75 (3) 375 (4) 1125

14. If $A = \begin{bmatrix} \operatorname{cosec}\alpha & \tan\alpha \\ \cot\alpha & -\sin\alpha \end{bmatrix}$, then A is a/an ______________.

(1) singular matrix. (2) scalar matrix.
(3) symmetric matrix. (4) non-singular matrix.

15. Which of the following statements is true?

(1) A singular matrix has an inverse.
(2) If a matrix doesn't have multiplicative inverse, it need not be a singular matrix.
(3) If a, b are non-zero real numbers, then $\begin{bmatrix} a+b & a-b \\ b-a & a+b \end{bmatrix}$ is a non-singular matrix.
(4) $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$ is a singular matrix.

16. What is the condition that is to be satisfied for the identity $(P + Q)(P - Q) = P^2 - Q^2$ to be true for any two square matrices P and Q?

(1) The identity is always true.
(2) $PQ \neq QP$.
(3) Both PQ and QP are not null matrices.
(4) P, Q and PQ are symmetric.

17. Solve the simultaneous equations:

$2x - 3y = 11$ and $5x + 4y = 16$

(1) $x = 5, y = -1/3$ (2) $x = 2, y = 2/3$
(3) $x = -1, y = 4$ (4) $x = 4, y = -1$

18. If I is a 2×2 identity matrix, then $|(3I)^{30}|^{-1} =$

(1) $\frac{1}{3^{30}}$ (2) $\frac{1}{3^{60}}$ (3) 3^{30} (4) 3^{60}

19. If A is a 2×2 square matrix, such that det A = 9, then det(9A) =

(1) $\frac{1}{3^{30}}$ (2) 9 (3) 81 (4) 729

20. Which of the following statement (s) is true?

(1) Inverse of a square matrix is not unique.
(2) If A and B are two square matrices, then $(AB)^T = A^T B^T$.
(3) If A and B are two square matrices, then $(AB)^{-1} = A^{-1} B^{-1}$.
(4) If A is a non-singular square matrix, then its inverse can be uniquely expressed as sum of a symmetric and a skew-symmetric matrix.

21. If A and B are two square matrices such that AB = A and BA = B, then find $(A^{2006} B^{2006})^{-1}$.

(1) $A^{-1} B^{-1}$ (2) $B^{-1} A^{-1}$
(3) AB (4) Cannot be determined

22. If $A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}$, then find the determinant value of AB.

(1) 10 (2) 20 (3) 12 (4) 15

23. If the trace of the matrix A is 4 and the trace of matrix B is 7, then find the trace of matrix AB.

(1) 4 (2) 7
(3) 28 (4) Cannot be determined

24. If the trace of the matrix A is 5, and the trace of the matrix B is 7, then find trace of the matrix (3A + 2B).

(1) 12 (2) 29 (3) 19 (4) None of these

25. If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then find A^n. (where $n \in N$)

(1) $\begin{bmatrix} 3n & 0 \\ 0 & 3n \end{bmatrix}$ (2) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(3) $3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (4) $I_{2 \times 2}$

26. If A is a 2 × 2 scalar matrix and 7 is the one of the elements in its principal diagonal, then the inverse of A is

(1) $\begin{bmatrix} \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} \end{bmatrix}$ (2) $\begin{bmatrix} 0 & -7 \\ -7 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ (4) $\begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{bmatrix}$

27. A_1, A_2, A_3 .... A_n and B_1, B_2, B_3 .... B_n are non-singular square matrices order n such that $A_1B_1 = I_n, A_2B_2 = In, A_3B_3 = In$ ----- $A_n\,B_n = I_n$, then $(A_1\,A_2\,A_3$ ------ $A_n)^{-1}$ =

(1) $B_1\,B_2\,B_3$ ------ B_n
(2) $B_1^{-1}\,B_2^{-1}\,B_3^{-1}$ ----- B_n^{-1}
(3) $B_n\,B_{n-1}\,B_{n-2}$ ------- B_1
(4) $B_n^{-1}\,B_{n-1}^{-1}\,B_{n-2}^{-1}$ ------ B_1^{-1}

28. If $A = \begin{bmatrix} 5 & 6 \\ 9 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ p & 3 \end{bmatrix}$ and AB = BA, then p =

(1) $\frac{9}{2}$ (2) $\frac{-2}{9}$ (3) $\frac{-9}{2}$ (4) $\frac{2}{9}$

29. The inverse of a scalar matrix A of order 2 × 2, where one of the principal diagonal elements is 5 is

(1) 5I (2) I (3) $\frac{1}{5}I$ (4) $\frac{1}{25}I$

30. If $A = \begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$, then AA^{-1} = __________.

(1) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Concept Application Level—2

31. If $A = \begin{pmatrix} 4 & 22 \\ -1 & -6 \end{pmatrix}$, then find $A + A^{-1}$.

(1) $\begin{bmatrix} 8 & -11 \\ -1 & -6 \end{bmatrix}$ (2) $\begin{bmatrix} 7 & 33 \\ 1/2 & -4 \end{bmatrix}$

(3) $\begin{bmatrix} 7 & 33 \\ -3/2 & -8 \end{bmatrix}$ (4) $\begin{bmatrix} 7 & 33 \\ -3/2 & -4 \end{bmatrix}$

32. If $A = \begin{bmatrix} 4 & p \\ 3 & -4 \end{bmatrix}$ and $A - A^{-1} = 0$, then p = __________.

(1) 4 (2) 3 (3) –5 (4) 5

33. If $\begin{pmatrix} 11 & -4 \\ 8 & -3 \end{pmatrix} \begin{pmatrix} -x & 4 \\ -8 & y \end{pmatrix} = -\begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$, then find $2x - y$.

(1) -5 (2) 5 (3) 0 (4) 14

34. If $\begin{bmatrix} a^{x} \\ a^{-x} \end{bmatrix} [1 \quad 2] = \begin{bmatrix} p & a^{-2} \\ q & \log_2 2 \end{bmatrix}$, $(a > 0)$, then $a^{p-q} =$ ________.

(1) $2^{3/2}$ (2) $2^{-3/2}$ (3) 1 (4) $4^{3/2}$

35. If the matrix $\begin{bmatrix} 2^{a} & 32 \\ 36 & 12^{b} \end{bmatrix}$ is singular and if $k = \dfrac{2a}{ca+1}$, then find c.

(1) $\dfrac{2}{3}$ (2) $\dfrac{3}{2}$ (3) $\dfrac{4}{3}$ (4) $\dfrac{3}{4}$

36. If $A = \begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}$, then find $(A^{12345})^{-1}$.

(1) A^{T} (2) A
(3) I (4) Cannot be determined

37. The inverse of a diagonal matrix, whose principal diagonal elements are *l*, m is__________.

(1) $\begin{bmatrix} \dfrac{1}{\ell} & 0 \\ 0 & \dfrac{1}{m} \end{bmatrix}$ (2) $\begin{bmatrix} \ell & 0 \\ 0 & m \end{bmatrix}$ (3) $\begin{bmatrix} \ell^{2} & 0 \\ 0 & m^{2} \end{bmatrix}$ (4) $\begin{bmatrix} 2\ell & 0 \\ 0 & 2m \end{bmatrix}$

38. If $\begin{bmatrix} 4^{b} & 288 \\ 72 & 18^{a} \end{bmatrix}$ is a singular matrix and $2b = a + \dfrac{1}{c}$, then c is ___________.

(1) 4 (2) $\dfrac{1}{4}$ (3) $\dfrac{1}{6}$ (4) 6

39. If $A = \begin{bmatrix} x^{2} & y \\ 5 & -4 \end{bmatrix}$ and $A = A^{-1}$, then find $\begin{bmatrix} x^{3} & y+x \\ 1 & 2x^{2}+y \end{bmatrix}^{-1}$

(1) $\dfrac{1}{41}\begin{bmatrix} 8 & 1 \\ -1 & 5 \end{bmatrix}$ (2) $\dfrac{1}{41}\begin{bmatrix} 5 & 8 \\ 1 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$ (4) $\dfrac{1}{41}\begin{bmatrix} 5 & 1 \\ -1 & 8 \end{bmatrix}$

40. If A is a non – singular square matrix such that $A^{2} - 7A + 5I = 0$, then $A^{-1} =$ ____________.

(1) $7A - I$ (2) $\dfrac{7}{5}I - \dfrac{1}{5}A$ (3) $\dfrac{7}{5}I + \dfrac{1}{5}A$ (4) $\dfrac{A}{5} - \dfrac{7}{5}$

41. If A = a $\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$ nd B = $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, then find $B^{-1}.A^{-1}$

(1) $\begin{bmatrix} \frac{-1}{2} & \frac{3}{4} \\ \frac{-1}{2} & \frac{-5}{4} \end{bmatrix}$ (2) $\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{-5}{4} & \frac{3}{4} \end{bmatrix}$

(3) $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-5}{4} & \frac{3}{4} \end{bmatrix}$ (4) $\begin{bmatrix} \frac{1}{2} & \frac{-3}{4} \\ \frac{-1}{2} & \frac{-5}{4} \end{bmatrix}$

42. If P = $\begin{bmatrix} \sec\alpha & \tan\alpha \\ -\cot\alpha & \cos\alpha \end{bmatrix}$ and Q = $\begin{bmatrix} -\cos\alpha & \tan\alpha \\ -\cot\alpha & -\sec\alpha \end{bmatrix}$, then $2P^{-1} + Q =$

(1) $\begin{bmatrix} \cos\alpha & -\tan\alpha \\ \cot\alpha & \sec\alpha \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} -\cos\alpha & \tan\alpha \\ -\cot\alpha & -\sec\alpha \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

43. If A is a skew symmetric matrix such that AB = aI, then find $(A^{-1})^T$.

(1) –1B (2) $(-aB^T)$ (3) $\frac{B}{a}$ (4) $-\frac{B}{a}$

44. If A = $\begin{bmatrix} 8 & -7 \\ 9 & -8 \end{bmatrix}$, then $\left(A^{2007}\right)^{-1} =$ __________.

(1) I (2) 2A (3) A (4) 2007I

45. If the matrix $\begin{pmatrix} 10 & -9 \\ 5x+7 & 5 \end{pmatrix}$ is non-singular, then the range of x.

(1) $\frac{113}{45}$ (2) $R - \left\{\frac{-113}{45}\right\}$ (3) $R - \left\{\frac{113}{45}\right\}$ (4) $\frac{-113}{45}$

Concept Application Level—3

46. The number of integral values of x for which the determinant of the matrix $\begin{bmatrix} 5x+14 & -2 \\ 7x+8 & x \end{bmatrix}$ is always less than 1 is

(1) 3 (2) 4
(3) 5 (4) 6

47. If $A = \begin{bmatrix} x^2 & y^3 \\ \log_{1024} a & -9 \end{bmatrix}$, $a = 16^{25}$ and if $A = A^{-1}$, then $\begin{bmatrix} x^2 & y \\ 1 & x^2 + y \end{bmatrix}^{-1} =$

(1) $\frac{1}{65}\begin{bmatrix} 7 & 2 \\ -1 & 9 \end{bmatrix}$ (2) $\frac{1}{65}\begin{bmatrix} 7 & -2 \\ 1 & 9 \end{bmatrix}$

(3) $\frac{1}{65}\begin{bmatrix} 7 & 2 \\ 1 & 9 \end{bmatrix}$ (4) $\frac{1}{65}\begin{bmatrix} 9 & 2 \\ -1 & 7 \end{bmatrix}$

48. If $A = \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 5 & 5 \end{pmatrix}$ and $A^n = \begin{pmatrix} 5^{200} & 5^{200} \\ 0 & 0 \end{pmatrix}$, then find n.

(1) 100 (2) 50 (3) 25 (4) None of these

49. If $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008} = \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010}$, then $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2009}$ is _____.

(1) $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008} + \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010}$ (2) $\frac{1}{2}\left[\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2010} - \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}^{2008}\right]$

(3) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (4) $\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$

50. If $A = \begin{pmatrix} 0 & 2009 \\ -2009 & 0 \end{pmatrix}$, then find $\left[A^{2009} + (A^T)^{2009}\right]$.

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (3) $\begin{pmatrix} 0 & 2009 \\ -2009 & 0 \end{pmatrix}$ (4) None of these

KEY

Very short answer type questions

1. James Joseph Sylvester

2. scalar **3.** -2

4. $n \times 1$ **5.** $\lambda = 3$

6. 4. **7.** 0

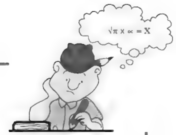
8. Multiplicative Inverse, A^{-1}.

9. 4×3

10. 5 and 1

11. Given matrix is singular matrix.

12. skew-symmetric



13. Non-singular

14. $B^{-1}A^{-1}$

15. A

16. $\begin{bmatrix} 1 & -3 & -4 \\ -5 & 7 & -8 \end{bmatrix}$

17. 3×2

18. 3×3

19. $\begin{bmatrix} 2 & -1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$

20. $\dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

21. – 13.

22. $\dfrac{1}{K}B$

23. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

24. 1

25. $\dfrac{1}{k}A^{-1}$

26. $\sqrt{bc}$

27. $A^2 + 2AB + B^2$

28. x, y = 3, 5 respectively.

29. $(sx + 2y)_{1 \times 1}$

30. Additive Inverse (–A).

Short answer type questions

31. $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

32. $A = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}; B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$

33. a = –1/2, b = –3/5, c = 4/6 and d = 1/7

34. $\begin{bmatrix} 10 & 0 \\ -2 & 5 \end{bmatrix}$

35. $\begin{bmatrix} -13 & -5 \\ 9 & 25 \end{bmatrix}$

36. $[-2 \quad 7/5]$

37. Marks obtained by Jack = 164.
Marks obtained by Jill = 157

38. $A = \begin{bmatrix} 4 & 5 \\ 4 & 0 \end{bmatrix}; B = \begin{bmatrix} -1 & 3 \\ 7 & 6 \end{bmatrix}$

39. $= \begin{bmatrix} -193 \\ 232 \\ -78 \end{bmatrix}$

41. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^{-1} = B = \begin{bmatrix} -3 & -7 \\ -4 & -9 \end{bmatrix}$

42. p = 0.

43. $\begin{bmatrix} 9/2 & -10 \\ 0 & 5/2 \end{bmatrix}$

44. $\begin{bmatrix} 3 & 7 \\ 3 & 16 \end{bmatrix}$

45. the orders possible for A are 1 × 18, 2 × 9, 3 × 6, 6 × 3, 9 × 2, 18 × 1the orders possible for B are 1 × 19 and 19 × 1

Essay type questions

46. 5I

47. x = 1, y = –3.

48. $(AB)^{-1} = B^{-1}.A^{-1}$

49. x = 1, y = – 3.

key points for selected questions

Short answer type questions

31. (i) Elements of A are a_{11}, a_{12}, a_{21} and a_{22}.
(ii) Find a_{11}, a_{12}, a_{21} and a_{22} by using a_{ij}.

32. (i) First of all find $(A + B^T)^T$.
(ii) Then add the equations and get A^T.
(iii) Then find A and B.

33. (i) AI = A
(ii) Then equate the corresponding elements

34. (i) $f(B) = B^2 - 4B + 5I$
(ii) Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

35. (i) First of all find B + C by adding corresponding element of B and C.
(ii) Find the product of A and (B + C).

36. (i) Take the matrix A as [a b]
(ii) Then multiply A and $\begin{bmatrix} -3 & 4 \\ 5 & 10 \end{bmatrix}$
(iii) Then equate the corresponding elements of product matrix with elements of [13, 6].

37. (i) First of all write matrices for the given information.
(ii) Then compute the product.

38. (i) Add the given matrix equations and get A.
(ii) Subtract one from the other and find B.

39. Refer the concept of multification

40. (i) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A^{-1}

$$= \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(ii) Then find $A - A^{-1}$ and 12I

41. (i) First of all find AB.
(ii) If AB = I, then $A^{-1} = B$.

42. (i) First of all find AB and BA.
(ii) Then equate the corresponding elements.

43. Substitute A in 2X = 4A + I and then find X

44. (i) Multiply each element of A by 2, for 2A.
(ii) Similarly find 3B and 4C.
(iii) Then add corresponding elements.

45. (i) Find the factors of 18 and 19
(ii) The number of different possible orders of matrix is equal to the number of factors of the total number of any elements.

Essay type questions

46. (i) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A^{-1}

$$= \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(ii) Find $10A^{-1}$.
(iii) Then perform $A + 10A^{-1}$.

47. (i) Express the given equations in the matrix form.
(ii) If AX = B, then write B_1 by replacing first column of A by B and write B_2 by replacing second column of A by B.
(iii) $x = \frac{|B_1|}{|A|}$ and $y = \left|\frac{B_2}{A}\right|$

48. (i) First of all find AB, $(AB)^{-1}$, A^{-1}, B^{-1}, $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$
(ii) Then check which of the following is true $(AB)^{-1} = A^{-1}B^{-1}$ (or) $(AB)^{-1} = B^{-1}A^{-1}$.

49. (i) Express the given equations in the matrix form.
(ii) If A X = B, then $X = A^{-1}B$

50. (i) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then A^{-1}

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(ii) Then perform $A + 23A^{-1}$ and 6I

Concept Application Level—1,2,3

1. 1	**2.** 4
3. 2	**4.** 4
5. 1	**6.** 4
7. 3	**8.** 2
9. 3	**10.** 4
11. 2	**12.** 1
13. 4	**14.** 4
15. 3	**16.** 4
17. 4	**18.** 2
19. 4	**20.** 4
21. 2	**22.** 2
23. 4	**24.** 2
25. 3	**26.** 4
27. 3	**28.** 1
29. 3	**30.** 3
31. 3	**32.** 3
33. 1	**34.** 2
35. 1	**36.** 2
37. 1	**38.** 2
39. 4	**40.** 2
41. 3	**42.** 2
43. 4	**44.** 3
45. 2	**46.** 2
47. 1	**48.** 1
49. 4	**50.** 2

Concept Application Level—1,2,3

Key points for select questions

1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

2. Apply matrix multiplication concept and then equate the corresponding elements.

3. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

4. If, multiplicative inverse of A exists, then determinant of A $\neq 0$.

5. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A^{-1}

$= \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6. If, multiplicative inverse of A does not exist, then $|A| = 0$.

7. Apply matrix multiplication concept.

8. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

9. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

10. Apply the properties of skew symmetric.

11. $\det A = \det A^T$

12. Write the given data in matrix form.

13. $\det(KA) = k^2|A|$ (where A is 2×2 matrix)

14. Find determinant of A.

16. If P, Q and PQ are symmetric, then PQ = QP.

17. Write the equations in matrix form.

18. If A is an $n \times n$ matrix, then $|KA| = k^n|A|$ and $\det I = 1$.

19. If A is an $n \times n$ matrix, then $|KA| = k^n|A|$.

20. Recall the properties of matrices.

21. (i) If $AB = A$, $BA = B$ then $A^2 = A$; $B^2 = B$.
(ii) $(AB)^{-1} = B^{-1}A^{-1}$.

22. $\det(AB) = \det A . \det B$.

23. Trace $(AB) \neq \text{trace}(A) . \text{trace}(B)$.

24. Trace $(kA + mB) = k\,(\text{trace } A) + m\,(\text{trace } B)$.

25. (i) $A = 3I$
(ii) $A^n = 3^n I^n$.

26. (i) Refer the definition of scalar matrix and write A.
(ii) $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

27. (i) $AB = I \Rightarrow A^{-1} = B$.
(ii) $(AB)^{-1} = B^{-1} A^{-1}$.

28. Find AB and BA.

29. Write matrix A and find A^{-1}.

30. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

31. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

32. $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

33. Apply matrix multiplication concept.

34. Multiply the left side matirces and equate the corresponding elements.

35. (i) If the matrix is singular, then its determinant is zero. Using this condition we can obtain the values of a and b.
(ii) Substitute the values of a and b in $b = \dfrac{2a}{ca+1}$, then find the value of 'c'.

36. (i) Calculate A^2.
(ii) Find A^2, A^3 .... and observe.

37. The inverse of diagonal matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$

38. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a singular matrix, then $ad - bc = 0$.

39. (i) Use $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A = A^{-1}$, then $a = d$ and $ad - bc = -1$.
(ii) Substitute the values of x and y. Find the inverse of that matrix.

40. Premultiply the given equation with A^{-1} and use the relation $A^{-1} A = I, A^{-1}I = A^{-1}$.

41. First find AB, after that find $(AB)^{-1}$.
We know that $(AB)^{-1} = B^{-1} A^{-1}$.

42. Find P^{-1} and then proceed.

43. (i) A is skew symmetric matrix, $A^T = -A$.
(ii) According to the problem, the inverse of A is $\dfrac{1}{a}B$.

44. Calculate A^2, then find $(A^{2007})^{-1}$

45. Deteminant is non-zero.

46. (i) Find the determinant of matrix.
(ii) Solve the inequation.

47. (i) Put $a = 16^{25}$ in $\log_{1024} a$ and simplify.
(ii) Then, find x and y values using the relation $A = A^{-1}$.
(iii) Find the inverse of the given matrix.

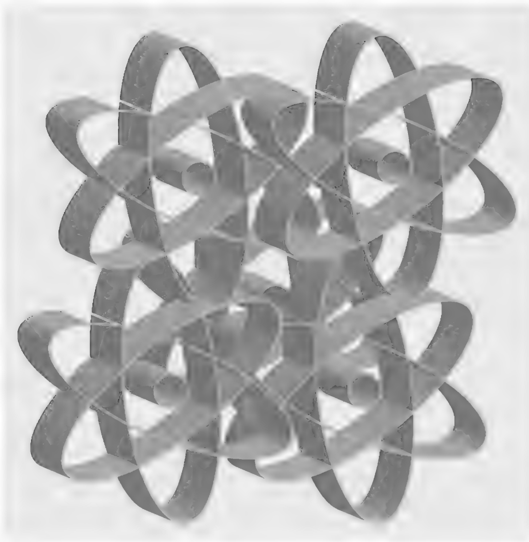
48. (i) Simplify the matrix A
(ii) find $A^2, A^3, \ldots\ldots A^n$

49. (i) Let $= A = \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$
(ii) Calculate A^2

50. Find A^2 then calculate A^{2009}.

CHAPTER 11

Remainder and Factor Theorems



INTRODUCTION

A real valued function f(x) of the form $a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$, $(a_0 \neq 0)$ is called as a polynomial of degree n, where n is a non-negative integer. Here $a_0, a_1, \ldots, a_n$ are the coefficients of various powers of x.

Example

(i) $4x^6 + 5x^5 + x^4 + x^2 - 1$ is a polynomial in x of degree 6.

(ii) $2x^3 + x^2 + 1$ is a polynomial in x of degree 3.

NOTE: A constant is considered to be a polynomial of zero degree.

In earlier classes we have learnt the different operations on polynomials like addition, subtraction, multiplication and division. Here we shall learn two important theorems on polynomials.

Remainder theorem

If p(x) is any polynomial and 'a' is any real number, then the remainder when p(x) is divided by (x – a) is given by p(a).

Proof

Let q(x) and r(x) be the quotient and the remainder respectively when p(x) is divided by x – a.

∴ By division algorithm

Dividend = quotient × divisor + remainder

i.e., $p(x) = q(x)\,(x - a) + r(x)$

If x = a, then

$p(a) = q(a)\,(a - a) + r(a) \Rightarrow r(a) = p(a)$

i.e., $p(x) = (x - a)\,q(x) + p(a)$

Thus the remainder is p(a)

NOTE:

1. If $p(a) = 0$, we say that 'a' is a zero of the polynomial $p(x)$.
2. If $p(x)$ is a polynomial and 'a' is a zero of $p(x)$, then $p(x) = (x - a)\, q(x)$.
3. If $p(x)$ is divided by $ax + b$, then the remainder is given by $p\left(\frac{-b}{a}\right)$
4. If $p(x)$ is divided by $ax - b$, then the remainder is given by $p\left(\frac{b}{a}\right)$

Example

Find the remainder when the polynomial $p(z) = z^3 - 3z + 2$ is divided by $z - 2$

Solution

Given $p(z) = z^3 - 3z + 2$

The remainder when $p(z)$ is divided by $z - 2$ is given by $p(2)$.

Now, $p(2) = (2)^3 - 3\,(2) + 2$

$= 8 - 6 + 2 = 4$

Hence, when $p(z)$ is divided by $z - 2$ the remainder is 4.

Factor theorem

If $p(x)$ is a polynomial of degree n (≥ 1) and a be any real number such that $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Proof

Let $q(x)$ be the quotient and $(x - k)$ $(k \in R)$ be a factor of $p(x)$

Given $p(a) = 0$

$\therefore$ By division algorithm

Dividend = quotient × divisor + remainder

$p(x) = q(x)\,(x - k) + p(a)$

$\Rightarrow p(x) = q(x)\,(x - k)$ ($\because$ $p(a) = 0$)

Therefore $(x - k)$ is a factor of $f(x)$, which is possible only if $f(k) = 0$

Hence $(x - a)$ is a factor of $p(x)$ ($\because$ $p(a) = 0$)

NOTE:

1. If $p(-a) = 0$, then $(x + a)$ is a factor of $p(x)$.
2. If $p\left(\frac{-b}{a}\right) = 0$, then $(ax + b)$ is a factor of $p(x)$.
3. If $p\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ is a factor of $p(x)$.
4. If sum of all the coefficients of a polynomial is zero, then $(x - 1)$ is one of its factors.
5. If sum of the coefficients of odd powers of x is equal to the sum of the coefficients of even powers of x, then one of the factors of the polynomial is $(x + 1)$.

Examples

(i) Determine whether $x - 3$ is a factor of $f(x) = x^2 - 5x + 6$

Given $f(x) = x^2 - 5x + 6$

Now $f(3) = (3)^2 - 5\,(3) + 6$

$= 9 - 15 + 6 .= 0 \Rightarrow f(3) = 0$

Hence, by factor theorem we can say that $(x - 3)$ is a factor of $f(x)$

(ii) Determine whether $(x - 1)$ is a factor of $x^3 - 6x^2 + 11x - 6$

Let $f(x) = x^3 - 6x^2 + 11x - 6$

Now $f(1) = (1)^3 - 6\,(1)^2 + 11\,(1) - 6$

$= 1 - 6 + 11 - 6 = 0 \Rightarrow f(1) = 0$

Hence, by factor theorem we can say that $(x - 1)$ is a factor of $f(x)$

Factorization of polynomials using factor theorem

(i) Factorize $x^2\,(y - z) + y^2\,(z - x) + z^2\,(x - y)$

Let us assume the given expression as a polynomial in x, say f (x)

$f(x) = x^2\,(y - z) + y^2\,(z - x) + z^2\,(x - y)$

Now put $x = y$ in the given expression

$\Rightarrow f(y) = y^2\,(y - z) + y^2\,(z - y) + z^2\,(y - y)$

$= y^3 - zy^2 + y^2z - y^3 + 0 = 0 \Rightarrow f(y) = 0$

$\Rightarrow x - y$ is a factor of the given expression

Similarly if we consider the given expression as a polynomial in y we get $y - z$ is a factor of the given expression and we also get $z - x$ is a factor of the expression when we consider it as an expression in z.

Let $x^2\,(y - z) + y^2\,(z - x) + z^2\,(x - y) = k(x - y)\,(y - z)\,(z - x)$

For $x = 0$, $y = 1$ and $z = 2$, we get

$0^2\,(1 - 2) + 1^2\,(2 - 0) + 2^2\,(0 - 1) = k(0 - 1)\,(1 - 2)\,(2 - 0)$

$\Rightarrow -2 = -2k \Rightarrow k = 1$

$\therefore$ the factors of the given expression are $x - y$, $y - z$ and $z - x$

(ii) Use factor theorem to factorize $x^3 + y^3 + z^3 - 3xyz$

Given expression is $x^3 + y^3 + z^3 - 3xyz$

Consider the expression as a polynomial in variable x say f(x)

i.e., $f(x) = x^3 + y^3 + z^3 - 3xyz$

Now $f[-(y + z)] = [-(y + z)]^3 + y^3 + z^3 - 3\,[-(y + z)]\,yz$

$= -(y + z)^3 + y^3 + z^3 + 3yz\,(y + z)$

$= -(y + z)^3 + (y + z)^3 = 0 \Rightarrow f[-(y + z)] = 0$

$\Rightarrow$ According to factor theorem $x - [-(y + z)]$ i.e., $x + y + z$ is a factor of $x^3 + y^3 + z^3 - 3xyz$

Now using the long division method we get the other factor as

$x^2 + y^2 + z^2 - xy - yz - zx$

$\therefore x^3 + y^3 + z^3 - 3xyz = (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

Horner's process for synthetic division of polynomials

When a polynomial $f(x) = p_0x^n + p_1x^{n-1} + --- + p_{n-1}x + p_n$ is divided by a binomial $x - \alpha$, let the quotient be Q (x) and remainder be r.

We can find quotient Q(x) and remainder r by using Horner's synthetic division process as explained below.

α	p_0	p_1	p_2	---- p_{n-1}	p_n	--- I st row
left (corner)		$q_0\alpha$	$q_1\alpha$	---- $q_{n-2}\alpha$	$q_{n-1}\alpha$	--- II nd row
	q_0	q_1	q_2	---- q_{n-1}	r	--- III rd row

Step 1: Write all the coefficients $p_0, p_1, p_2, - - - - - , p_n$ of the given polynomial f(x) in the order of descending powers of x as in the first row. When any term in f(x) (as seen with descending powers of x) is missing we write zero for its coefficient.

Step 2: Divide the polynomial f(x) by $(x - \alpha)$ by writing α in the left corner as shown above $(x - \alpha = 0 \Rightarrow x = \alpha)$

Step 3: Write the first term of the third row as $q_0 = p_0$ then multiply q_0 by α to get $q_0\,\alpha$ and write it under p_1, as the first element of the second row

Step 4: Add $q_0\,\alpha$ to p_1 to get q_1, the second element of the third row

Step 5: Again multiply q_1 with α to get $q_1\alpha$ and write $q_1\,\alpha$ under p_2 and add $q_1\,\alpha$ to p_2 to get q_2 which is the third element of the third row

Step 6: Continue this process till we obtain q_{n-1} in the third row. Multiply q_{n-1} with α and write $q_{n-1}\,\alpha$ under p_n and add $q_{n-1}\,\alpha$ to p_n to get r in third row as shown above

In the above process the elements of the third row i.e., $q_0, q_1, q_2, - - - , q_{n-1}$ are the coefficients of the quotient Q(x) in the same order of descending powers starting with x^{n-1}

$\therefore Q(x) = q_0x^{n-1} + q_1x^{n-2} + - - - - + q_{n-2}\,x + q_{n-1}$

and the remainder is r i.e., the last element of the third row

NOTE: If the remainder r = 0 then α is one of the roots of f(x) = 0 or $x - \alpha$ is a factor of f(x)

Example

Factorize $x^4 - 10x^2 + 9$

Let $p(x) = x^4 - 10x^2 + 9$

Here sum of coefficients = 0, and also

sum of coefficients even powers of x = sum of coefficients of odd powers of x

$\therefore$ (x – 1) and (x + 1) are the factors of p(x).

Multiplier of x – 1 is 1 and x + 1 is –1

$\therefore$ The quotient is $x^2 - 9$

Hence $p(x) = (x - 1)(x + 1)(x^2 - 9)$

$\Rightarrow p(x) = (x - 1)(x + 1)(x - 3)(x + 3)$

1	1	0	–10	0	9
	0	1	1	–9	–9
–1	1	1	–9	–9	0
	0	–1	0	9	
	1	0	–9	0	

Problems based on factor and remainder theorems

Examples

1. Find the value of a if $ax^3 - (a + 1)x^2 + 3x - 5a$ is divisible by (x –2).

Solution

Let $p(x) = ax^3 - (a + 1)x^2 + 3x - 5a$

If p(x) is divisible by (x – 2), then its remainder is zero i.e., p(2) = 0

$\Rightarrow a(2)^3 - (a + 1)(2)^2 + 3(2) - 5a = 0$

$\Rightarrow 8a - 4a - 4 + 6 - 5a = 0$

$\Rightarrow -a + 2 = 0$

$\Rightarrow a = 2$

$\therefore$ The required value of a is 2.

2. If the polynomial $x^3 + ax^2 - bx - 30$ is exactly divisible by $x^2 - 2x - 15$. Find a and b and also the third factor.

Solution

Let $p(x) = x^3 + ax^2 - bx - 30$

Given p(x) is exactly divisible by $x^2 - 2x - 15$ i.e., $(x - 5)(x + 3)$

$\Rightarrow$ p(x) is divisible by $(x + 3)$ and $(x - 5)$

$\therefore p(-3) = 0$ and $p(5) = 0$

Consider $p(-3) = 0$

$\Rightarrow (-3)^3 + a(-3)^2 - b(-3) - 30 = 0$

$\Rightarrow -27 + 9a + 3b - 30 = 0$

$\Rightarrow 9a + 3b - 57 = 0$

$\Rightarrow 3a + b - 19 = 0 \rightarrow (1)$

Now consider $p(5) = 0$

i.e., $5^3 + a(5)^2 - b(5) - 30 = 0$

$\Rightarrow 125 + 25a - 5b - 30 = 0$

$\Rightarrow 25a - 5b + 95 = 0$

$\Rightarrow 5a - b + 19 = 0 \rightarrow (2)$

Adding (1) and (2), we get

$8a = 0$

$\Rightarrow a = 0$

Substituting a in (1), we get $b = 19$

$\therefore$ The required values of a and b are 0 and 19 respectively

$\Rightarrow p(x) = x^3 + 0(x^2) - 19x - 30$

i.e., $p(x) = x^3 - 19x - 30$

Thus, the third factor is $x + 2$.

–3	1	0	–19	–30
	0	–3	9	30
5	1	–3	–10	0
	0	5	10	
	1	2	0	

3. Find the linear polynomial in x which when divided by $(x - 3)$ leaves 6 as remainder and is exactly divisible by $(x + 3)$.

Solution

Let the linear polynomial be $p(x) = ax + b$

Given $p(3) = 6$ and $p(-3) = 0$

$\Rightarrow a(3) + b = 6$ and $a(-3) + b = 0$

$\Rightarrow 3a + b = 6 \rightarrow (1)$ and $-3a + b = 0 \rightarrow (2)$

Adding (1) and (2),

$2b = 6 \Rightarrow b = 3$

Substituting the value of b in (1), we get $a = 1$

$\therefore$ The required linear polynomial is $x + 3$.

4. A quadratic polynomial in x leaves remainders as 4 and 7 respectively when divided by $(x + 1)$ and $(x - 2)$. Also it is exactly divisible by $(x - 1)$. Find the quadratic polynomial.

Solution

Let the quadratic polynomial be $p(x) = ax^2 + bx + c$

Given $p(-1) = 4$, $p(2) = 7$ and $p(1) = 0$

$p(-1) = a(-1)^2 + b(-1) + c = 4$

$\Rightarrow a - b + c = 4 \rightarrow (1)$

Now $p(1) = 0$ and $p(2) = 7$

$\therefore\ a(1)^2 + b(1) + c = 0$ and

$a(2)^2 + b(2) + c = 7$

$\Rightarrow a + b + c = 0 \rightarrow (2)$

$4a + 2b + c = 7 \rightarrow (3)$

Subtracting (2) from (1), we have

$2b = -4$

$\Rightarrow b = -2$. Subtracting (2) from (3), we have

$3a + b = -7$

$\Rightarrow 3a - 2 = 7$ $(\because b = -2)$

$\Rightarrow 3a = 9 \Rightarrow a = 3$

Substituting the values of a and b in (1), we get $c = -1$

Hence, the required quadratic polynomial is $3x^2 - 2x - 1$

5. Find a common factor of the quadratic polynomials $3x^2 - x - 10$ and $2x^2 - x - 6$.

Solution

Consider $p(x) = 3x^2 - x - 10$ and $q(x) = 2x^2 - x - 6$

Let $(x - k)$ be a common factor of $p(x)$ and $q(x)$

$\therefore\ p(k) = q(k) = 0$

$\Rightarrow 3k^2 - k - 10 = 2k^2 - k - 6$

$\Rightarrow k^2 - 4 = 0$

$\Rightarrow k^2 = 4$

$\Rightarrow k = \pm 2$

$\therefore$ The required common factor is $(x - 2)$ or $(x + 2)$.

6. Find the remainder when x^{999} is divided by $x^2 - 4x + 3$

Solution

Let $q(x)$ and $mx + n$ be the quotient and the remainder respectively when x^{999} is divided by $x^2 - 4x + 3$.

$\therefore\ x^{999} = (x^2 - 4x + 3)\ q(x) + mx + n$

If $x = 1$,

$1^{999} = (1 - 4 + 3)\ q(x) + m(1) + n$

$\Rightarrow 1 = 0 \times q(x) + m + n$

$\Rightarrow m + n = 1 \rightarrow (1)$

If $x = 3$,

$3^{999} = (3^2 - 4(3) + 3)\ q(x) + 3m + n$

$\Rightarrow 3^{999} = 0 \times q(x) + 3m + n$

$\Rightarrow 3m + n = 3^{999} \rightarrow (2)$

Subtracting (1) from (2) we get

$2m = 3^{999} - 1$

$m = \frac{1}{2}(3^{999} - 1)$

Substituting m in (1), we have

$$n = 1 - \frac{1}{2}(3^{999} - 1) = 1 - \frac{1}{2}3^{999} + \frac{1}{2} = \frac{3}{2} - \frac{1}{2}3^{999}$$

$$n = \frac{3}{2}\ (1 - 3^{998})$$

$\therefore$ The required remainder is $\frac{1}{2}(3^{999} - 1)\ x + \frac{3}{2}(1 - 3^{998})$.

7. Find the remainder when x^5 is divided by $x^3 - 4x$.

Solution

Let q(x) be the quotient and $\ell x^2 + mx + n$ be the remainder when x^5 is divided by $x^3 - 4x$ i.e., $x(x - 2)\ (x + 2)$

$\therefore\ x^5 = (x^3 - 4x)\ q(x) + \ell x^2 + mx + n$

Put $x = 0$

$\Rightarrow 0 = 0 \times q(x) + \ell(0) + m(0) + n$

$\Rightarrow n = 0$

Put $x = 2$

$\Rightarrow 2^5 = (8 - 8)\ q(x) + \ell(2)^2 + m(2) + n$

$\Rightarrow 32 = 4\ell + 2m + n$

$\Rightarrow 4\ell + 2m = 32\ (\because\ n = 0)$

$\Rightarrow 2\ell + m = 16$ ----- (1)

Put $x = -2$

$(-2)^5 = (-8 + 8)\ q(x) + \ell(-2)^2 + m\ (-2) + n$

$\Rightarrow -32 = 4\ell - 2m + n$

$\Rightarrow 4\ell - 2m = -32\ (\because\ n = 0)$

$\Rightarrow 2\ \ell - m = -16 \dashrightarrow (2)$

Adding (1) and (2),

$4\ell = 0$

$\Rightarrow \ell = 0$

Substituting ℓ in (1), we get

$2(0) + m = 16$

$\Rightarrow m = 16$

$\therefore$ The required remainder is $0(x^2) + 16x + 0$ i.e., $16x$

test your concepts

Very short answer type questions

1. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots\ldots + a_n$ $(a_0 \neq 0)$ be a polynomial of degree n. If $x + 1$ is one of its factors, then ______.
2. If a polynomial f(x) is divided by $(x + a)$, then the remainder obtained is ______.
3. If $a - b$ is a factor of $a^n - b^n$, then n is ______.
4. If $f(x) = x^3 + 2$ is divided by $x + 2$, then the remainder obtained is ______.
5. The condition for which $ax^2 + bx + a$ is exactly divisible by $x - a$ is ______.
6. If $x + 1$ is a factor of $x^m + 1$, then m is ___________.
7. The remainder when $f(x) = x^3 + 5x^2 + 2x + 3$ is divided by x is ______.
8. The remainder when $(x - a)^2 + (x - b)^2$ is divided by x is ______.
9. The remainder when $x^6 - 4x^5 + 8x^4 - 7x^3 + 3x^2 + 2x - 7$ is divided by $x - 1$ is _________.
10. For two odd numbers x and y, if $x^3 + y^3$ is divisible by 2^k, $k \in N$, then $x + y$ is divisible by 2^k. [True/False]
11. One of the factors of $2x^{17} + 3x^{15} + 7x^{23}$ is __________ $(x^{17} / x^{15} / x^{23})$
12. If $(x - 2)^2$ is the factor of an expression of the form $x^3 + bx + c$, then the other factor is ___________.
13. What should be added to $3x^3 + 5x^2 - 6x + 3$ to make it exactly divisible by $x - 1$?
14. The remainder when $2x^6 - 5x^3 - 3$ is divided by $x^3 + 1$ is _________.
15. The remainder when f(x) is divided by g(x) is $f\left(-\frac{3}{2}\right)$, then g(x) is necessarily $2x + 3$. [True/False]
16. Find the remainder when the polynomial $x^2 + 13x + 11$ is divided by $x - 1$.
17. Find the value of the polynomial $a^2 - \frac{1}{6}a + \frac{3}{2}$ when $a = \frac{1}{2}$.
18. The polynomial $7x^2 - 11x + a$ when divided by $x + 1$ leaves a remainder of 8. Then find the value of 'a'.
19. If $x + 2$ is a factor of f(x) and $f(x) = x^3 + 4x^2 + kx - 6$, then find the value of k.

20. Find the values of a if $x^3 - 5x(a-1) - 3(x+1) + 5a$ is divisible by $x - a$.

21. Find the value of a if $x - a$ is a factor of the polynomial $x^5 - ax^4 + x^3 - ax^2 + 2x + 3a - 2$.

22. Find the remainder when $x^3 + 3px + q$ is divided by $(x^2 - a^2)$ without actual division.

23. The remainder obtained when $x^2 + 3x + 1$ is divided by $(x - 5)$ is ______.

24. If the polynomial $3x^4 - 11x^2 + 6x + k$ is divided by $x - 3$, it leaves a remainder 7. Then the value of k is ______.

25. $(7x - 1)$ is a factor of $7x^3 + 6x^2 - 15x + 2$ (True/False)

26. If $ax^2 + bx + c$ is exactly divisible by $2x - 3$, then the relation between a, b and c is _________.

27. If $x^2 + 5x + 6$ is a factor of $x^3 + 9x^2 + 26x + 24$, then find the remaining factor.

28. If $(2x - 1)$ is a factor of $2x^2 + px - 2$, then the other factor is ______.

29. The expression $x^{m^n} - 1$ is divisible by $x + 1$, only if M is (even/odd) ___________.

30. If $x + m$ is one of the factors of the polynomial $x^2 + mx - m + 4$, then the value of m is ______.

Short answer type questions

31. For what values of m and n is $2x^4 - 11x^3 + mx + n$ is divisible by $x^2 - 1$?

32. Find a linear polynomial which when divided by $(2x + 1)$ and $(3x + 2)$ leaves remainders 3 and 4 respectively.

33. Prove that $x^m + 1$ is a factor of $x^{mn} - 1$ if n is even.

34. The remainders of a polynomial f(x) in x are 10 and 15 respectively when f(x) is divided by $(x - 3)$ and $(x - 4)$. Find the remainder when f(x) is divided by $(x - 3)(x - 4)$.

35. If x^{555} is divided by $x^2 - 4x + 3$, then find its remainder.

36. If $(x^2 - 1)$ is a factor of $ax^3 - bx^2 - cx + d$, then find the relation between a and c.

37. When $x^4 - 3x^3 + 4x^2 + p$ is divided by $(x - 2)$, the remainder is zero. Find the value of p.

38. Find the common factors of the expressions $a_1 x^2 + b_1 x + c_1$ and $a_2 x^2 + b_2 x + c_1$ where $c_1 \neq 0$.

39. If $(x - 3)$ is a factor of $x^2 + q$ (where $q \in Q$), then find the remainder when $(x^2 + q)$ is divided by $(x - 2)$.

40. If $p + q$ is a factor of the polynomial $p^n - q^n$, then n is

41. The expression $x^{4005} + y^{4005}$ is divisible by_______.

42. The value of a for which $x - 7$ is a factor of $x^2 + 11x - 2a$ is ______.

43. If a polynomial f(x) is divided by $(x - 3)$ and $(x - 4)$ it leaves remainders as 7 and 12 respectively, then find the remainder when f(x) is divided by $(x - 3)(x - 4)$.

44. Find the remainder when $5x^4 - 11x^2 + 6$ is divided by $5x^2 - 6$.

45. If $f(x - 2) = 2x^2 - 3x + 4$, then find the remainder when f(x) is divided by $(x - 1)$.

Essay type questions

46. Factorize $x^4 - 2x^3 - 9x^2 + 2x + 8$ using remainder theorem.

47. Find the remainder when x^{29} is divided by $x^2 - 2x - 3$.

48. If $x^2 - 2x - 1$ is a factor of $px^3 + qx^2 + 1$, (where p, q are integers) then find the value of $p + q$.

49. If $x^2 - x + 1$ is a factor of $x^4 + ax^2 + b$, then the values of a and b are respectively ______.

50. If $\ell x^2 + mx + n$ is exactly divisible by $(x - 1)$ and $(x + 1)$ and leaves a remainder 1 when divided by $x + 2$, then find m and n.

CONCEPT APPLICATION

Concept Application Level—1

1. The value of a for which the polynomial $y^3 + ay^2 - 2y + a + 4$ in y has $(y + a)$ as one of its factors is ______.

(1) $\frac{-3}{4}$ (2) $\frac{4}{3}$ (3) $\frac{3}{4}$ (4) $\frac{-4}{3}$

2. If the expression $2x^3 - 7x^2 + 5x - 3$ leaves a remainder of $5k - 2$ when divided by $x + 1$, then find the value of k.

(1) 3 (2) – 3 (3) 5 (4) – 5

3. Find the remainder when $x^{2003} + y^{6009}$ is divided by $x + y^3$.

(1) y^{4006} (2) 1
(3) 0 (4) Cannot be determined

4. Find the remainder when $x^6 - 7x^3 + 8$ is divided by $x^3 - 2$.

(1) – 2 (2) 2 (3) 7 (4) 1

5. If both the expressions $x^{1248} - 1$ and $x^{672} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ___________.

(1) 48 (2) 96 (3) 54 (4) 112

6. When $x^2 - 7x + 2$ is divided by $x - 8$, then the remainder is ______.

(1) 122 (2) 4 (3) 45 (4) 10

7. If $ax^2 + bx + c$ is exactly divisible by $4x + 5$, then

(1) $25a - 5b + 16c = 0$. (2) $25a + 20b + 16c = 0$.
(3) $25a - 20b - 16c = 0$. (4) $25a - 20b + 16c = 0$.

8. The expression $2x^3 + 3x^2 - 5x + p$ when divided by $x + 2$ leaves a remainder of $3p + 2$. Find p.

(1) –2 (2) 1 (3) 0 (4) 2

9. $3x - 4$ is a factor of ______.

(1) $18x^4 - 3x^3 - 28x^2 - 3x + 4$
(2) $3x^4 - 10x^3 - 7x^2 + 38x - 24$
(3) $9x^4 - 6x^3 + 5x^2 - 15$
(4) $9x^4 + 36x^3 + 17x^2 - 38x - 24$

10. Which of the following is a factor of $5x^{20} + 7x^{15} + x^9$?

(1) x^{20} (2) x^{15} (3) x^9 (4) x^{24}

11. If $(x + 3)^2$ is a factor of $f(x) = ex^3 + kx + 6$, then find the remainder obtained when f(x) is divided by $x - 6$.

(1) 1 (2) 0 (3) 5 (4) 4

12. The expression $x^{mn} + 1$ is divisible by $x + 1$, only if

(1) n is odd. (2) m is odd. (3) both m and n are even. (4) Cannot say

13. If both the expressions $x^{1215} - 1$and $x^{945} - 1$, are divisible by $x^n - 1$, then the greatest integer value of n is ______.

(1) 135 (2) 270 (3) 945 (4) None of these

14. If $(x - 2)$ is a factor of $x^2 + bx + 1$ (where $b \in Q$), then find the remainder when $(x^2 + bx + 1)$ is divided by $2x + 3$.

(1) 7 (2) 8 (3) 1 (4) 0

15. When $x^3 + 3x^2 + 4x + a$ is divided by $(x + 2)$, the remainder is zero. Find the value of a.

(1) 4 (2) 6 (3) –8 (4) –12

16. If $(x + 1)$ and $(x - 1)$ are the factors of $ax^3 + bx^2 + cx + d$, then which of the following is true?

(1) $a + b = 0$ (2) $b + c = 0$ (3) $b + d = 0$ (4) None of these

17. Find the remainder when x^5 is divided by $x^2 - 9$.

(1) $81x$ (2) $81x + 10$ (3) $3^5x + 3^4$ (4) None of these

18. The remainder when $x^{45} + x^{25} + x^{14} + x^9 + x$ divided by $x^2 - 1$ is ______.

(1) $4x - 1$ (2) $4x + 2$ (3) $4x + 1$ (4) $4x - 2$

19. For what values of a and b is the expression $x^4 + 4x^3 + ax^2 - bx + 3$ a multiple of $x^2 - 1$?

(1) $a = 1, b = 7$ (2) $a = 4, b = -4$ (3) $a = 3, b = -5$ (4) $a = -4, b = 4$

20. When the polynomial $p(x) = ax^2 + bx + c$ is divided by $(x - 1)$ and $(x + 1)$, the remainders obtained are 6 and 10 respectively. If the value of p(x) is 5 at $x = 0$, then the value of $5a - 2b + 5c$ is ______.

(1) 40 (2) 44 (3) 21 (4) 42

21. If $p - q$ is a factor of the polynomial $p^n - q^n$, then n is ______.

(1) a prime number (2) an odd number (3) an even number (4) All the above

22. When the polynomial $f(x) = ax^2 + bx + c$ is divided by x, $x - 2$ and $x + 3$, remainders obtained are 7, 9 and 49 respectively. Find the value of $3a + 5b + 2c$.

(1) –2 (2) 2 (3) 5 (4) –5

23. If $f(x + 1) = 2x^2 + 7x + 5$, then one of the factors of f(x) is ______.

(1) $2x + 3$ (2) $2x^2 + 3$ (3) $3x + 2$ (4) None of these

24. If $(x - p)$ and $(x - q)$ are the factors of $x^2 + px + q$, then the values of p and q are respectively ______.

(1) $1, -2$ (2) $2, -3$

(3) $\frac{-1}{3}, \frac{-2}{3}$ (4) None of these

25. Let $f\left(x - \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, find the remainder when f(x) is divided by $x - 3$.

(1) $\frac{82}{9}$ (2) $\frac{8}{3}$ (3) 10 (4) 11

26. If $(x - 2)^2$ is a factor of $f(x) = x^3 + px + q$, then find the remainder when f(x) is divided by $x - 1$.

(1) 4 (2) -4 (3) -5 (4) 5

27. A quadratic polynomial in x leaves remainders 4, 4 and 0 respectively when divided by $(x - 1)$, $(x - 2)$ and $(x - 3)$. Find the quadratic polynomial.

(1) $-2x^2 + 6x + 3$ (2) $-2x^2 + 6x$ (3) $-2x^2 + 6x + 5$ (4) $-2x^2 + 6x - 5$

28. If $f(x + 3) = x^2 + x - 6$, then one of the factors of f(x) is ______.

(1) $x - 3$ (2) $x - 4$ (3) $x - 5$ (4) $x - 6$

29. If $(x - 1)^2$ is a factor of $f(x) = x^3 + bx + c$, then find the remainder when f(x) is divided by $(x - 2)$.

(1) 2 (2) -3 (3) 4 (4) -4

30. For what values of m and n, the expression $2x^2 - (m + n)x + 2n$ is exactly divisible by $(x - 1)$ and $(x - 2)$?

(1) $m = 5, n = 2$ (2) $m = 3, n = 4$ (3) $m = 4, n = 2$ (4) $m = 2, n = 4$

Concept Application Level—2

31. The ratio of the remainders when the expression $x^2 + bx + c$ is divided by $(x - 3)$ and $(x - 2)$ respectively is 4 : 5. Find b and c, if $(x - 1)$ is a factor of the given expression.

(1) $b = \frac{-11}{3}, c = \frac{14}{3}$ (2) $b = \frac{-14}{3}, c = \frac{11}{3}$ (3) $b = \frac{14}{3}, c = \frac{-11}{3}$ (4) None of these

32. If the polynomials $f(x) = x^2 + 9x + k$ and $g(x) = x^2 + 10x + \ell$ have a common factor, then $(k - \ell)^2$ is equal to ______.

(1) $9\ell - 10k$ (2) $10\ell - 9k$ (3) Both (1) and (2) (4) None of these

33. When f(x) is divided by $(x - 2)$, the quotient is Q(x) and the remainder is zero. And when f(x) is divided by $[Q(x) - 1]$, the quotient is $(x - 2)$ and the remainder is R(x). Find the remainder R(x).

(1) $x + 2$ (2) $-x + 2$ (3) $x - 2$ (4) Cannot be determined

34. Find the values of m and n, if $(x - m)$ and $(x - n)$ are the factors of the expression $x^2 + mx - n$.

(1) $m = -1, n = -2$ (2) $m = 0, n = 1$ (3) $m = \frac{-1}{2}, n = \frac{1}{2}$ (4) $m = -1, n = 2$

35. Let $f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2}$, find the remainder when f(x) is divided by 2x + 1.

(1) $\frac{-7}{4}$ (2) $\frac{9}{4}$ (3) $\frac{-9}{4}$ (4) $\frac{11}{4}$

36. A polynomial f(x) leaves remainders 10 and 14 respectively when divided by (x – 3) and (x – 5). Find the remainder when f(x) is divided by (x – 3) (x – 5).

(1) 2x + 6 (2) 2x – 4 (3) 2x + 4 (4) 2x – 6

37. If f(x + 3) = x^2 – 7x + 2, then find the remainder when f(x) is divided by (x + 1).

(1) 8 (2) – 4 (3) 20 (4) 46

38. A polynomial f(x) when divided by (x – 5) and (x – 7) leaves remainders 6 and 16 respectively. Find the remainder when f(x) is divided by (x – 5) (x – 7).

(1) 5x + 7 (2) 5x – 7 (3) 5x + 19 (4) 5x – 19

39. A polynomial p(x) leaves remainders 75 and 15 respectively, when divided by (x – 1) and (x + 2). Then the remainder when f(x) is divided by (x – 1) (x + 2) is ______.

(1) 5(4x + 11) (2) 5(4x – 11) (3) 5(3x + 11) (4) 5(3x – 11)

40. The leading coefficient of a polynomial f(x) of degree 3 is 2006. Suppose that f(1) = 5, f(2) = 7 and f(3) = 9. Then find f(x).

(1) 2006 (x – 1) (x – 2) (x – 3) + 2x + 3
(2) 2006 (x – 1) (x – 2) (x – 3) + 2x + 1
(3) 2006 (x – 1) (x – 2) (x – 3) + 2x – 1
(4) 2006 (x – 2) (x – 3) (x – 1) – (2x – 3)

41. The ratio of the remainders when the expression x^2 + ax + b is divided by (x – 2) and (x – 1) respectively is 4 : 3. Find a and b if (x + 1) is a factor of the expression.

(1) 9, –10 (2) –9, 10 (3) 9, 10 (4) –9, –10

42. If $x^3 - ax^2 + bx - 6$ is exactly divisible by $x^2 - 5x + 6$, then $\frac{a}{b}$ is ______.

(1) $\frac{6}{11}$ (2) $\frac{-6}{11}$ (3) $\frac{1}{3}$ (4) $-\frac{1}{3}$

43. If f(x) = x^2 + 5x + a and g(x) = x^2 + 6x + b have a common factor, then which of the following is true?

(1) $(a-b)^2 + 5(a-b) + b = 0$
(2) $(a+b)^2 + 5(a+b) + a = 0$
(3) $(a+b)^2 + 6(a+b) + b = 0$
(4) $(a-b)^2 + 6(a-b) + b = 0$

44. If $ax^4 + bx^3 + cx^2 + dx$ is exactly divisible by $x^2 - 4$, then $\frac{a}{c}$ is ______.

(1) $\frac{1}{4}$ (2) $\frac{-1}{4}$ (3) $\frac{-1}{8}$ (4) $\frac{1}{8}$

45. If $x^2 + x + 1$ is a factor of $x^4 + ax^2 + b$, then the values of a and b respectively are

(1) 2, 4 (2) 2, 1 (3) 1, 1 (4) None of these

Concept Application Level—3

46. Find the remainder when x^{33} is divided by $x^2 - 3x - 4$.

(1) $\left(\dfrac{4^{33}-1}{5}\right)x + \left(\dfrac{4^{33}-4}{5}\right)$ (2) $\left(\dfrac{4^{33}+1}{5}\right)x + \left(\dfrac{4^{33}-4}{5}\right)$

(3) $\left(\dfrac{4^{33}-4}{5}\right)x + \left(\dfrac{4^{33}+1}{5}\right)$ (4) $\left(\dfrac{4^{33}+4}{5}\right)x + \left(\dfrac{4^{33}-1}{5}\right)$

47. If $6x^2 - 3x - 1$ is a factor of $ax^3 + bx - 1$ (where a, b are integers), then find the value of b.

(1) 1 (2) 3 (3) –5 (4) –7

48. If the polynomials $f(x) = x^2 + 6x + p$ and $g(x) = x^2 + 7x + q$ have a common factor, then which of the following is true?

(1) $p^2 + q^2 + 2pq + 6p - 7q = 0$ (2) $p^2 + q^2 - 2pq + 7p - 6q = 0$

(3) $p^2 + q^2 - 2pq + 6p - 7q = 0$ (4) $p^2 + q^2 + 2pq + 7p - 6q = 0$

49. A polynomial of degree 2 in x, when divided by (x + 1), (x + 2) and (x + 3), leaves remainders 1, 4 and 3 respectively. Find the polynomial.

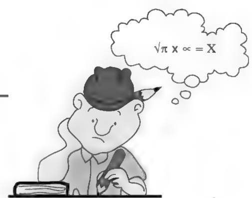
(1) $\dfrac{1}{2}(x^2 + 9x + 6)$ (2) $\dfrac{1}{2}(x^2 - 9x + 6)$ (3) $\dfrac{-1}{2}(x^2 - 9x + 6)$ (4) $\dfrac{-1}{2}(x^2 + 9x + 6)$

50. When a third degree polynomial f(x) is divided by (x – 3), the quotient is Q(x) and the remainder is zero. Also when f(x) is divided by [Q(x) + x + 1], the quotient is (x – 4) and remainder is R(x). Find the remainder R(x).

(1) $Q(x) + 3x + 4 + x^2$ (2) $Q(x) + 4x + 4 - x^2$

(3) $Q(x) + 3x + 4 - x^2$ (4) Cannot be determined

KEY



Very short answer type questions

1. $a_1 + a_3 + a_5 + \ldots\ldots\ldots = a_0 + a_2 + a_4 + \ldots\ldots\ldots\ldots$

2. f (–a)

3. $n \in N$

4. – 6

5. $a = 0$ or $a^2 + b + 1 = 0$

6. odd

7. 3

8. $a^2 + b^2$

9. – 4

10. True

11. x^{15}

12. x + 4

13. –5

14. 4

15. False

16. 25

17. $\dfrac{5}{3}$

18. -10 **19.** 1

20. 1 and 3 **21.** $\frac{2}{5}$

22. $(a^2 + 3p)x + q$. **23.** 41

24. -155 **25.** True

26. $9a+6b+4c = 0$ **27.** $(x + 4)$.

28. $x + 2$ **29.** even number

30. 4

Short answer type questions

31. $m = 11$ and $n = -2$

32. $-6x$

34. $5(x-1)$

35. $\frac{1}{2}(3^{555}-1)x+\frac{3}{2}(1-3^{554})$

36. $a = c$ **37.** -8

38. $\left(x + \frac{b_1 - b_2}{a_1 - a_2}\right)$ **39.** -5

40. 42 **41.** $x + y$

42. 63 **43.** $5x - 8$

44. 0 **45.** 13

Essay type questions

46. $(x - 1)(x + 1)(x + 2)(x - 4)$.

47. $\left(\frac{3^{29}+1}{4}\right)x+\left(\frac{3^{29}-3}{4}\right)$

48. -3

49. 1, 1

50. $m = 0, n = -1/3$

key points for selected questions

Very short answer type questions

16. Put $x = 1$ in $x^2 + 13x + 11$, the result obtained is the required remainder.

17. Substitute $a = \frac{1}{2}$ in $a^2 - \frac{a}{6}+\frac{3}{2}$ and then simplify.

18. (i) Let $f(x) = 7x^2 - 11x + a$.
(ii) Use $f(-1) = 8$ and solve for a.

19. Use, $f(-2) = 0$ and solve for k.

20. (i) Consider given polynomial as f(x)
(ii) Use, $f(a) = 0$ and solve for a

21. (i) Consider the given polynomial as f(x).
(ii) Use, $f(a) = 0$ and solve for a.

22. (i) Let $f(x) = x^3 + 3px + q$ and divisor is $x^2 - a^2$.
(ii) By division rule, $f(x) = Q(x)(x^2 - a^2) + (\ell x + n)$ where $(\ell x + n)$ is the required remainder.
(iii) Put $x = a$ and $x = -a$, and frame the equations.
(iv) Solve the equations to get p and q.

23. Use remainder theorem

24. Use remainder theorem

25. Use factor theorem

26. Use factor theorem

27. (i) Let $f(x) = x^3 + 9x^2 + 26x + 24$.
(ii) Factor is $(x^2 + 5x + 6)$ i.e., $(x + 2)(x + 3)$.

(iii) By using Horner's method, get coefficients of the quotient when f(x) is divided by (x + 2).
(iv) Again get coefficients of the new quotient when the previous quotient is divided by (x + 3).
(v) If the coefficients are a and b, then the remaining factor is (ax + b).

28. Find p using factor theorem

29. Use factor theorem

30. Use factor theorem

Short answer type questions

31. (i) Let $f(x) = 2x^4 - 11x^3 + mx + n$.
(ii) Divisor is $x^2 - 1$ i.e., $(x + 1)(x - 1)$.
(iii) Use, $f(1) = 0$ and $f(-1) = 0$ and frame equations in m and n.
(iv) Then solve the equations for m and n.

32. (i) Let $f(x) = ax + b$ and divisors are $(2x + 1)$ and $(3x + 2)$.
(ii) By the remainder theorem, $f\left(-\frac{1}{2}\right) = 3$, $f\left(-\frac{2}{3}\right) = 4$.
(iii) Solve two equations for a and b.

33. (i) Let $f(x) = (x^m)^n - 1$.
(ii) As divisor is $x^m - 1$, remainder is $f(1) = 0$.

34. (i) Let $f(x) = x^2 + ax + b$
(ii) Given $f(3) = 10$ and $f(4) = 15$
(iii) Then assume f(x) as $f(x) = Q(x)(x - 3)(x - 4) + (\ell x + n)$
(iv) Put x = 3 and x = 4, then get two equations in ℓ and n.
(v) Then solve the equations.

35. (i) Let $f(x) = x^{555}$
(ii) By division rule, $f(x) = Q(x)(x^2 - 4x + 3) + (ax + b)$.
(iii) Factorize $(x^2 - 4x + 3)$.
(iv) Then substitute the zeroes of the factors in the equation which is mentioned in step (ii).
(v) Then solve two equations for a and b.

38. (i) Let $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_1$.
(ii) Let $(x - k)$ be the common factor of f(x) and g(x).
(iii) Then equate f(k) and g(k) to get k.

40. Use remainder theorem.

41. $x^n + y^n$ is divisible by x + y, if n is odd.

42. Use factor theorem

44. Use remainder theorem

Essay type questions

46. (i) Let $f(x) = x^4 - 9x^2 + 2x + 8$
(ii) As sum of the coefficients of f(x) is zero, $(x - 1)$ in a factor of f(x)
(iii) As sum of the coefficient even powers of x is equal to the odd power of x., $(x + 1)$ is also a factor of f(x).
(iv) Then apply Horner's method to get the remaining factors.

49. Use factor theorem.

50. (i) $f(1) = 0$, $f(-1) = 0$ and $f(-2) = 1$.
(ii) Solve for ℓ, m and n.

KEY

Concept Application Level—1,2,3

1. 4	**2.** 2	**9.** 1	**10.** 3
3. 3	**4.** 1	**11.** 2	**12.** 2
5. 2	**6.** 4	**13.** 1	**14.** 1
7. 4	**8.** 4	**15.** 1	**16.** 3
		17. 1	**18.** 3

19. 4 20. 2

21. 3 22. 1

23. 1 24. 1

25. 4 26. 4

27. 2 28. 3

29. 3 30. 3

31. 2 32. 1

33. 3 34. 4

35. 1 36. 3

37. 4 38. 1

39. 1 40. 1

41. 4 42. 1

43. 4 44. 2

45. 3 46. 2

47. 3 48. 2

49. 4 50. 3

Concept Application Level—1,2,3

Key points for select questions

1. Use factor theorem.
2. Use remainder theorem.
3. Use remainder theorem.
4. Use remainder theorem.
5. The greatest possible value of n is the HCF of 1278 and 672.
6. Use remainder theorem.
7. Use factor theorem.
8. Use remainder theorem
9. Use factor theorem.
10. $5x^{20} + 7x^{15} + x^9 = x^9(5x^{11} + 7x^6 + 1)$
11. Since the coefficient of x^2 is zero, the sum of the roots is zero.
12. Use factor theorem.
13. Largest possible value of n is the HCF of 1215 and 945.

17. Use division algorithm.

18. Use division algorithm.

19. $(x + 1)$ and $(x - 1)$ are the factors of the given expression.

20. $P(1) = 6, P(-1) = 10$ and $P(0) = 5$.

21. Use division algorithm.

22. $f(0) = 7, f(2) = 9$ and $f(-3) = 49$.

23. Put $x = x - 1$ in $f(x + 1)$ to get $f(x)$.
    (i) Write $2x^2 + 7x + 5$ in terms of $x + 1$.
    (ii) Replace $x + 1$ by x.
    (iii) Apply remainder theorem.

24. (i) $x^2 + px + q = (x - p)(x - q)$.
    (ii) Compare the terms in L.H.S and R.H.S

25. (i) $f\left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)^2 + 2$.
    (ii) Replace $\left(x - \frac{1}{x}\right)$ with x.
    (iii) Use remainder theorem to obtain remainder.

26. (i) Since the coefficient of x^2 is 0, the sum of the roots is '0'. ⇒ Third root is -4.
    (ii) Apply remainder theorem for $f(x) = (x - 2)^2 (x + 4)$.

27. (i) Let $f(x) = ax^2 + bx + c$. $f(1) = 4; f(2) = 4; f(3) = 0$
    (ii) Solve for a, b, and c.

28. (i) Put $x = x - 3$ in $f(x + 3)$ to get $f(x)$.
    (ii) Apply factor theorem.

29. (i) Coefficient of x^2 is 0, therefore sum of roots is 0.
    $\therefore$ Third root $= -2$.
    (ii) Apply factor theorem.
    (iii) To obtain the remainder, use the remainder theorem.

30. (i) Take the given polynomial as $f(x)$.
    (ii) $f(1) = 0, f(2) = 0$.

31. $\frac{f(3)}{f(2)} = \frac{4}{5}$ and $f(1) = 0$.

32. (i) Let the common factor be $x - a$ and find $f(a)$, and $g(a)$.
    (ii) Obtain the value of a in terms of k and ℓ.

33. Dividend = Divisor × Quotient + Remainder.

34. (i) $x^2 + mx - n = (x - m)(x - n)$.
    (ii) Equate the corresponding terms.

35. (i) $f\left(x+\frac{1}{x}\right) = \left(x+\frac{1}{x}\right)^2 - 2.$

(ii) Replace $x + \frac{1}{x}$ by x.

(iii) Put $x = 1/2$.

36. (i) $f(3) = 10, f(5) = 14$

(ii) Dividend = Divisor × Quotient + Remainder.

39. (i) $f(1) = 75, f(-2) = 15$.

(ii) Dividend = Divisor × Quotient + Remainder.

40. Verify from the options whether $f(1) = 5$, $f(2) = 7$ and $f(3) = 9$ by using remainder theorem.

41. $\frac{f(2)}{f(1)} = \frac{4}{0}$ and $f(-1) = 0$.

42. (i) $x^2 - 5x + 6 = (x - 2)(x - 3)$

(ii) $f(2) = 0, f(3) = 0$.

43. (i) Let the common factor be $(x - a)$, then $f(a) = g(a)$, obtain value of 'a'.

(ii) Substitute value of 'a' in f(x).

44. $f(2) = 0$ and $f(-2)= 0$.

45. $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$.

48. (i) Let the common factor be $(x - a)$, then make $f(a) = g(a)$, and get the value of 'a'.

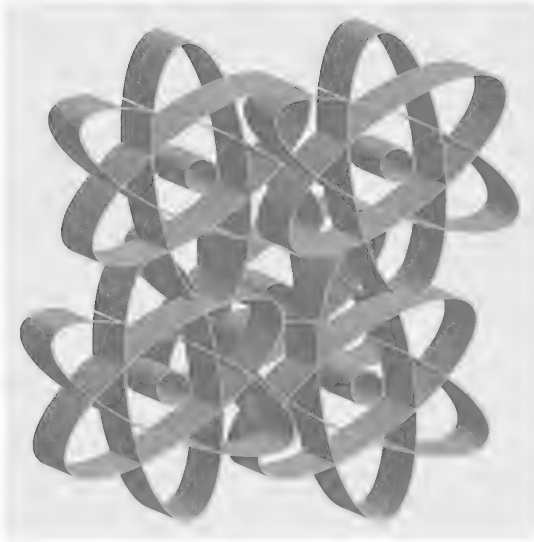
(ii) Substitute value of 'a' in f(x).

49. Let $f(x) = ax^2 + bx + c$, given $f(-1) = 1$, $f(-2) = 4$ and $f(-3) = 3$.

50. Dividend = Divisor × Quotient + Remainder.

CHAPTER 12

Statistics



INTRODUCTION

The word 'statistics' is derived from the Latin word 'status' which means political state.

Political states had to collect information about their citizens to facilitate Governance and plan for the development. Then, in course of time, statistics came to mean a branch of mathematics which deals with the collection, classification, presentation and analysis of numerical data.

In this chapter, we shall learn about the classification of data viz., Grouped Data and Ungrouped Data, Measures of central tendency and their properties.

Data

The word **data** means information in the form of numerical figures or a set of given facts.

For example, the percentage of marks scored by 10 pupils of a class in a test are:

36, 80, 65, 75, 94, 48, 12, 64, 88 and 98.

The set of these figures is the data related to the marks obtained by 10 pupils in a class test.

Types of data

Statistics is basically the study of numerical data. It includes methods of collection, classification, presentation, analysis of data and inferences from data. Data as such can be qualitative or quantitative in nature. If one speaks of honesty, beauty, colour etc., the data is qualitative while height, weight, distance, marks etc., are quantitative. Data can also be classified as raw data and grouped data.

(a) Raw data

Data obtained from direct observation is called raw data.

The marks obtained by 10 students in a monthly test is an example of raw data or ungrouped data.

In fact, very little can be inferred from this data. So, to make this data clearer and more meaningful, we group it into ordered intervals.

(b) Grouped data

To present the data in a more meaningful way, we condense the data into convenient number of classes or groups, generally not exceeding 10 and not less than 5. This helps us in perceiving at a glance, certain salient features of data.

Some basic definitions

Before getting into the details of tabular representation of data, let us review some basic definitions:

(i) **Observation**
Each numerical figure in a data is called an observation.

(ii) **Frequency**
The number of times a particular observation occurs is called its frequency.

Tabulation or presentation of data

A systematical arrangement of the data in a tabular form is called tabulation or presentation of the data. This grouping results in a table called the frequency table which indicates the number of scores within each group. Many conclusions about the characteristics of the data, the behaviour of variables etc., can be drawn from this table.

The quantitative data that is to be analysed statistically can be divided into three categories:

(i) Individual series
(ii) Discrete series and
(iii) Continuous series

(i) **Individual series**
Any raw data that is collected forms an individual series.

☛ *Examples*

(i) The weights of 5 students:

32, 40, 65, 48 and 54 (in kg)

(ii) Percentage marks obtained by 10 students in a test:

48, 59, 63, 72, 48, 72, 84, 98, 90 and 60

(ii) **Discrete series**
A discrete series is formulated from raw data by taking the frequency of the observations into consideration.

☛ *Example*

Given below is the data showing the number of computers in 12 families of a locality:

1, 1, 2, 3, 2, 1, 4, 3, 2, 2, 1, 1

Arranging the data in the ascending order:

1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4

To count, we can use tallymarks. We record tallymarks in bunches of five, the fifth one crossing the other four diagonally i.e., ~~IIII~~

Thus, we may prepare a frequency table as below.

[redacted]		
1	~~IIII~~	5
2	IIII	4
3	II	2
4	I	1

(iii) **Continuous series**

When the data contains large number of observations, we put them into different groups called class intervals such as 1–10, 11–20, 21–30, etc.

Here, 1 – 10 means data whose values lie between 1 and 10 including both 1 and 10.

This form is known as **inclusive form**. Also, 1 is called the **lower limit** and 10 is called the **upper limit**.

Example

Given below are the marks (out of 50) obtained by 30 students in an examination.

43	19	25	32	48
17	29	9	15	50
7	24	20	37	44
22	2	50	27	25
18	42	16	1	33
25	35	45	35	28

Taking class intervals 1–10, 11–20, 21–30, 31–40 and 41–50, we construct a frequency distribution table for the above data.

First, we write the marks in the ascending order as:

1	2	7	9	15	16	17	18	19	20
22	24	25	25	25	27	28	29	32	33
35	35	37	42	43	44	45	48	50	50

Now, we can prepare the frequency distribution table as below.

[redacted]		
1 – 10	IIII	4
11 – 20	IIII I	6
21 – 30	IIII III	8
31 – 40	IIII	5
41 – 50	IIII II	7

Now, with this idea, let us review some more concepts, about tabulation.

Class interval

A group into which the raw data is condensed is called a class-interval.

Each class is bounded by two figures, which are called the class limits. The figure on the L.H.S. is called the lower limit and the figure on the R.H.S. is called the upper limit of the class. Thus 0–10 is a class with lower limit being 0 and the upper limit being 10.

Class boundaries

In an exclusive form, the lower and upper limits are known as class boundaries or true lower limit and true upper limit of the class respectively.

Thus, the boundaries of 15 – 25 in exclusive form are 15 and 25.

The boundaries in an inclusive form are obtained by subtracting 0.5 to the lower limit and adding 0.5 to the upper limit.

Thus, the boundaries of 15 – 25 in the inclusive form are 14.5 – 25.5.

Class size

The difference between the true upper limit and the true lower limit is called the class size.
Hence, in the above example, the class size = 25 – 15 = 10.

Class mark or mid-value

Class mark = $\frac{1}{2}$ (upper limit + lower limit)

Thus, the class mark of 15 – 25 is $\frac{1}{2}(25 + 15) = 20$.

Statistical graphs

The information provided by a numerical frequency distribution is easily understood when represented by diagrams or graphs. The diagrams act as visual aids and leave a lasting impression on the mind. This enables the investigator to make quick conclusions about the distribution.

There are different types of graphs or diagrams to represent statistical data. Some of them are:

1. Bar chart or bar graph (for unclassified frequency distribution)
2. Histogram (for classified frequency distribution)
3. Frequency polygon (for classified frequency distribution)
4. Frequency curve (for classified frequency distribution)
5. Cumulative frequency curve (for classified frequency distribution).
   (a) Less than cumulative frequency curve.
   (b) Greater than cumulative frequency curve.

Bar graph

The important features of bar graphs are:

1. Bar graphs are used to represent unclassified frequency distributions.
2. Frequency of a value of a variable is represented by a bar (rectangle) whose length (i.e., height) is equal (or proportional) to the frequency.

3. The breadth of the bar is arbitrary and the breadth of all the bars are equal. The bars may or may not touch each other.

Example

Represent the following frequency distribution by a bar graph:

Value of variable	2	4	6	8	10
Frequency	5	8	4	2	7

Solution

Either of the following bar graphs (fig a or fig b) may be used to represent the above frequency distribution. The first graph takes value of the variable along the x-axis and the frequency along the y-axis, whereas the second one takes the frequency along the x-axis and the value of the variable on the y-axis.

All the rectangles (bars) should be of same width and uniform spaces should be left between any two consecutive bars.

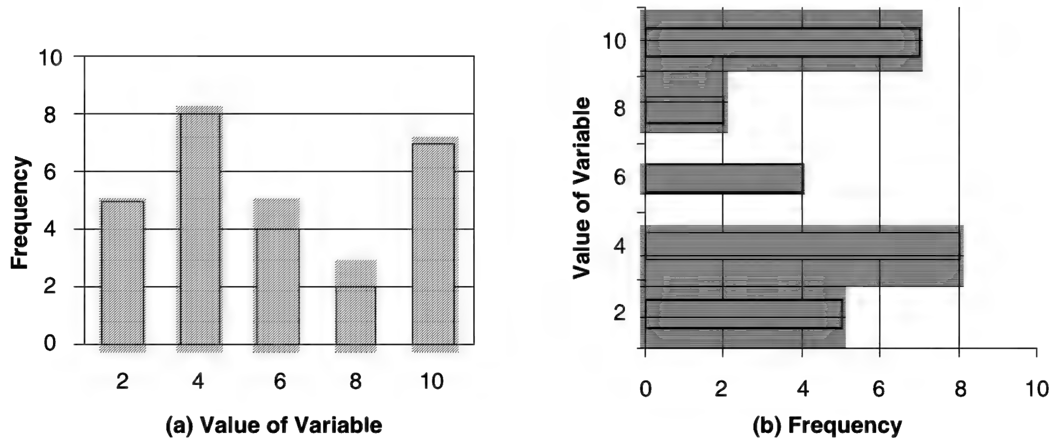


Figure 12.1

Histograms

Classified or grouped data is represented graphically by histograms. A histogram consists of rectangles each of which has its breadth proportional to the size of concerned class interval and its height proportional to the corresponding frequency. In a histogram, two consecutive rectangles have a common side.

Hence, in a histogram, we do the following:

(i) We represent class boundaries along the x-axis.
(ii) Along the y-axis, we represent class frequencies
(iii) We construct rectangles with bases along the X-axis and heights along the Y-axis.

Example

Construct a histogram for the frequency distribution below:

Class Interval	20-30	30-40	40-50	50-60	60-70
Frequency	5	8	3	7	4

Solution

Here, the class intervals are continuous.

The following histogram is drawn according to the method described above.

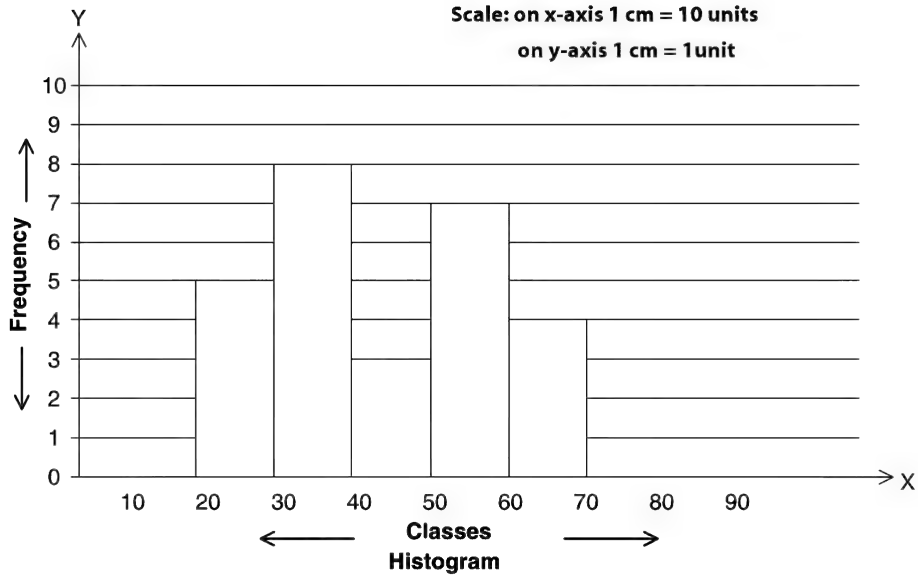


Figure 12.2

Remarks

The following points may be noted.

1. A link mark (—/\/—) made on the horizontal axis, between the vertical axis and first vertical rectangle, if there is a gap between 0 and the lower boundary of first class interval.
2. We may shade all rectangles. A heading for the histogram may also be given.

Important observations

1. If the class intervals are discontinuous, the distribution has to be changed into continuous intervals and then the histogram has to be drawn.
2. Bar graphs are used for unclassified frequency distributions, where as histograms are used for classified frequency distribution. The breadths of rectangles in a bar graph are arbitary, while those in histogram are determined by class size.

Frequency polygon

Frequency polygons are used to represent classified or grouped data graphically. It is a polygon whose vertices are the midpoints of the top sides of the rectangles, forming the histogram of the frequency distribution.

To draw a frequency polygon for a given frequency distribution, the mid-values of the class intervals are taken on X-axis and the corresponding frequencies on Y-axis and the points are plotted on a graph sheet. These points are joined by straight line segments which form the frequency polygon.

Example

Construct a frequency polygon for the following data:

Class Interval	12–17	18–23	24–29	30–35	36–41	Total
Frequency	10	7	12	8	13	50

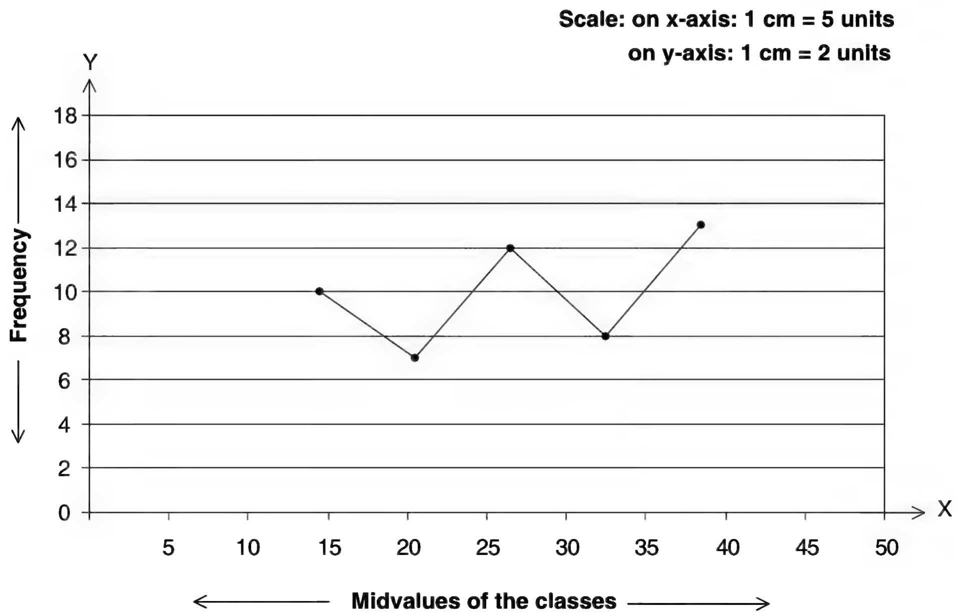
Solution

Here the class intervals are discontinuous.

Hence, first we convert the class intervals to continuous class intervals and then find mid-points of each class intervals. We do this by adding 0.5 to each upper limit and subtracting 0.5 from each lower limit.

[redacted]			
12–17	11·5–17·5	14·5	10
18–23	17·5–23·5	20·5	7
24–29	23·5–29·5	26·5	12
30–35	29·5–35·5	32·5	8
36–41	35·5–41·5	38·5	13

Now, taking the mid-values of class intervals on the X-axis and the corresponding frequencies on the Y-axis, we draw a frequency polygon as below.



Frequency Polygon

Figure 12.3

Frequency curve

Frequency curves are used to represent classified or grouped data graphically.

As the class-interval in a frequency distribution decreases, the points of the frequency polygon become closer and closer and then the frequency polygon tends to become a frequency curve. So, when the number of scores in the data is sufficiently large and the class-intervals become smaller (ultimately tending to zero), the limiting form of frequency polygon becomes frequency curve.

Example

Draw a frequency curve for the data given below

Mid-values	5	10	15	20	25	30	35	40
Frequency	2	4	7	5	10	12	6	4

Solution

For the given data; a table, showing the mid-values of classes and frequencies is made.

Now, taking the mid-values of the classes along the X-axis and the corresponding frequency along the Y-axis, we mark the points obtained from the above table in a graph sheet and join them with a smooth curve, which gives the frequency curve as shown below.

[redacted]	
5	2
10	4
15	7
20	5
25	10
30	12
35	6
40	4

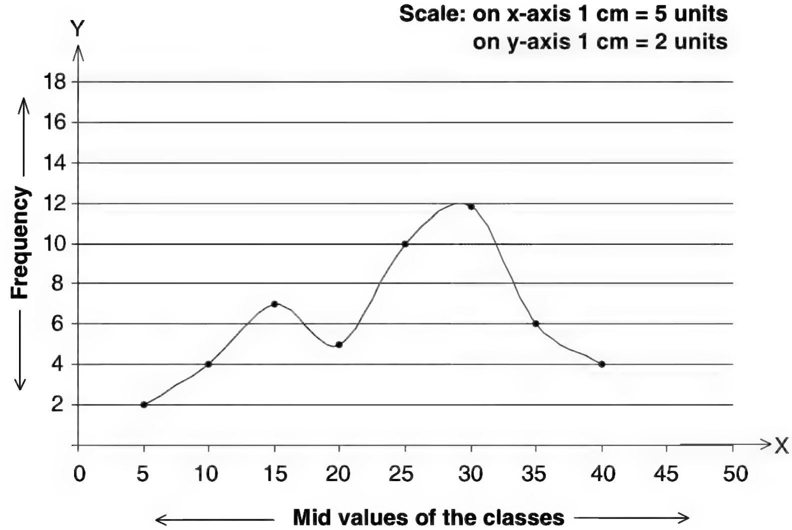


Figure 12.4

Cumulative frequency curves

The curves drawn for cumulative frequencies, less than or more than the true limits of the classes of a frequency distribution are called cumulative frequency curves. The curve drawn for the "less than cumulative

frequency distribution" is called the "less than cumulative frequency curve" and the curve drawn for the greater than cumulative frequency distribution is called the "greater than cumulative frequency curve".

From these curves, we can find the total frequency above or below a particular value of the variable.

Example

For the given distribution, draw the less than and greater than cumulative frequency curves.

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	2	4	5	7	17	12	6	4	3

Solution

Less than cumulative frequency distribution:

[redacted]		
20	2	2
30	4	6
40	5	11
50	7	18
60	17	35
70	12	47
80	6	53
90	4	57
100	3	60

Greater than cumulative frequency distribution:

[redacted]		
10	2	60
20	4	58
30	5	54
40	7	49
50	17	42
60	12	25
70	6	13
80	4	7
90	3	3

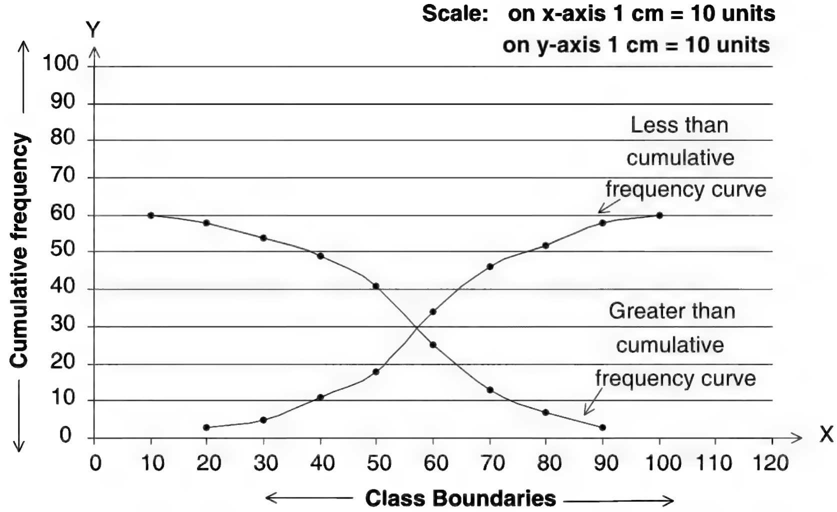


Figure 12.5

Measures of central tendencies for ungrouped data

Till now, we have seen that the data collected in statistical enquiry or investigation is in the form of raw data. If the data is very large, the user cannot get much information from such data. For this reason, the data is grouped together to obtain some conclusions.

The measure of central tendency is a value which represents the total data i.e., it is the value in a data around which the values of all the other observations tend to concentrate.

The most commonly used measures of central tendency are:

1. Arithmetic mean
2. Median
3. Mode

These measures give an idea about how the data is clustered or concentrated.

Arithmetic mean or mean (A.M)

The arithmetic mean (or simply the mean) is the most commonly used measure of central tendency.

(i) **Arithmetic mean for raw data**

The arithmetic mean of a statistical data is defined as the quotient obtained when the sum of all the observations or entries is divided by the total number of items.

If $x_1, x_{2,\ldots.} x_n$ are the n items, then

$$\text{A.M.} = \frac{x_1 + x_2 + \ldots.. + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \text{ or briefly } \frac{\sum x}{n}$$

A.M. is usually denoted by $\overline{x}$

Example

Find the mean of the first 10 natural numbers.

Solution

Given data is 1, 2, 3, . . . 10

$\therefore$ Arithmetic mean (A.M.) $= \dfrac{\text{sum of observations}}{\text{total number of observations}} = \dfrac{1+2+3+\ldots+10}{10} = \dfrac{55}{10} = 5.5$

(ii) **Mean of discrete series**

Let $x_1, x_2, x_3 \ldots x_n$ be n observations with respective frequencies $f_1, f_2, \ldots f_n$.

This can be considered as a special case of raw data where the observation x_1 occurs f_1 times; x_2 occurs f_2 times, and so on.

$\therefore$ The mean of the above data $= \dfrac{f_1x_1 + f_2x_2 + \ldots + f_nx_n.}{f_1 + f_2 + \ldots + f_n}$

It can also be represented by $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$

Weighted arithmetic mean

When the variables $x_1, x_2, \ldots\ldots\ldots, x_n$ do not have same importance, and the weights $w_1, w_2, \ldots, w_n$ are given to each of the variables, the weighted arithmetic is given by $\bar{X}_w = \dfrac{\sum x_i w_i}{\sum w_i}$.

Example

The salaries of 100 workers of a factory are given below.

[redacted]	
6000	40
8000	25
10000	12
12000	10
15000	8
20000	4
25000	1
Total	100

Find the mean salaries of workers of the factory.

Solution

The mean $\bar{x}$ is given by

$$\bar{x} = \frac{(6000 \times 40) + (8000 \times 25) + (10000 \times 12) + (12000 \times 10) + (15000 \times 8) + (20000 \times 4) + (25000 \times 1)}{40+25+12+10+8+4+1}$$

$\Rightarrow \bar{x} = \text{Rs } 9050$

i.e., The mean salary of the workers is Rs 9050.

Some important results about A.M.

1. The algebraic sum of deviations taken about the mean is zero. i.e., $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$
2. The value of the mean depends on all the observations.
3. The A.M. of two numbers a and b is $\frac{a+b}{2}$
4. Combined mean: If $\bar{x}_1$ and $\bar{x}_2$ are the arithmetic Means of two series with n1 and n2 observations respectively, then the combined mean is:

   $$\bar{x}_c = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

   The above result can be extended to any number of groups of data.
5. If $\bar{x}$ is the mean of x1, x2, . . . xn, then the mean of x1 + a, x2 + a, x3 + a, . . . xn + a is $\bar{x}$ + a, for all values of a.
6. If $\bar{x}$ is the mean of x1, x2, . . . xn, then the mean of ax1, ax2, . . . axn is a $\bar{x}$ and that of $\frac{x_1}{a}, \frac{x_2}{a}, \ldots \frac{x_n}{a}$ is $\frac{\bar{x}}{a}$
7. The mean of the first *n* natural numbers is $\left(\frac{n+1}{2}\right)$
8. The mean of the squares of the first *n* natural numbers $= \frac{(n+1)(2n+1)}{6}$
9. The mean of the cubes of the first *n* natural numbers $= \frac{n(n+1)^2}{4}$

Median

Another measure of the central tendency of a given data is the median.

Definition

If the values x_i in the raw data are arranged either in the increasing or decreasing order of magnitude, then the middle-most value in this arrangement is called the median.

Thus, for the raw (ungrouped) data, the median is computed as follows:

(i) The values of the observations are arranged in the order of magnitude.

(ii) The middle most value is taken as the median. Hence, depending on the number of observations (odd or even), we determine median as follows.

(a) When the number of observations(n) is odd, then the median is the value of $\left(\frac{n+1}{2}\right)$ th observation.

(b) If the number of observations (n) is even, then the median is the mean of $\left(\frac{n}{2}\right)$ th observation and $\left(\frac{n}{2}+1\right)$ th observation.

Example

Find the median of the following data: 2, 7, 3, 15, 12, 17 and 5

Solution

Arranging the given numbers in the ascending order, we have 2, 3, 5, 7, 12, 15, 17

Here, middle term is 7

$\therefore$ Median = 7.

Example

Find the median of the data 5, 8, 4, 12, 16 and 10.

Solution

Arranging the given data in ascending order we have 4, 5, 8, 10, 12, 16.

As the given number of values is even, we have two middle values they are $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations. They are 8 and 10.

$\therefore$ Median of the data = Average of 8 and 10.

$= \frac{8+10}{2} = 9$

Some important facts about median

1. The median does not take into consideration all the items.
2. The sum of absolute deviations taken about the median is the least.
3. The median can be calculated graphically while the mean cannot be.
4. The median is not effected by extreme values.
5. The sum of deviations taken about median is less than the sum of absolute deviations taken from any other observation in the data.

Mode

The third measure of central tendency of a data is the mode.

The most frequently found value in the data is called the *mode.*

This is the measure which can be identified in the simplest way.

Example

Find the mode of 0, 5, 2, 7, 2, 1, 1, 3, 2, 4, 5, 7, 5 ,1 and 2.

Solution

Among the observations given, the most frequently found observation is 2. It occurs 4 times.

$\therefore$ Mode = 2

NOTE:

1. For a given data, the mode may or may not exist. In a series of observations, if no item occurs more than once, then the mode is said to be ill-defined.
2. If the mode exists for a given data, it may or may not be unique.
3. Data having unique mode is uni-model while data having two modes is bi-model.

Properties of mode

1. It can be calculated graphically.
2. It is not effected by extreme values.
3. It can be used for open-ended distribution and qualitative data.

Emperical relationship among mean, median and mode

For a moderately symmetric data, the above three measures of central tendency can be related by the formula, Mode = 3 Median – 2 Mean

Example

Find the mode when median is 12 and mean is 16 of a data.

Solution

Mode = 3 Median – 2 Mean

= (3 × 12) – (2 × 16) = 36 – 32 = 4

Observations

1. For a symmetric distribution,
   Mean = Median = Mode
2. Given any two of the mean, median and mode the third can be calculated.
3. This formula is to be applied in the absence of sufficient data.

Measure of central tendencies for grouped data

We studied the measure of central tendencies of ungrouped or raw data. Now we study the measures of central tendencies (mean, median and mode) for grouped data.

Mean of grouped data

If the frequency distribution of 'n' observations of a variable x has k classes, x_i is the mid-value and f_i is the frequency of ith class, then the mean $\bar{x}$ of grouped data is defined as

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \dots\dots\dots\dots + f_kx_k}{f_1 + f_2 + \dots\dots\dots + f_k} = \frac{\sum_{i=1}^{k} f_ix_i}{\sum_{i=1}^{k} f_i}$$

(or) simply, $\bar{x} = \frac{\sum f_ix_i}{N}$ where $N = \sum_{i=1}^{k} f_i$

In grouped data, it is assumed that the frequency of each class is concentrated at its midvalue.

Example

Calculate the arithmetic mean (A.M.) of the following data

Percentage of marks	0-20	20-40	40-60	60-80	80-100
No. of students	2	12	13	15	8

Solution

Let us write the tabular form as given below:

[redacted]			
0-20	2	10	20
20-40	12	30	360
40-60	13	50	650
60-80	15	70	1050
80-100	8	90	720
	$\Sigma f_i = N = 50$		$\Sigma f_i x_i = 2800$

$$\therefore \text{ Mean } = \bar{x} = \frac{\sum fx}{N} = \frac{2800}{50} = 56$$

Short-cut method for finding the mean of group data (deviation method)

Sometimes, when the frequencies are large in number, the calculation of mean using above formula is cumbersome. This can be simplified if the class interval of each class of grouped data is the same. Under the assumption of equal class interval, we get the following formula for the mean of grouped data.

$$\bar{x} = A + \frac{1}{N}\left(\sum_{i=1}^{k} f_i u_i\right) \times c$$

where, A = assumed value from among mid-values

C = length of class interval

K = number of classes of the frequency distribution

N = Sum of frequencies = $\sum_{i=1}^{k} f_i$

$u_i = \frac{x_i - A}{C}$, i = 1, 2, 3, ...k and

x_i = midvalue of the ith class

u_i is called the deviation or difference of the midvalue of the ith class from the assumed value, divided by the class interval.

Using this method the previous example can be worked out as follows:

[redacted]

0-20	2	10	−3	−6
20-40	12	30	−2	−24
40-60	13	50	−1	−13
60-80	15	70 (A)	0	0
80-100	8	90	1	8
	N= 50			$\Sigma f_i u_i = -35$

Here, A = 70; N = 50; C = 20; $\Sigma f_i u_i = -35$

$$\therefore \text{ A.M.} = A + \frac{1}{N}(\Sigma f_i u_i) \times c = 70 + \frac{1}{50}(-35) \times 20 = 70 - 14 = 56.$$

Median of grouped data

Before finding out how to obtain the median of grouped data, we first review what a median class is

If 'n' is the number of observations, then from the cumulative frequency distribution, the class in which $\left(\frac{n}{2}\right)^{th}$ observation lies is called the median class.

Formula for calculating median:

$$\text{Median (M)} = L + \frac{\frac{n}{2} - F}{f}(C)$$

Where, L = Lower boundary of median class

i.e., class in which $\left(\frac{n}{2}\right)$th observation lies.

N= Sum of frequencies

F = cumulative frequency of the class just preceding the median class.

f = frequency of median class.

C = length of class interval.

Example

Given below is the data showing weights of 40 students in a class. Find its median.

Weight	45	46	47	48	49	50	51	52	53
No. of students	6	2	3	4	7	4	7	4	3

Solution

To find the median, we prepare less than cumulative frequency table as given below.

[redacted]		
45	6	6
46	2	8
47	3	11
48	4	15
49	7	22
50	4	26
51	7	33
52	4	37
53	3	40

Here n = 40 which is even

∴ Median = value of 40/2 or the 20th observation

From the column of cumulative frequency, the value of 20th observation is 49.

∴ Median = 49 kg

NOTE: In the above example, we do not have any class interval. As there is no class interval, we can not use the formula.

Example

Find the median of the following data.

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	7	6	5	8	9

Solution

To find the median, we prepare the following table:

[redacted]		
0-10	7	7
10-20	6	13 (F)
20-30	5 (f)	18
30-40	8	26
40-50	9	35
Total	35	

Median $= L + \dfrac{\left(\dfrac{n}{2} - F\right)}{f} \times C$ Here n = 35 $\Rightarrow \dfrac{n}{2} = \dfrac{35}{2} = 17 \cdot 5$

This value appears in the class 20-30.

L = lower boundary of median class (20 – 30) = 20

F = 13; f = 5 C = 10 (class length)

$$\therefore \text{Median} = 20 + \frac{\left(\frac{35}{2} - 13\right)}{5} \times 10 = 20 + \frac{9}{2 \times 5} \times 10 = 29$$

Mode of grouped data

The formula for determining the mode of grouped data is $L_1 + \dfrac{\Delta_1 C}{\Delta_1 + \Delta_2}$

where L_1 = lower boundary of modal class (class with highest frequency)

$\Delta_1 = f - f_1$ and $\Delta_2 = f - f_2$ where f is the frequency of model class.

f_1 = frequency of previous class of the model class.

f_2 = frequency of next class of the model class.

Rewriting the formula,

$$\text{Mode} = L_1 + \frac{(f - f_1)C}{(f - f_1) + (f - f_2)}$$

$$\text{Mode} = L_1 + \frac{(f - f_1)C}{2f - (f_1 + f_2)}$$

Example

The following information gives the monthly salaries of 100 employees. Find the mode of the data.

Salary (Rs)	2000-3499	3500-4999	5000-6499	6500-7999	8000-9499
Number of Persons	35	37	21	12	5

Solution

Here the given classes are not continuous

Hence we first rewrite it as shown below.

2000–3499	1999·5–3499·5	35 (f_1)
3500–4999	3499·5–4999·5	37 (f)
5000–6499	4995·5–6499·5	29 (f_2)
6500–7999	6499·5–7999·5	12
8000–9499	7999·5–9499·5	5

From the above table it can be known that the maximum frequency occurs in the class interval 3500–4999.

$\therefore\ f = 37; f_1 = 35; f_2 = 29;$

$L_1 = 3499{\cdot}5; C = 1500$

$$\therefore \text{Mode} = L_1 + \frac{(f - f_1)C}{(f - f_1) + (f - f_2)} = 3499\cdot5 + \frac{1500(2)}{2+8} = 3499{\cdot}5 + 300 = 3799{\cdot}5$$

Range

The difference between the maximum and the minimum values of the given observations is called the range of the data.

Given $x_1, x_2 \ldots\ldots x_n$ (n individual observations)

Range = (Maximum value) – (Minimum Value)

Example

Find the range of $\{2, 7, 6, 4, 3, 8, 5, 12\}$.

Solution

Arranging the given data in the ascending order

We have; $\{2, 3, 4, 5, 6, 7, 8, 12\}$

$\therefore$ Range = (Maximum value) – (Minimum value) = 12 – 2 = 10.

Note: The range of the class interval is the difference of the actual limits of the class.

Calculation of variance and standard deviation for raw data

Standard deviation (S.D.) is reffered to as root mean squared deviation about the mean. It is denoted by the symbol σ and read as sigma.

Variance is denoted by σ^2 which is the square of standard deviation.

$\therefore$ Varience = $(\text{S.D.})^2$ or S.D. = $\sqrt{\text{varience}}$

$$\text{S.D. } (\sigma) = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots\ldots\ldots + (x_n - \bar{x})^2}{n}}$$

where $x_1, x_2, \ldots\ldots\ldots\ldots x_n$ are n observations with mean as $\bar{x}$.

Alternatively the above formula can also be written as

$$\text{S.D. } (\sigma) = \sqrt{\frac{\sum x_i^{\,2}}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

Example

Calculate variance and standard deviation of the following data:

10, 12, 8, 14, 16.

Solution

$$\text{A.M. } \bar{x} = \frac{10 + 12 + 8 + 14 + 16}{5} = \frac{60}{5} = 12$$

$$\text{Varience} = \frac{(10-12)^2 + (12-12)^2 + (8-12)^2 + (14-12)^2 + (16-12)^2}{5}$$

$$= \frac{4+0+16+4+16}{5} = \frac{40}{5} = 8$$

$$\text{S.D.}(\sigma) = \sqrt{\text{variance}} = \sqrt{8}$$

Calculation of variance and S.D. for a grouped data

1. N = Σf = The sum of the frequencies
2. A.M. $(\overline{x}) = \dfrac{\sum fx}{N}$
3. D = Deviation from the A.M. = $(x - \overline{x})$
4. Standard deviation (S.D.) = $\sigma = \sqrt{\dfrac{\sum f D^2}{N}}$

Examples

1. Calculate S.D. for the given data.

f	1	2	3	4
x	5	10	15	20

Solution

[redacted]					
1	5	5	–10	100	100
2	10	20	–5	25	50
3	15	45	0	0	0
4	20	80	5	25	100
$\Sigma f = 10$		$\Sigma fx = 150$			$\Sigma fD^2 = 250$

$$\text{A.M.}(\overline{x}) = \frac{\sum fx}{N} = \frac{150}{10} = 15$$

$$\text{S.D.} = (\sigma) = \sqrt{\frac{\sum fD^2}{N}} = \sqrt{250/10} = 5$$

2. Find S.D. for the given data.

C.I.	0–10	10–20	20–30	30–40
Frequency	4	3	2	1

[redacted]						
0–10	4	5	20	–10	100	400
10–20	3	15	45	0	0	0
20–30	2	25	50	10	100	200
30–40	1	35	35	20	400	400
	$\Sigma f = 10$		$\Sigma fx = 150$			$\Sigma fD^2 = 1000$

$$\text{A.M.} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{150}{10} = 15$$

$$\text{S.D.} = \sigma = \sqrt{\frac{\sum f.D^2}{N}} = \sqrt{\frac{1000}{10}} = \sqrt{100} = 10$$

Coefficient of variation (C.V.)

For any given data, let S.D. be the Standard Deviation and A.M. be the Arithmetic Mean, then Coefficient ofVariation (C.V.) of the data is defined as

$$\text{C.V.} = \frac{\text{S.D.}}{\text{A.M.}} \times 100$$

This is a relative measure and helps in measuring the consistency. Smaller the C.V., greater is the consistency.

Example

In a series of observations, find the coefficient of variation given S.D. = 12.5 and A.M. = 50.

Solution

$$\text{C.V.} = \frac{\text{S.D.}}{\text{A.M.}} \times 100 = \frac{12.5}{50} \times 100 = 25$$

$\therefore$ Coefficient of Variation = 25.

Quartiles

In a given data, the observations that divide the given set of observations into four equal parts are called quartiles.

First quartile or lower quartile

When the given observations are arranged in ascending order, the observation which lies midway between the lower extreme and the median is called the first quartile or the lower quartile and is denoted as Q_1.

Third quartile or upper quartile

Of the data when the given observations are arranged in ascending order the observation that lies in midway between the median and the upper extreme observation is called the third quartile or the upper quartile and is denoted by Q_3.

We can find Q_1 and Q_3 for an ungrouped data containing n observations as follows.

1. We arrange the given n observations or items in ascending order then,

   Lower or first quartile, Q_1 is $\left(\frac{n}{4}\right)$th item or observation if n is even and $\left(\frac{n+1}{4}\right)$th item or observation when n is odd.

Example

Find Q_1 for the following data 8, 12, 7, 5, 16, 10, 21 and 19.

Solution

Arranging the given observations in ascending order

We have 5, 7, 8, 10, 12, 16,19, 21

Here n = 8 (n is even)

$\therefore$ first quartile, $Q_1 = \left(\frac{n}{4}\right)$th item $= \left(\frac{8}{4}\right)$th item = 2nd item of the data i.e., 7.

$\therefore$ $Q_1 = 7$.

Example

Find Q_1 of the observations 21, 12, 9, 6, 18, 16 and 5.

Solution

Arranging the observations in ascending order, we have 5, 6, 9, 12, 16, 18, 21.

Here n = 7 (odd)

$\therefore$ $Q_1 = \left(\frac{n+1}{4}\right)$ th item i.e., $\left(\frac{7+1}{4}\right)$ th item

= 2nd item.

$\therefore$ $Q_1 = 6$.

Example

The marks of 10 students in class are 38, 24, 16, 40, 25, 27, 17, 32, 22 and 26. Find Q_1.

Solution

The given observations when arranged in ascending order

we have 16, 17, 22, 24, 25, 26, 27, 32, 38, 40

Here n = 10 (even)

$\therefore$ $Q_1 = \left(\frac{n}{4}\right)$th item $= \left(2\frac{1}{2}\right)$th item of the data

$\therefore\ Q_1 = \text{2nd item} + \frac{1}{2}\text{(3rd – 2nd) item} = 17+\frac{1}{2}(22-17) = 17+\frac{5}{2} = 19\cdot5$

$\therefore\ Q_1 = 19\cdot5$

Third quartile: $Q_3 = \left(\frac{3n}{4}\right)$th item, when n is even. $= 3\left(\frac{n+1}{4}\right)$th item when n is odd.

Example

Find Q_3 for the following data 7, 16, 19, 10, 21 and 12.

Solution

Arrange the data in ascending order we have 7, 10, 12, 16, 19, 21.
Here n = 6 (even)

$\therefore\ Q_3 = 3\left(\frac{n}{4}\right)\text{th item} = 4\frac{1}{2}\text{th item}$

$\Rightarrow Q_3 = \left[\text{4th} + \frac{1}{2}\text{(5th – 4th)}\right]\text{item} = 28+\frac{1}{2}(32-28)$

$Q_3 = 30.$

Semi-inter quartile range or quartile deviation (Q·D.)

Quartile deviation is given by the formula, Q.D. $= \frac{Q_3 - Q_1}{2}$

Example

Find semi-inter quartile range of the following data.

X	2	5	6	8	9	10	12
Frequency	1	8	12	16	11	9	3

Solution

[redacted]		
2	1	1
5	8	9
6	12	21
8	16	37
9	11	48
10	9	57
12	3	60
	N = 60	

Here N = 60

$\therefore\ Q_1 = \left(\dfrac{N}{4}\right)$th item $= \left(\dfrac{60}{4}\right)$th item $=$ 15th item

$\therefore\ Q_1 = 6$ (as 15th item lies in 21 in the cumulative frequency)

$Q_3 = 3\left(\dfrac{N}{4}\right)$th item = 45th item

$\therefore\ Q_3 = 9$ (as 45th item lies in 48 in the cumulative frequency)

Semi inter quartile range (Q.D.) $= \dfrac{Q_3 - Q_1}{2} = \dfrac{9-6}{2} = 1\cdot 5$

Note: For an individual data, the second quartile Q_2 coincides with median.

$\Rightarrow Q_2$ = Median of the data

Estimation of median and quartiles from ogive

1. Prepare the cumulative frequency table with the given data.
2. Draw ogive.
3. Let, total number of observations = sum of all frequencies = N.
4. Mark the points A, B and C on Y-axis, corresponding to $\dfrac{N}{4}, \dfrac{N}{2}$ and $\dfrac{3N}{4}$ respectively.
5. Mark three points (P, Q, R) on ogive corresponding to $\dfrac{3N}{4}, \dfrac{N}{2}$ and $\dfrac{N}{4}$ respectively.
6. Draw vertical lines from the points R, Q and P to meet x-axis Q_1, M and Q_3 respectively.
7. Then, the abscissas of Q_1, M and Q_3 gives lower quartile, median and upper quartile respectively.

Example

The following table shows the distribution of the weights of a group of students.

Weight in Kgs	30–35	35–40	40–45	45–50	50–55	55–60	60–65
No.of students	5	6	7	5	4	3	2

Solution

[redacted]

30–35	5	5
35–40	6	11
40–45	7	18
45–50	5	23
50–55	4	27
55–60	3	30
60–65	2	32
	N = 32	

N = 32

$\Rightarrow \frac{N}{4} = 8$, $\frac{N}{2} = 16$ and $\frac{3N}{4} = 24$

From the graph,

Lower quartile (Q_1) = 38

Upper quartile (Q_3) = 52

Median (M) = 44

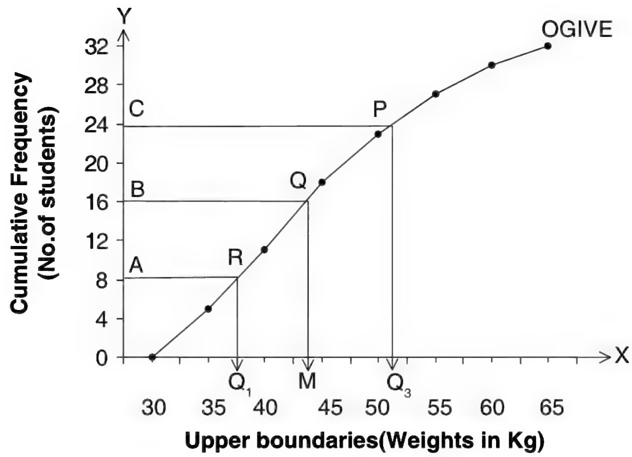


Figure 12.6

Estimation of mode from histogram

1. Draw a histogram to represent the given data.
2. From the upper corners of the highest rectangle, draw segments to meet the opposite corners of adjacent rectangles, diagonally as shown in the given example. Mark the intersecting point as P.
3. Draw PM perpendicualr to X-axis, to meet X-axis at M.
4. Abscissa of M gives the Mode of the data.

Example

Estimate mode of the following data from the histrogram.

C.I.	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	10	16	17	20	15	13	12

From the graph, Mode (M) = 34

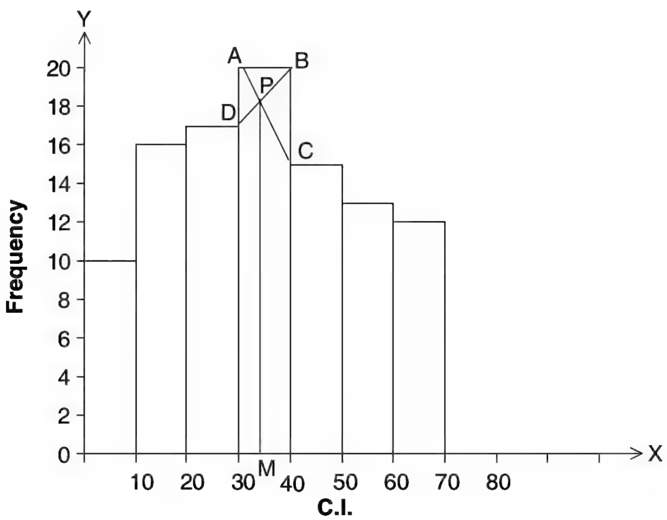


Figure 12.7

test your concepts

Very short answer type questions

1. The class mark of a class is 25 and if the upper limit of that class is 40, then its lower limit is _______.
2. Consider the data: 2, 3, 2, 4, 5, 6, 4, 2, 3, 3, 7, 8, 2, 2. The frequency of 2 is ________.
3. 1–5, 6–10, 11–15 ... are the classes of a distribution, the upper boundary of the class 1–5 is ________.
4. 0–10, 10–20, 20–30 ... are the classes, the lower boundary of the class 20–30 is ________.
5. The mid-value of 20–30 is _____.
6. If 1–5, 6–10, 11–15... are the classes of a frequency distribution then the size of the class is _____.
7. A class interval of a data has 15 as the lower limit and 25 as the size. Then the class mark is ________.
8. In a histogram, the _____ of all rectangles are equal. (width/length/area)
9. The sum of 12 observations is 600, then their mean is _____.
10. If the lower boundary of the class is 25 and the size of the class is 9 then the upper boundary of the same class _____.
11. If 1–5, 6–10, 11–15, 16–20... are the classes of a frequency distribution, then the lower boundary of the class 11–15 is _____.
12. Arithmetic mean of first n natural numbers is _____.
13. The width of a rectangle in a histrogram represents frequency of the class. (True/False)
14. If 16 observations are arranged in ascending order, then median is

    $\dfrac{(8\text{th observation} + 9\text{th observation})}{2}$. (True/False)
15. The mean of x, y, z is y, then $x + z = 2y$. (True/False)
16. Range of the scores 25, 33, 44, 26, 17 is _____.
17. Upper quartile of the data 4, 6, 7, 8, 9 is _____.
18. 2(Median – Mean) = Mode – Mean. (True/False)
19. Lower quartile of the data 5, 7, 8, 9, 10 is _____.
20. Consider the data; 2, x, 3, 4, 5, 2, 4, 6, 4 where $x > 2$. The mode of the data is ________.
21. Find the mean and the median of the data 10, 15, 17, 19, 20 and 21.
22. Find the semi-inter quartile range of the data 32, 33, 38, 39, 36, 37, 40, 41, 47, 34 and 49.
23. Find the mean of first 726 natural numbers.
24. Find the range of the data 14, 16, 20, 12, 13, 4, 5, 7, 29, 32 and 6.

25. Find the mean of the observations 425, 430, 435, 440, 445,…………., 495. (difference between any two consecutive given observation is equal)

26. The mean of 10 observations is 15·5. By an error, one observation is registered as 13 instead of 34. Find the actual mean.

27. Observations of the certain data are $\frac{x}{8}, \frac{x}{4}, \frac{x}{2}, x, \frac{x}{16}$ where $x > 0$. If median of the given data is 8, then find the mean of the given data.

28. The mean of 12 observations is 14. By an error one observation is registered as 24 instead of –24. Find the actual mean.

29. The mean weight of 20 students is 25 kg and the mean weight of another 10 students is 40 kg. Find the mean weight of the 30 students.

30. Find the variance and standard deviation of the scores 7, 8, 9, 10 and 16.

Short answer type questions

31. Tabulate the given data by taking

Class intervals: 1–10, 11–20, 21–30, 31–40

Data: 9, 10, 8, 6, 7, 4, 3, 2, 16, 28, 22, 36, 24, 18, 27, 35, 19, 29, 23, 34.

32. If the mean and the median of a unimodel data are 34·5 and 32·5, then find the mode of the data.

33. The heights of 100 students in primary classes is classified as below, find the median.

Height (in cm)	81	82	83	84	85
Number of students	22	14	26	23	15

34. The weight (in kg) of 25 children of 9th class is given, find the mean weight of the children.

Weight (in kg)	40	41	42	43	44	45
Number of children	3	4	6	2	5	5

35. If the mean of the following data is 5·3, then find the missing frequency y of the following distribution.

x	4	8	6	7
f	11	2	3	y

36. The mean of the data is 15. If each observation is divided by 5 and 2 is added to each result, then find the mean of the observations so obtained.

37. Draw the histogram for the following distribution.

Marks	0-10	10-20	20-30	30-40	40-50
Number of students	3	4	8	9	6

38. Find the mode of the following data.

Class interval	1-5	6-10	11-15	16-20	21-25
Frequency	3	4	10	6	7

39. Six faced balanced dice is rolled 20 times and the frequency distribution of the integers obtained is given below. Find the inter quartile range.

Integer	1	2	3	4	5	6
Frequency	3	4	2	5	4	2

40. Construct a less than cumulative frequency curve and a greater than cumulative frequency curve and answer the following

Daily wages (in Rs)	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Number of persons	5	12	17	36	20	10	8	2

(i) Find the number of persons who got Rs 60 and more than Rs 60.
(ii) Find the number of persons who got Rs 90 and less than Rs 90.

41. Draw the frequency polygon for the following distribution.

Class interval	0-5	5-10	10-15	15-20	20-25
Frequency	8	12	20	16	4

42. Find the median of the following data.

Class interval	0-20	20-40	40-60	60-80	80-100
Frequency	8	10	12	9	9

43. Construct a greater than cumulative frequency curve.

Class interval	5-9	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49
Frequency	1	5	10	19	25	21	15	3	1

44. Draw a histogram of the following data on graph paper and estimate the mode.

Percentage of marks	0-20	20-40	40-60	60-80	80-100
Number of students	10	12	16	14	8

45. Find the coefficient of variation of the following dies create series.

Scores	1	2	3	4	5
Frequency	0	4	3	2	1

Essay type questions

46. If the mean of the following table is 30, find the missing frequencies.

Class interval	0–15	15–30	30–45	45–60	Total
Frequency	10	a	b	8	60

47. Calculate the A.M. of the following data by short cut method.

Marks	0–10	10–20	20–30	30–40	40–50
Number of students	3	4	6	8	9

48. For the following frequency distribution, construct a less than cumulative frequency curve. And also find Q_1, Q_2, Q_3 by using graph.

Class interval	0–9	10–19	20–29	30–39	40–49	50–59	60–69
Frequency	4	3	5	6	1	2	1

49. Find the standard deviation of the following descrete series.

Scores	1	2	3	4	5
Frequency	0	4	3	2	1

50. Find the variance and S.D. for the given frequency distribution.

Class interval	1–5	6–10	11–15	16–20
Frequency	4	1	2	3

CONCEPT APPLICATION

Concept Application Level—1

1. If the arithmetic mean of the first n natural numbers is 15, then n is _____.

(1) 15 (2) 30 (3) 14 (4) 29

2. If the arithmetic mean of 7, 8, x, 11, 14 is x, then x is _____.

(1) 9 (2) 9·5 (3) 10 (4) 10·5

3. Find the mode of the data 5, 3, 4, 3, 5, 3, 6, 4, 5.

(1) 5 (2) 4 (3) 3 (4) Both (1) and (3)

4. The median of the data 5, 6, 7, 8, 9, 10 is ______.

(1) 7 (2) 8 (3) 7.5 (4) 8.5

5. If a mode exceeds a mean by 12 then the mode exceeds the median by _________.

(1) 4 (2) 8 (3) 6 (4) 10

6. If the less than cumulative frequency of a class is 50 and that of the previous class is 30, then the frequency of that class is ______.

(1) 10 (2) 20 (3) 40 (4) 30

7. If the median of the data, $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ is a, then find the median of the data x_3, x_4, x_5, x_6. (where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8$)

(1) a (2) $\frac{a}{2}$ (3) $\frac{a}{4}$ (4) Cannot say

8. The mode of the data 6, 4, 3, 6, 4, 3, 4, 6, 5 and x can be

(1) only 5 (2) both 4 and 6 (3) both 3 and 6 (4) 3, 4 or 6

9. If the greater than cumulative frequency of a class is 60 and that of the next class is 40, then find the frequency of that class.

(1) 10 (2) 20 (3) 50 (4) Cannot say

10. If the difference between the mode and median is 2, then the difference between the median and mean is ______ (in the given order).

(1) 2 (2) 4 (3) 1 (4) 0

11. In a series of observations, S.D. is 7 and mean is 28. find the coefficient of variation.

(1) 4 (2) 1/4 (3) 25 (4) 12.5

12. If the S.D. of x_1, x_2, x_3 ....x_n is 5, then find S.D. of $x_1 + 5, x_2 + 5, x_3 + 5$ ....$x_n + 5$.

(1) 0 (2) 10 (3) 5 (4) Cannot say

13. In a series of observations, coefficient of variation is 16 and mean is 25. find the variance

(1) 4 (2) 8 (3) 12 (4) 16

14. If the S.D. of y_1, y_2, y_3 ....,y_n is 6, then variance of $(y_1 - 3), (y_2 - 3), (y_3 - 3)$, ...$(y_n - 3)$ is _______.

(1) 6 (2) 36 (3) 3 (4) 27

15. Lower quartile, upper quartile and interquartile range respectively are Q_1, Q_3 and Q. If the average of Q, Q_1 and Q_3 is 40 and semi-interquartile range is 6, then find the lower quartile.

(1) 24 (2) 36 (3) 48 (4) 60

Direction for questions 16 and 17: These questions are based on the following data.

The weights of 20 students in a class are given below.

Weight (in kg)	31	32	33	34	35
Number of students	6	3	5	2	4

16. Find the median of the above frequency distribution.

(1) 32·5
(2) 33
(3) 33·5
(4) Cannot say

17. The interquartile range of the above frequency distribution is ______.

(1) 4 (2) 3 (3) 2 (4) 1

18. If the average of a, b, c and d is the average of b and c, then which of the following is necessarily true?

(1) (a + d) = (b + c)
(2) (a + b) = (c + d)
(3) (a – d) = (b – c)
(4) $\frac{(a+b)}{4} = \frac{(b+c)}{2}$

19. Find the interquartile range of the data 3, 6, 5, 4, 2, 1 and 7.

(1) 4 (2) 3 (3) 2 (4) 1

20. If the mean of the lower quartile and upper quartile is 10 and the semi-interquartile range is 5, then the lower quartile and the upper quartile respectively are _____ and _____.

(1) 2, 12 (2) 3, 13 (3) 4, 14 (4) 5, 15

21. The lower quartile of the data 5, 3, 4, 6, 7, 11, 9 is ______.

(1) 4 (2) 3 (3) 5 (4) 6

22. Find the arithmetic mean of the first 567 natural numbers.

(1) 284
(2) 283.5
(3) 283
(4) None of these

23. If a < b < c < d and a, b, c, d are non zero integers, the mean of a, b, c, d is 0 and the median of a, b, c, d is 0, then which of the following is correct?

(1) b = –c
(2) a = –d
(3) Both (1) and (2)
(4) None of these

24. The mean of 16 observations is 16. And if one observation 16 is deleted and three observations 5, 5 and 6 are included, then find the mean of the final observations.

(1) 16 (2) 15.5 (3) 13.5 (4) None of these

25. If the average wage of 50 workers is Rs 100 and the average wage of 30 of them is Rs 120, then the average wage of the remaining workers is ______.

(1) Rs 80 (2) Rs 70 (3) Rs 85 (4) Rs 75

26.

x	2	4	6	8
f	3	5	6	y

The mean of the above data is 5.5. Find the missing frequency (y) in the above distribution.

(1) 6 (2) 8 (3) 15 (4) 11

27. If L = 44·5, N = 50, F = 15, f = 5 and C = 20, then find the median from of given data.

(1) 84·5 (2) 74·5 (3) 64·5 (4) 54·5

28. If $L = 39{\cdot}5$, $\Delta_1 = 6$, $\Delta_2 = 9$ and $c = 10$, then find the mode of the data.

(1) 45·5 (2) 43·5 (3) 46·5 (4) 44·5

29. The average weight of 55 students is 55 kg and the average weight of another 45 students is 45 kg. Find the average weight of all the students.

(1) 48 kg (2) 50 kg (3) 50.5 kg (4) 52.25 kg

30. If the mean of 26, 19, 15, 24 and x is x, then find the median of the data.

(1) 23 (2) 22 (3) 20 (4) 21

Concept Application Level—2

Direction for questions 31 and 32: These questions are based on the following data.

The heights of 31 students in a class are given below.

Height (in cm)	126	127	128	129	130	131	132
Number of Students	7	3	4	2	5	6	4

31. Find the median of the above frequency distribution.

(1) 129 (2) 130 (3) 128 (4) 131

32. Find the semi-interquartile range of the above frequency distribution.

(1) 1 (2) 2 (3) 3 (4) 4

33. The mean and median of the data a, b and c are 50 and 35 respectively, where $a < b < c$. If $c - a = 55$, then find $(b - a)$.

(1) 8 (2) 7 (3) 3 (4) 5

34. If $a < b < 2a$ and the mean and the median of a, b and 2a are 15 and 12 respectively, then find a.

(1) 7 (2) 11 (3) 10 (4) 8

35. The variance of $6x_i + 3$ is 30, then find standard deviation of x_i

(1) $\frac{5}{\sqrt{6}}$ (2) $\sqrt{\frac{5}{6}}$ (3) 30 (4) $\sqrt{30}$

36. The frequency distribution of the marks obtained by 28 students in a test carrying 40 marks is given below.

Marks	0-10	10-20	20-30	30-40
Number of Students	6	x	y	6

If the mean of the above data is 20, then find the difference between x and y.

(1) 3 (2) 2 (3) 1 (4) 0

Direction for questions 37 and 38: These questions are based on the following data (figure).

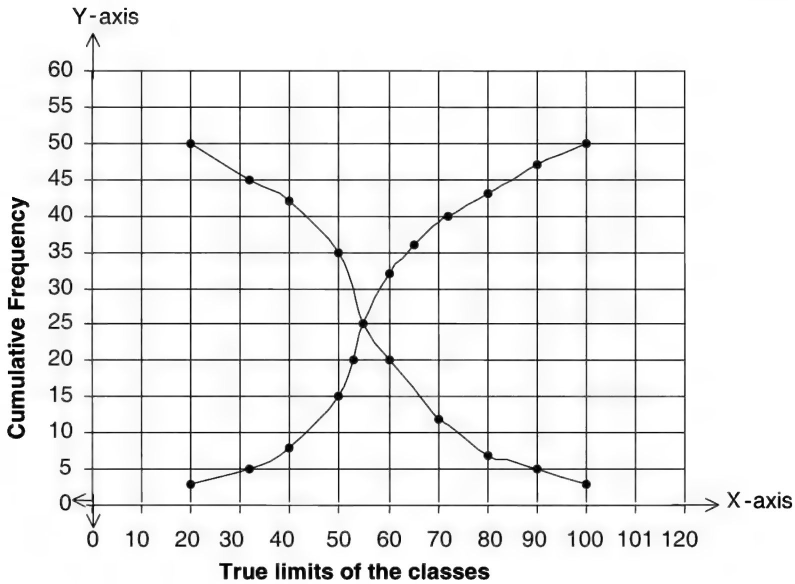


Figure 12.8

The given figure represents the percentage of marks on X–axis and the number of students on Y–axis.

37. Find the number of students who scored less than or equal to 50% of marks.

(1) 35 (2) 15 (3) 20 (4) 30

38. Find the number of students who scored greater than or equal to 90% of marks.

(1) 47 (2) 45 (3) 5 (4) 10

39. Mode for the following distribution is 22 and $10 > y > x$. find y.

Class interval	0-10	10-20	20-30	30-40	40-50	total
Frequency	5	8	10	x	y	30

(1) 2 (2) 5 (3) 3 (4) 4

40. Find the variance of the scores 2, 4, 6, 8 and 10.

(1) 2 (2) 4 (3) 6 (4) 8

41. If $A = 55{\cdot}5$, $N = 100$, $C = 20$ and $\sum f_i d_i = 60$, then find the mean from the given data.

(1) 67.5 (2) 57.5 (3) 77.5 (4) 47.5

42. If the standard deviation of $2x_i + 3$ is 8, then the variance of $\frac{3x_i}{2}$ is

(1) 24 (2) 36 (3) 6 (4) 18

43. Mode for the following distribution is 17.5 and x is less than 6. Find x.

Class interval	0–5	5–10	10–15	15–20	20–25
Frequency	5	2	3	6	x

(1) 3 (2) 2 (3) 4 (4) 5

Direction for questions 44 and 45: These questions are based on the following data: consider the following distribution table:

Class interval	0–6	6–12	12–18
Frequency	2	4	6

44. Find the coefficient of variation for the given distribution:

(1) $\frac{200\sqrt{6}}{11}$ (2) $\frac{200\sqrt{3}}{11}$ (3) $\frac{500}{11}$ (4) $\frac{200\sqrt{5}}{11}$

45. Find the variance for the given distribution:

(1) 24 (2) 12 (3) 20 (4) 25

Concept Application Level—3

46. The arithmetic mean of the following data is 7. Find (a + b).

x	4	6	7	9
f	a	4	b	5

(1) 4
(2) 2
(3) 3
(4) Cannot be determined

Direction for questions 47 and 48: The questions are based on the following data.

The performance of four students in annual report is given below:

[redacted]

Dheeraja	75	11.25
Nishitha	65	5.98
Sindhuja	48	8.88
Akshitha	44	5.28

47. Who is more consistent than the others?

(1) Dheeraja (2) Nishitha (3) Sindhuja (4) Akshita

48. Who is less consistent than the others?

(1) Dheeraja (2) Nishitha

(3) Sindhuja (4) Akshitha

49. If the mean of the squares of first n natural numbers is 105, then find the median of the first n natural numbers.

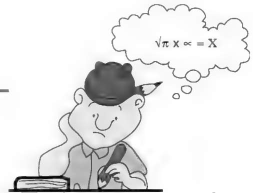
(1) 8 (2) 9 (3) 10 (4) 11

50. Range of the scores 18, 13, 14, 42, 22, 26 and x is 44 ($x > 0$). Find the sum of the digits of x.

(1) 16 (2) 14

(3) 12 (4) Cannot be determined

KEY



Very short answer type questions

1. 10 **2.** 5

3. 5.5 **4.** 20

5. 25 **6.** 5

7. 27.5 **8.** width

9. 50 **10.** 34

11. 10·5 **12.** $\frac{(n+1)}{2}$

13. False **14.** True

15. True **16.** 27

17. 8.5 **18.** False

19. 6 **20.** 4

21. Mean = 17.
Median = 18

22. 3.5 **23.** 363·5

24. 28. **25.** 460

26. 17.6 **27.** 12.4

28. 10 **29.** 30 kg

30. Variance = 10 S.D. = $\sqrt{10}$

Short answer type questions

31.

Class Interval	Tally Marks	Frequency
1–10	~~IIII~~ III	8
11–20	III	3
21–30	~~IIII~~ I	6
31–40	III	3

32. 28·5 **33.** 83 cm

34. 42.68 kg **35.** 4

36. 5 **37.**

38. 13.5 **39.** 3

40. (i) 40 (ii) 108 **42.** 50

44. 53 **45.** $33\frac{1}{3}$

Essay type questions

46. 18, 24 **47.** 30.33

48. **49.** 1

50. Variance = 41
S.D. = $\sqrt{41}$

key points for selected questions

Very short answer type questions

21. (i) Mean $= \dfrac{\text{Sum of the scores}}{\text{Number of scores}}$.

(ii) Median is the middle most term when the scores are arranged in either ascending or descending order.

(iii) If there are two middle most terms, then the median is average of the two terms.

22. (i) Arrange the given numbers in ascending order.

(ii) Then find N

(iii) Then find $Q_1 \left(\dfrac{N+1}{4}\text{th term}\right)$ and Q_3

$\left(\dfrac{3N+1}{4}\text{th term}\right)$

(iv) $QD = \dfrac{Q_3 - Q_1}{2}$

23. Mean of first n natural numbers is $\dfrac{(n+1)}{2}$.

24. (i) First of all write the highest score and least the score of the data.

(ii) Range = HS – LS

25. Mean of the observations which are in A.P. is the average of the first term and the last term.

26. (i) Sum of the observations is the product of mean and the number of observations.

(ii) Then subtract the excess value from the sum.

(iii) Then find the actual mean.

27. (i) First of all arrange the given scores in the ascending order.

(ii) Find x, by using median = 8

(iii) Then find the scores.

(iv) Now find the mean of the data

28. (i) Sum of the observations is the product of mean and the number of observations.

(ii) Then subtract the excess value from the sum.

(iii) Then find actual mean.

29. (i) First of all find sum of weight of 20 and 10 students respectively.

(ii) Then find the sum of weights of 30 students.

(iii) Now, find the mean weight of 30 students.

30. (i) First of all find mean ($\overline{x}$) of the given observations.

(ii) Variance

$$= \frac{(x_1-\overline{x})^2 + (x_2-\overline{x})^2 + \ldots + (x_n-\overline{x})^2}{n}$$

(iii) $SD = \sqrt{\text{Variance}}$

Short answer type questions

31. Find frequency of each class by using tally marks, and then tabulate.

32. Mode = 3 Median – 2 Mean.

33. (i) First of all write LCF of the given frequencies.

(ii) Then find N i.e., Σf_i.

(iii) Then find which observation is median.

34. Weighted mean is $\dfrac{\Sigma W_i\, x_i}{\Sigma x_i}$.

35. (i) Mean of descrete series is $\dfrac{\Sigma f_i\, x_i}{\Sigma f_i}$.

(ii) Given that, mean = 5.3

36. Mean $(ax_i + b) = a$ Mean $(x_i) + b$.

37. (i) Take classes on X – axis and number of students on Y – axis.

(ii) On Y – axis, take 1 cm = 1 student and draw the histogram.

38. (i) First of all convert the class intervals into class boundaries.

(ii) Identify the model class i.e., which has highest frequency.

(iii) Write f, f_1 and f_2 i.e., frequencies of model class, previous class of the model class and next class of the model class respectively.

(iv) Mode = $L_1 + \frac{(f-f_1)C}{2f-(f_1+f_2)}$ where L_1 is the lower boundary of model class.

39. (i) First of all find LCF.

(ii) Then find N i.e., Σf_i.

(iii) Then $\frac{N}{4}$ th term and $\frac{3N}{4}$ th terms which are Q_1 and Q_3 respectively.

(iv) Inter quartile range = $Q_3 - Q_1$.

40. (i) For less than cumulative frequency curve, take upper boundaries on X axis and LCF on Y–axis.

(ii) For greater than cumulative frequency curve, take lower boundaries on Y–axis and GCF on Y–axis.

41. (i) Take mid-values on X–axis.

(ii) Take frequencies on Y–axis.

(iii) On X–axis, take 1 cm = 5 (mid-values).

(iv) On Y–axis, take 1 cm = 2 (frequency).

42. (i) First of all write LCF of the given frequencies.

(ii) Then find N i.e., Σf_i.

(iii) Then find $\frac{N}{2}$.

(iv) Now find the lower boundary (L) of median class

(v) Find frequency (f) of median class.

(vi) Then find cumulative frequency (F) of the class just preceding the median class

(vii) Median = $L + \frac{\left(\frac{N}{2}-F\right)}{f} \times C$.

43. For greater than cumulative frequency curve, take lower boundaries on X-axis and GCF on Y-axis.

44. (i) First of all draw the histogram for the given data.

(ii) Consider three consecutive bars in which middle one must have lengthiest bar in the diagram.

(iii) And mark four vertices and draw diagrams (for this refer the study material)

45. CV = $\frac{SD \times 100}{AM}$

Essay type questions

46. (i) First of all write mid-values of the classes.

(ii) Then assume one of the mid-values as A.

(iii) Then find μi, by using $\mu i = \frac{x_i - A}{C}$.

(iv) Then find $f_i \mu_i$.

(v) AM = $A + \frac{1}{N}(\Sigma f_i \mu_i) \times C$

47. (i) First of all write mid-values of the classes

(ii) Then assume one of the mid-values as A

(iii) Then find u_i, by using $u_i = \frac{x_i - A}{C}$

(iv) Then find $f_i \times u_i$.

(v) AM = $A + \frac{1}{N}(\Sigma f_i \times u_i) \times C$

48. For less than cumulative frequency curve, take upper boundaries on X–axis and LCF on Y–axis

(ii) Then mark $\frac{N}{4}, \frac{N}{2}$ and $\frac{3N}{4}$ on Y–axis.

(iii) Then find Q_1, Q_2, Q_3 on X-axis corresponding to $\frac{N}{4}, \frac{N}{2}$ and $\frac{3N}{4}$ respectively.

49. (i) First of all find mean ($\bar{x}$).

(ii) Then find $D_i = (\bar{x}_i - \bar{x})$

(iii) Now find $\Sigma f_i D_i^2$.

(iv) SD = $\sqrt{\Sigma f_i D^2{}_i / N}$

50. (i) First of all find the mid-points (xi) of classes

(ii) Then find mean ($\bar{x}$)

(iii) Then find $D_i = x_i - \bar{x}$.

(iv) Then find $\Sigma f_i D_i^2$

SD = $\sqrt{\frac{\Sigma f_i D_i^2}{N}}$

KEY

Concept Application Level—1,2,3

1. 4	**2.** 3
3. 4	**4.** 3
5. 2	**6.** 2
7. 1	**8.** 4
9. 2	**10.** 3
11. 3	**12.** 3
13. 4	**14.** 2
15. 3	**16.** 2
17. 2	**18.** 1
19. 1	**20.** 4
21. 1	**22.** 1
23. 3	**24.** 4
25. 2	**26.** 1
27. 1	**28.** 2
29. 3	**30.** 4
31. 1	**32.** 2
33. 4	**34.** 2
35. 2	**36.** 4
37. 2	**38.** 3
39. 2	**40.** 4
41. 1	**42.** 2
43. 1	**44.** 4
45. 3	**46.** 4
47. 2	**48.** 3
49. 2	**50.** 3

Concept Application Level—1,2,3

Key points for select questions

1. Arithmatic mean of first n natural numbers is $\dfrac{n+1}{2}$.

2. Arithmatic mean

$$= \frac{\text{Sum of the quantities}}{\text{Number of the quantities}}$$

3. An observation which has more frequency in the data is called the mode of the data.

4. If the number of observations is even, then the median of the data is the average of $\left(\dfrac{n}{2}\right)$th and $\left(\dfrac{n}{2}+1\right)$th observations.

5. Use 'Empirical formula'.

6. Frequency of a particular class = cumulative frequency of that class – cumulative frequency of the previous class.

7. If the number of observations is even, then the median of the data is the average of $\left(\dfrac{n}{2}\right)$th and $\left(\dfrac{n}{2}+1\right)$th observations.

8. An observation which has more frequency in the data is called the mode of the data.

9. Frequency of a particular class = (cumulative frequency of that class)–(cumulative frequency of the next class).

10. Use Empirical formulae.

11. Coefficient of variation

$$= \frac{\text{Standard deviation}}{\text{mean}} \times 100$$

12. S.D. does not alter when each term is incresed by fixed constant.

13. Coefficient of variation

$$= \frac{\text{Standard deviation}}{\text{Mean}} \times 100$$

14. S.D. does not alter when each term is decreased by fixed constant.

15. Semi interquartile range $= \dfrac{Q_3 - Q_1}{2}$

16. Find less than cumulative frequency then find median by using formulae

17. Find less than cumulative frequency then find inter quartile range by using formulae.

18. Average = $\dfrac{\text{Sum of the quantities}}{\text{Number of the quantities}}$

19. $Q = Q_3 - Q_1$

20. Semi-interquartile range = $\dfrac{Q_3 - Q_1}{2}$

21. Write the data in the asceding order.

Lower quartile is $\left(\dfrac{n+1}{4}\right)$th observation.

22. Arithmetic mean of first 'n' natural numbers is $\dfrac{n+1}{2}$.

23. Average = $\dfrac{\text{Sum of the quantities}}{\text{Number of the quantities}}$

24. Mean = $\dfrac{\text{Sum of the observations}}{\text{Number of the observations}}$

25. Average = $\dfrac{\text{Sum of the quantities}}{\text{Number of the quantities}}$

26. (i) Apply mean concept for discrete series.

(ii) Find $\Sigma f x$ and Σf.

(iii) Now, A.M. = $\dfrac{\sum fx}{\sum f}$.

27. Median = $L + \left[\dfrac{\left(\dfrac{N}{2} - F\right)C}{f}\right]$

28. Mode = $L_1 + \left[\dfrac{(f - f_1)C}{2f - (f_1 + f_2)}\right]$

29. Average = $\dfrac{\text{Sum of the quantities}}{\text{Number of the quantities}}$

30. Arithmatic mean

$= \dfrac{\text{Sum of the quantities}}{\text{Number of the quantities}}$

31. Find less than cumulative frequency then find median by using formulae

32. Find less than cumulative frequency then find inter quartile range by using formulae

33. (i) Since, a, b, c are in ascending order, b is the median.

(ii) A.M.

$= \dfrac{\text{Sum of all observations}}{\text{Total number of observations}}$

(iii) Using the above relation, write the relation between c and a and find the value of a and c. Finally find the values of b and a.

34. (i) Since $a < b < 2a$, Median = b = 12

(ii) Mean

$= \dfrac{\text{Sum of all observations}}{\text{Total number of observations}}$.

35. (i) If the variance of $(ax_i + b)$ is k, then SD of $(ax_i + b) = \sqrt{k}$.

(ii) SD of $x_i = \dfrac{\sqrt{k}}{a}$.

36. (i) Find the mid-values (x_i) of class intervals.

(ii) Find $\dfrac{\sum fx}{\sum f}$.

(iii) Equate $\dfrac{\sum fx}{\sum f}$ with mean and find the values of x and y.

37. In the given graph, one curve represents less than cumulative frequency, another curve represents greater than cumulative frequency.

38. Apply greater than cumulative frequency concept.

39. (i) 10 is the model class.

(ii) Using mode = $L + \dfrac{\Delta_1}{\Delta_1 + \Delta_2} \times C$,

we can get the value of x.

40. (i) SD of AP with common difference 'd' is $d\sqrt{\dfrac{n^2-1}{12}}$.

(ii) Variance = $(SD)^2$.

41. (i) Use Arithmetic mean formula for grouped data.

(ii) Mean of grouped data

$$= A + \frac{\Sigma f_i d_i}{N} \times C.$$

43. Find mode of the given data in terms of x and form an equation to find x.

45. (i) First calculate mean.

(ii) Find deviations about mean $(x_i - \overline{x})$.

$D = x_i - \overline{x}$, N = Sum of all frequencies, $S\,D = \sqrt{\frac{\Sigma f\ D^2}{N}}$

46. Use Arithmetic mean formulae for descreate series.

47. (i) Coefficient of variation $= \frac{SD}{Mean} \times 100$

(ii) Using the above, find CV of all the four members.

(iii) The member whose CV is least is more consistent.

48. (i) Find coefficient of variation then decide.

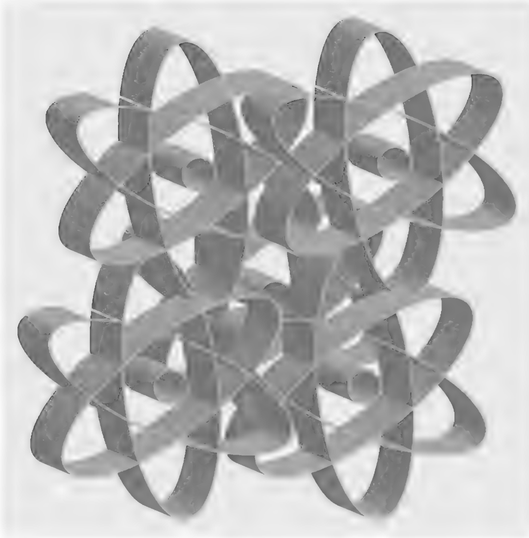
(ii) The one with highest CV is less consistent.

49. The mean of squares of n natural numbers $\frac{(n+1)\ (2n+1)}{6}$

50. Range is maximum value – minimum value.

CHAPTER 13

Geometry



INTRODUCTION

In this chapter, we shall learn about symmetry, similarity of geometrical figures in general, triangles in particular. We shall understand similarity through size transformation. Further we shall learn about concurrent lines, geometric centres in triangles, basic concepts of circles and related theorems. We shall also focus on some constructions related to polygons and circles. Finally we shall discuss the concept of locus.

Symmetry

Let us examine the following figures drawn on a rectangular piece of paper.

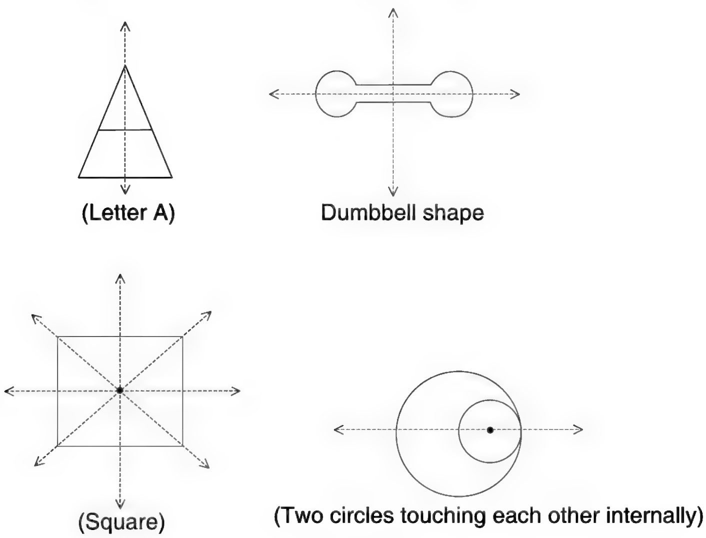


Figure 13.1

What do you infer? We can observe that when these figures are folded about the dotted lines, the two parts on either side of the dotted lines coincide. This property of geometrical figures is called symmetry.

In this chapter, we shall discuss two basic types of symmetry—**line symmetry and point symmetry**. Then we shall see how to obtain the image of a point, a line segment and an angle about a line.

Line symmetry

Trace a geometrical figure on a rectangular piece of paper as shown below:

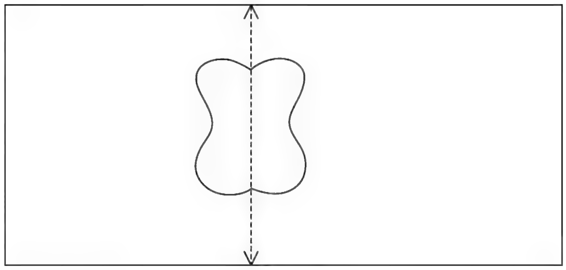


Figure 13.2

Now fold the paper along the dotted line. You will find that the two parts of the figure on either sides of the line coincide. Thus the line divides the figure into two identical parts. In this case, we say that the figure is symmetrical about the dotted line or Line Symmetric. Also, the dotted line is called the Line Of Symmetry or The Axis Of Symmetry.

So, a geometrical figure is said to be line symmetric or symmetrical about a line if there exists at least one line in the figure such that the parts of the figure on either sides of the line coincide when it is folded about the line.

Example

Consider the following figure.

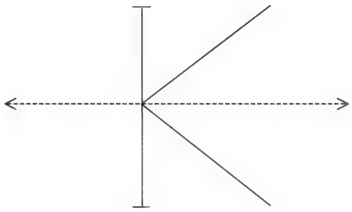


Figure 13.3

The above figure is symmetrical about the dotted line. Also, there is only one line of symmetry for the figure.

Example

Observe the figure given below.

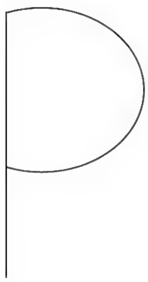


Figure 13.4

There is no line in the figure about which the figure is symmetric.

Example

Consider a rectangle as shown below.

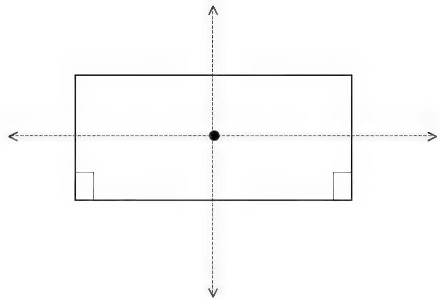


Figure 13.5

The rectangle is symmetrical about the two dotted lines. So, a rectangle has two lines of symmetry.

Example

Consider a figure in which 4 equilateral triangles are placed, one on each side of a square as shown below.

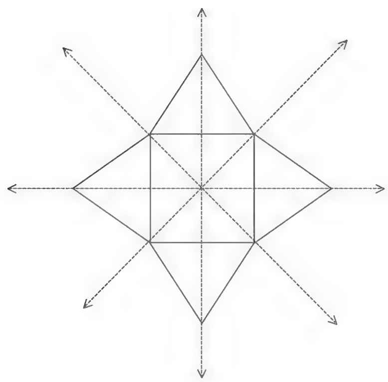


Figure 13.6

The figure is symmetrical about the dotted lines. It has 4 lines of symmetry.

Example

A circle has an infinite number of lines of symmetry, some of which are shown in the figure below.

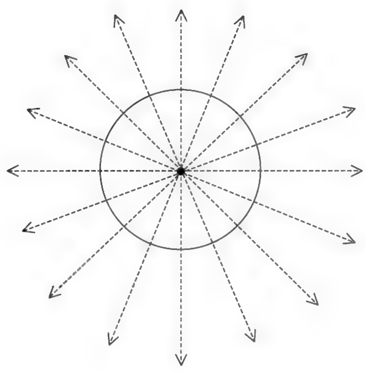


Figure 13.7

From the above illustrations, we observe the following:

(i) A geometrical figure may not have a line of symmetry, i.e., a geometric figure may not be line symmetric.
(ii) A geometrical figure may have more than one line of symmetry i.e., a geometrical figure may be symmetrical about more than one line.

Point symmetry

Trace the letter N on a rectangular piece of paper as shown below.

Let P be the mid-point of the inclined line in the figure. Now draw a line segment through the point P touching the two vertical strokes of N. We find that the point P divides the line segment into two equal parts. Thus, every line segment drawn through the point P and touching the vertical strokes of N is bisected at the point P. Also, when we rotate the letter N about the point P through an angle of 180°, we find that it coincides exactly with the initial position. This property of geometrical figures is called *Point Symmetry*. In this case, we say that the point P is the point of symmetry of the figure.

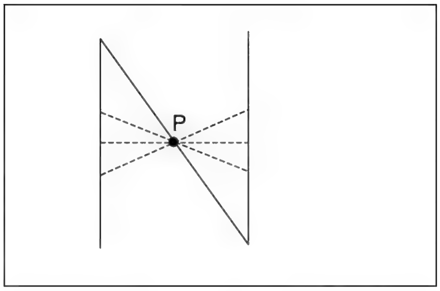


Figure 13.8

So, a geometrical figure is said to have symmetry about a point P if every line segment through the point P touching the boundary of the figure is bisected at the point P.

OR

A geometrical figure is said to have point symmetry if the figure does not change when rotated through an angle of 180°, about the point P.

Here, point P is called the centre of symmetry.

Illustrations

Example

Consider a rectangle ABCD. Let P be the point of intersection of the diagonals AC and BD. Then, the rectangle ABCD has the point symmetry about the point P.

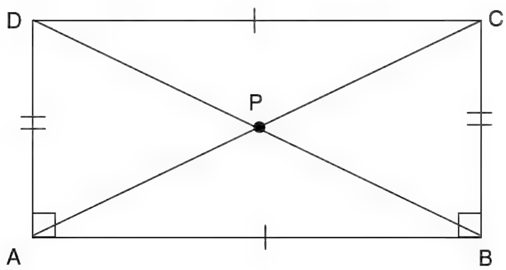


Figure 13.9

Example

The letter "S" has a point of symmetry about the turning point in S.

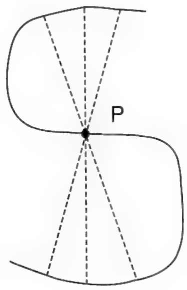


Figure 13.10

☛ *Example*

A circle has a point of symmetry. This point is the centre of the circle.

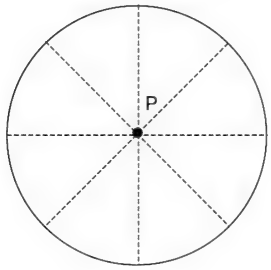


Figure 13.11

Image of a point about a line

Draw a line L and mark a point A on a rectangular piece of paper as shown below.

Draw AM perpendicular to L and produce it to B such that AM = MB. Then the point B is said to be the image of the point A about the line L.

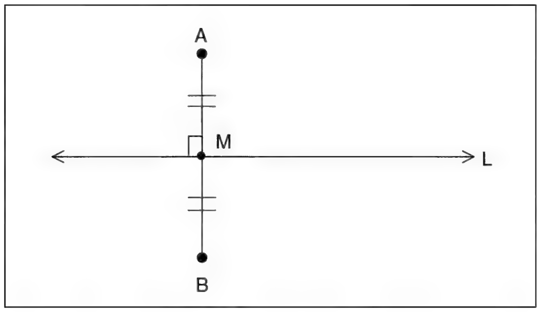


Figure 13.12

Image of a line segment about a line

Draw a line L and a line segment PQ on a rectangular piece of paper as shown below.

Draw perpendiculars PL and QM from the points P and Q respectively to the line ℓ and produce them to P^1 and Q^1 respectively, such that $PL = LP^1$ and $QM = MQ^1$. Join the points P^1, Q^1. Then the line segment P^1Q^1 is called the image of the line segment PQ about the line L.

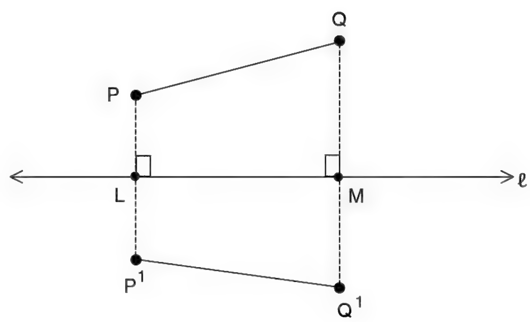


Figure 13.13

Image of an angle about a line

Draw a line L and an angle PQR on a piece of paper as shown below.

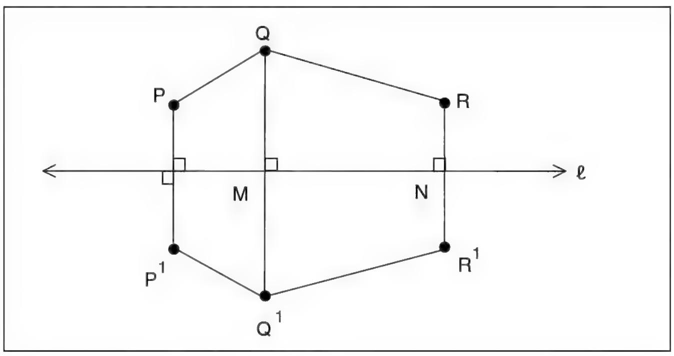


Figure 13.14

Draw perpendiculars PL, QM and RN from the points P, Q and R respectively to the line ℓ and produce them to P^1, Q^1 and R^1 respectively, such that $PL = LP^1$, $QM = MQ^1$ and $RN = NR^1$. Join the points P^1, Q^1 and Q^1, R^1. Then, the angle $P^1Q^1R^1$ is called the image of the angle PQR about the line L.

☛ *Example*

Determine the line of symmetry of a triangle ABC in which $\angle A = 40°$, $\angle B = 70°$ and $\angle C = 70°$.

Solution

The line which bisects $\angle A$ is the line of symmetry of the given triangle ABC.

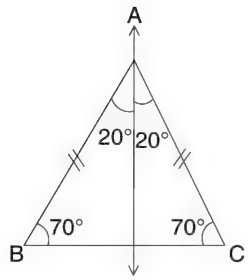


Figure 13.15

☛ *Example*

Determine the point of symmetry of a regular hexagon.

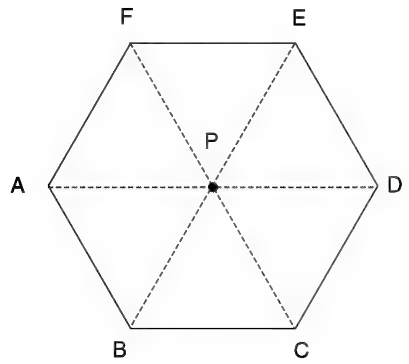


Figure 13.16

Solution

The point of intersection of the diagonals of regular hexagon is the required point of symmetry.

☛ *Example*

Complete adjacent figure so that X-axis and Y-axis are the lines of symmetry of the completed figure.

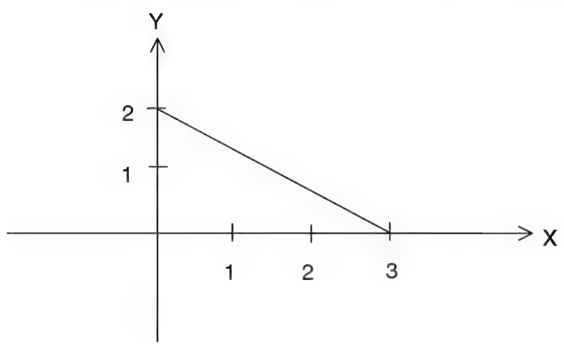


Figure 13.17

Solution

Required figure is a rhombus of side $\sqrt{13}$ units.

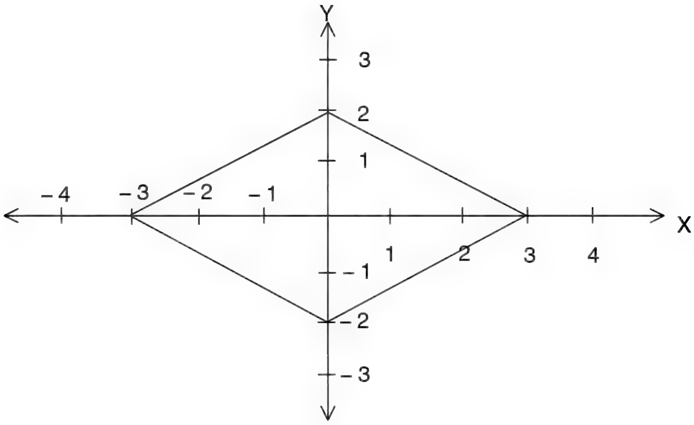
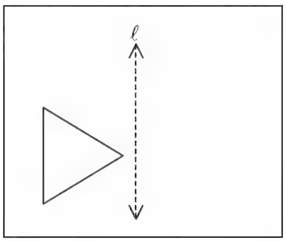


Figure 13.18

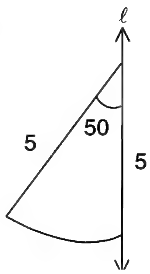
Example

Determine the images of the following figure about the given line.

(i)

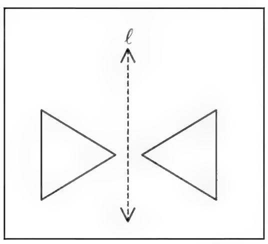


(ii)

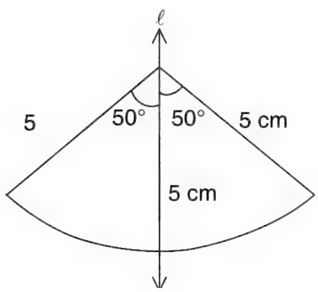


Solution

(i)



(ii)



Similarity

1. **Similarity of geometrical figures**

Two geometrical figures of same shape, but not necessarily of same size are said to be similar.

☛ *Examples*

1. Any two circles are similar.
2. Any two squares are similar.
3. Any two equilateral triangles are similar.

Two circles C_1 and C_2 are similar, since their shapes are the same.

C_1 and C_2 may have an equal area.

While considering the circles C_1 and C_3 they may not be of equal area, but their shapes are the same.

C_1 and C_2 are of the same shape and of same size.

C_1 and C_3 are of the same shape but of different sizes.

In both the cases, they are of the same shape.

Hence C_1, C_2 and C_3 are similar circles.

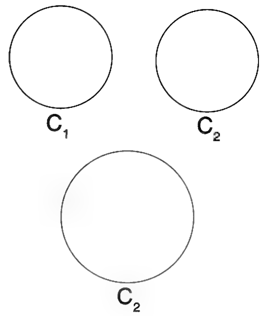


Figure 13.19

Similarity as a size transformation

Enlargement

Consider triangle ABC.

Construct triangle PQR whose sides are twice the corresponding sides of triangle ABC, as shown in the following figure.

Construction

First draw the triangle ABC. Take a point D outside the triangle. Join DA, DB and DC.

Now, extend DA, DB and DC to the points P, Q and R respectively such that DP = 2DA, DQ = 2DB and DR = 2DC

PQR is the enlarged image of ABC.

On verification, we find that PQ = 2AB

QR = 2BC and RP = 2AC.

It can be defined that triangle ABC has been enlarged by a scale factor of 2 about the center of the enlargement D to give image PQR.

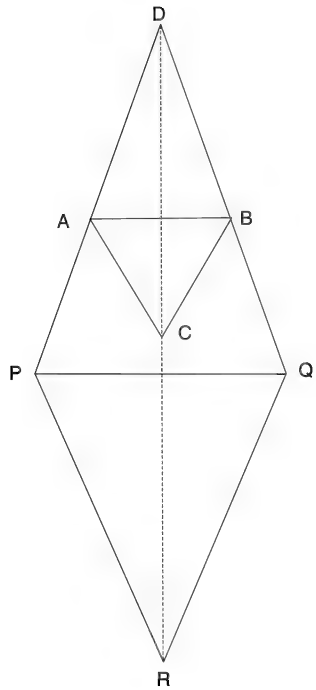


Figure 13.20

Reduction

Consider square ABCD. One can draw a square PQRS whose side is half the length of the side of ABCD.

Construction

Draw square ABCD.

Take a point E outside the square.

Join EA, EB, EC and ED. Mark the points P, Q, R and S on $\overline{EA}$, $\overline{EB}$, $\overline{EC}$ and $\overline{ED}$ such that

$$EP = \frac{1}{2}EA,\ EQ = \frac{1}{2}EB,\ ER = \frac{1}{2}EC \text{ and } ES = \frac{1}{2}ED$$

Square PQRS is the reduced image of square ABCD.

We find that,

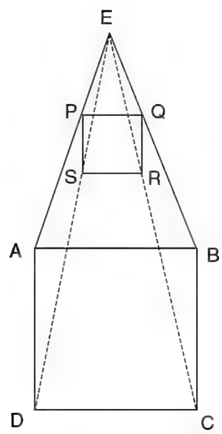


Figure 13.21

$PQ = \frac{1}{2}AB$, $QR = \frac{1}{2}BC$,

$RS = \frac{1}{2}CD$ and $SP = \frac{1}{2}AD$

ABCD has been reduced by a scale factor $\frac{1}{2}$ about the centre of reduction E to give the image PQRS.

Size transformation is the process which a geometrical figure is enlarged or reduced by a scale k, such that the image formed is similar to the given figure.

Properties of size transformation

1. The shape of the given figure remains the same.
2. If k is the scale factor of a given size transformation and $k > 1 \Rightarrow$ the image is enlarged and $k < 1 \Rightarrow$ the image is reduced.

   and $k = 1 \Rightarrow$ the image is identical to the original figure.
3. Each side of the given geometrical figure = k (The corresponding side of the given figure).
4. Area of the image = k^2 (Area of the given geometrical figure).
5. Volume of the image, if it is a 3-dimensional figure, is equal to k^3 (Volume of the original figure).

Model

The model of a plane figure and the given figure are similar to each other. If the model of a plane figure is drawn to the scale 1 : x, then scale factor, $k = \frac{1}{x}$.

1. Length of the model = k(Length of the original figure).
2. Area of the model = k^2(Area of the original figure)
3. Volume of the model = k^3(Volume of the original figure)

Map

If the map of a plane figure, is drawn to the scale 1 : x, then, scale factor, $k = \frac{1}{x}$.

1. Length in the map = k (Original length)
2. Area in the map = k^2 (Original area)

The discussion on similarity can be extended further as follows:
Two polygons are said to be similar to each other if

(i) their corresponding angles are equal and
(ii) the lengths of their corresponding sides are proportional.

Note: "~" is the symbol used for "is similar to".

If quadrilateral ABCD is similar to quadrilateral PQRS we denote this as $\square ABCD \sim \square PQRS$.
The relation 'is similar to' satisfies the following properties.

1. It is reflexive as every figure is similar to itself.
2. It is symmetric as A is similar to B then B is also similar to A.
3. It is transitive as, if A is similar to B and B is similar to C, then A is similar to C.

∴ The relation is 'similar to' is an equivalence relation.

Criteria for similarity of triangles

Similar triangles

In two triangles, if either the corresponding angles are equal or the ratio of corresponding sides are equal, then the two triangles are similar to each other.

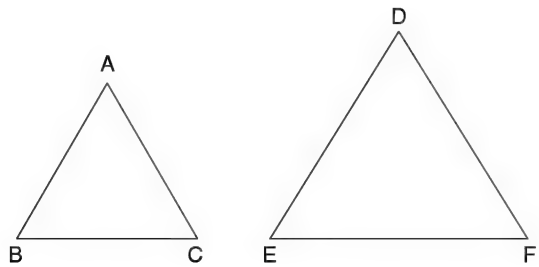


Figure 13.22

In ΔABC and ΔDEF if

1. $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$ or
2. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, then $\Delta ABC \sim \Delta DEF$ or ΔABC is similar to ΔDEF.

Three similarity axioms for triangles

1. **A.A. – Axiom or A.A.A. – Axiom**
   In two triangles, if corresponding angles are equal then the triangles are similar to each other.

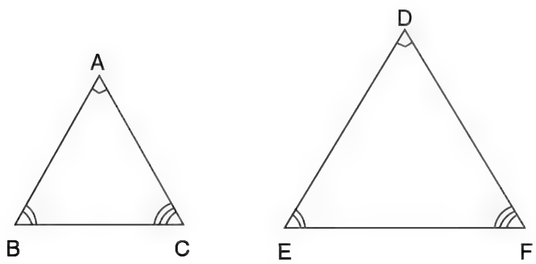


Figure 13.23

In ΔABC and ΔDEF, $\angle A = \angle D, \angle B = \angle E$ and $\angle C = \angle F$.
∴ $\Delta ABC \sim \Delta DEF$.

Note: In two triangles, if two pairs of corresponding angles are equal then the triangles are similar to each other, because the third pair corresponding angles will also be equal.

2. **S.S.S. – Axiom**
   In two triangles, if the corresponding sides are proportional, then the two triangles are similar to each other.

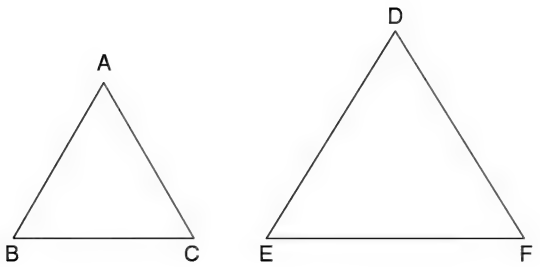


Figure 13.24

In ΔABC and ΔDEF, if $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$

$\Rightarrow \Delta ABC \sim \Delta DEF$.

3. **S.A.S. – Axiom**
   In two triangles, if one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

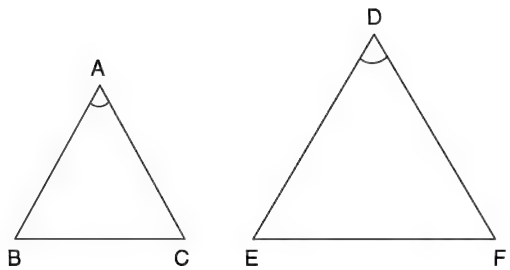


Figure 13.25

In ΔABC and ΔDEF,

If $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ and $\angle A = \angle D$, then $\Delta ABC \sim \Delta DEF$.

Right angle theorem

In a right angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, then the two right triangles formed on either side of the perpendicular are similar to each other and similar to the given triangle.

In the figure, ΔABC is right angled at A. $\overline{AD}$ is the perpendicular drawn from A to BC.

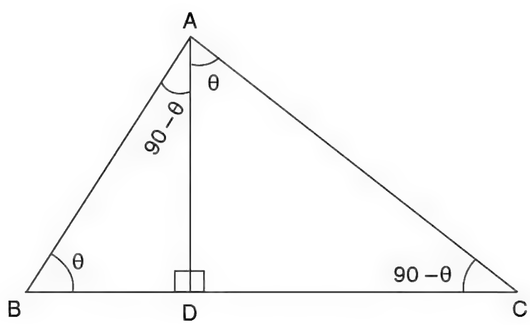


Figure 13.26

Let $\angle ABD = \theta$

Then, $\angle BAD = 90° - \theta$

$\angle DAC = \angle BAC - \angle BAD = 90° - (90° - \theta)$

$\Rightarrow \angle DAC = \theta$

$\therefore \angle DCA = 90° - \theta$.

In triangles BAD, DAC, and ABC, three corresponding angles are equal. Therefore, the triangle ABC is similar to triangle DAC or triangle DBA.
In triangles DBA and DAC,

$$\because \frac{BD}{AD} = \frac{AD}{DC}$$

$\Rightarrow AD^2 = BD \times DC$

AD is the mean proportional of BD and DC.

$\Rightarrow \quad AD^2 = BD \cdot DC$

Results on areas of similar triangles

The ratio of areas of the two similar triangles is equal to the ratio of the squares of any two corresponding sides of the triangles.

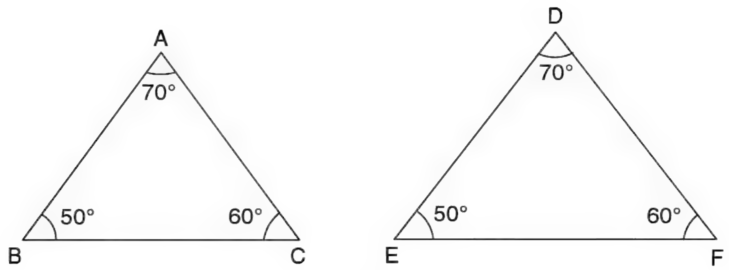


Figure 13.27

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2}$$

(a) The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding altitudes.

In the following figures, $\Delta ABC \sim \Delta DEF$ and AX, DY are the altitudes.

$$\text{Then, } \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AX^2}{DY^2}$$

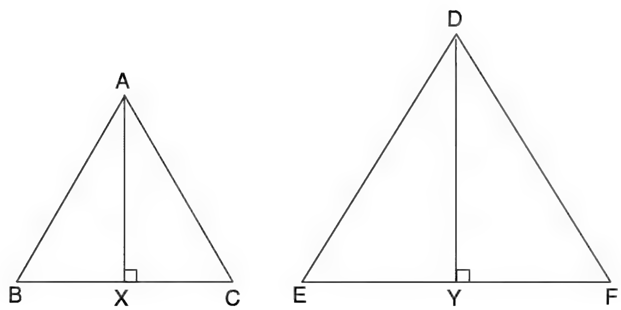


Figure 13.28

(b) The ratio of areas of two similar triangles is equal to the ratio of the squares on their corresponding medians.

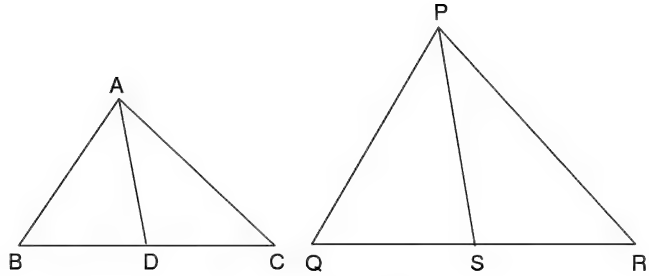


Figure 13.29

In the above figure, $\Delta ABC \sim \Delta PQR$ and AD and PS are medians.

Then, $\dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta PQR} = \dfrac{AD^2}{PS^2}$

(c) The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding angle bisector segments.

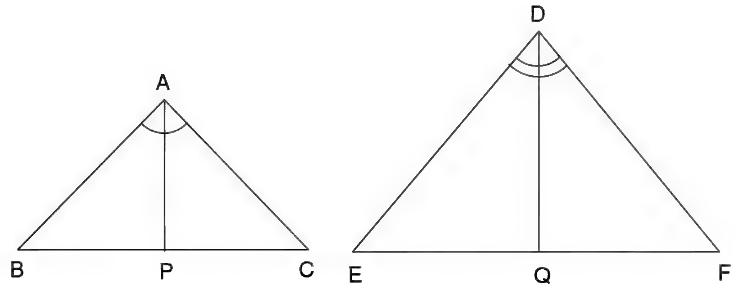


Figure 13.30

In the figure, $\Delta ABC \sim \Delta DEF$ and

AP, DQ are bisectors of $\angle A$ and $\angle D$ respectively, then

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{AP^2}{DQ^2}$$

Pythagoras theorem

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

To prove : $AC^2 = AB^2 + BC^2$

Construction: Draw BP perpendicular to AC.

Proof: In triangles APB and ABC,

$\angle APB = \angle ABC$ (right angles)

$\angle A = \angle A$ (common)

$\therefore$ Triangle APB is similar to triangle ABC.

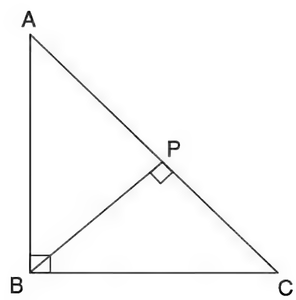


Figure 13.31

$$\Rightarrow \frac{AP}{AB} = \frac{AB}{AC}$$

$\Rightarrow AB^2 = (AP)(AC)$

Similarly, $BC^2 = (PC)(AC)$

$\therefore AB^2 + BC^2 = (AP)(AC) + (PC)(AC)$

$AB^2 + BC^2 = (AC)(AP + PC)$

$AB^2 + BC^2 = (AC)(AC)$

$\Rightarrow AC^2 = AB^2 + BC^2$

Hence proved.

The results in an obtuse triangle and an acute triangle are as follows:

In ΔABC, $\angle ABC$ is obtuse and AD is drawn perpendicular to BC, then

$AC^2 = AB^2 + BC^2 + 2BC \cdot BD$

ΔABC is an acute angled triangle, acute angle at B and AD is drawn perpendicular to BC, then $AC^2 = AB^2 + BC^2 - 2BC \cdot BD$

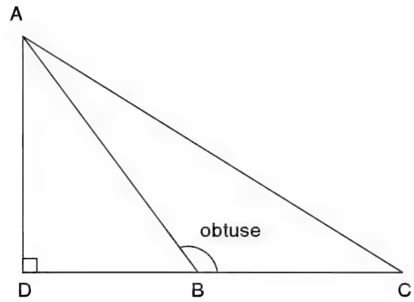


Figure 13.32

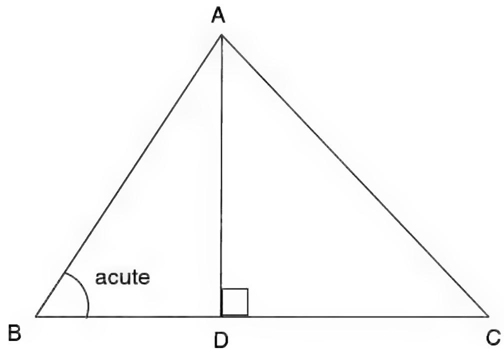


Figure 13.33

Appolonius theorem

In a triangle, the sum of the squares of two sides of a triangle is equal to twice the sum of the square of the median which bisects the third side and the square of half the third side.

Given: n ΔABC, AD is the median.

RTP: $AB^2 + AC^2 = 2(AD^2 + DC^2)$ or $2(AD^2 + BD^2)$

Construction: Draw AE perpendicular to BC.

Case (i) If $\angle ADB = \angle ADC = 90°$

According to Pythagoras theorem,

In $\Delta ABD, AB^2 = BD^2 + AD^2$ -----------(1)

In $\Delta ADC, AC^2 = CD^2 + AD^2$ -----------(2)

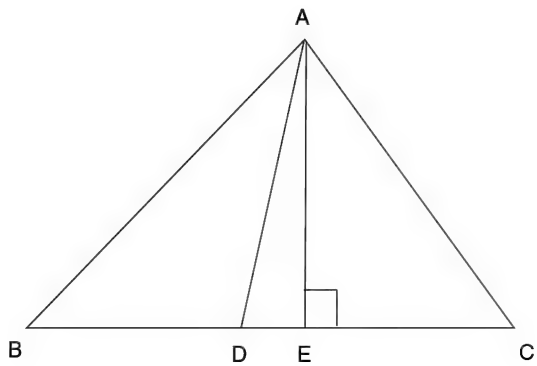


Figure 13.34

Adding (1) and (2), $AB^2 + AC^2 = BD^2 + CD^2 + 2AD^2$

$= 2BD^2 + 2AD^2$ [$\because$ CD = BD]

or $2[CD^2 + AD^2]$

Case (ii) If $\angle ADC$ is acute and $\angle ADB$ is obtuse.

In triangle ADB,

$AB^2 = AD^2 + BD^2 + 2 \times BD \times DE$

In triangle ADC,

$AC^2 = AD^2 + DC^2 - 2 \times CD \times DE$

But BD = CD

$\therefore$ $AB^2 + AC^2 = 2AD^2 + 2BD^2$ or $2(AD^2 + CD^2)$

Hence proved.

Basic proportionality theorem

In a triangle, if a line is drawn parallel to one side of a triangle, then it divides the other two sides in the same ratio.

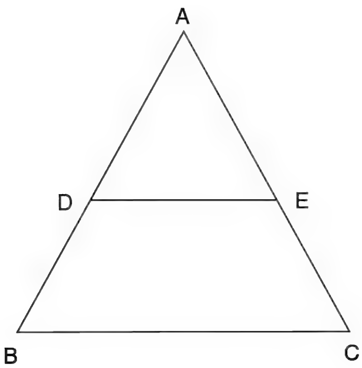


Figure 13.35

Given: In ΔABC, $\overline{DE} \parallel \overline{BC}$

(i) RTP: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

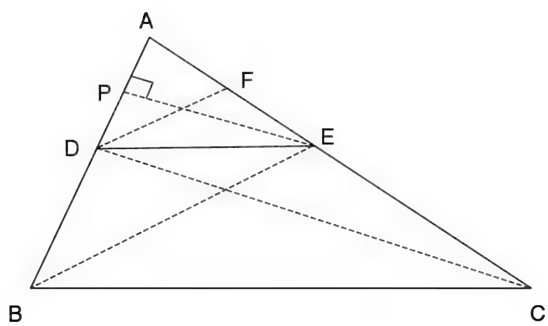


Figure 13.36

Construction: Draw EP $\perp$ AB and DF $\perp$ AC. Join $\overline{DC}$ and $\overline{BE}$.

Proof: $\dfrac{\text{Area of triangle ADE}}{\text{Area of triangle ADE}} = \dfrac{\frac{1}{2}\times AD\times PE}{\frac{1}{2}\times BD\times PE} = \dfrac{AD}{BD}$

and $\dfrac{\text{Area of triangle ADE}}{\text{Area of triangle CDE}} = \dfrac{\frac{1}{2}\times AE\times DF}{\frac{1}{2}\times EC\times DF} = \dfrac{AE}{EC}$

But area so of triangles BDE and CDE are equal. (Two triangles lying on the same base and between the same parallel lines are equal in area).

$\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$

NOTE: From the above result we can prove that $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ and $\dfrac{AB}{BD} = \dfrac{AC}{CE}$.

Converse of basic proportionality theorem

If a line divides two sides of a triangle in the same ratio then that line is parallel to the third side.

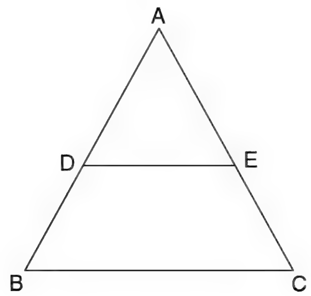


Figure 13.37

In the figure given, $\dfrac{AD}{DB} = \dfrac{AE}{EC} \Rightarrow \overline{DE} \parallel \overline{BC}$.

Vertical angle bisector theorem

The bisector of the vertical angle of a triangle divides the base in the ratio of the other two sides.
Given: In ΔABC, AD is the bisector of $\angle A$.

R.T.P: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

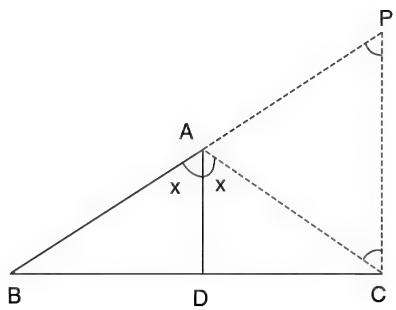


Figure 13.38

Construction: Draw CP parallel to AD to meet BA produced at P.

Proof: $\angle DAC = \angle ACP$ (alternate angles and $\overline{AD} \parallel \overline{CP}$)

$\angle BAD = \angle APC$ (corresponding angles)

But $\angle BAD = \angle DAC$ (given)

$\therefore\ \angle ACP = \angle APC$

In triangle APC,

AC = AP (sides opposite to equal angles are equal)

In triangle BCP,

$\frac{BD}{DC} = \frac{BA}{AP}$ (by basic proportionality theorem)

$\Rightarrow \frac{BD}{DC} = \frac{BA}{AC}$.

Hence proved.

Converse of vertical angle bisector theorem

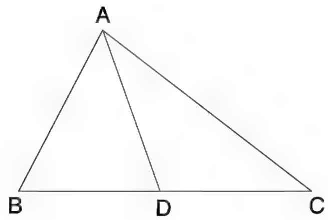


Figure 13.39

If a line that passes through a vertex of a triangle, divides the base in the ratio of the other two sides, then it bisects the angle.

In the adjacent figure, AD divides BC in the ratio $\frac{BD}{DC}$ and if $\frac{BD}{DC} = \frac{AB}{AC}$, then AD is the bisector of $\angle A$.

Concurrency–geometric centres of a triangle

Let us recall that if three or more lines pass through a fixed point, then those lines are said to be concurrent and that fixed point is called the point of concurrence. In this context, we recall different concurrent lines and their points of concurrence associated with a triangle, also called geometric centres of a triangle.

1. Circumcentre

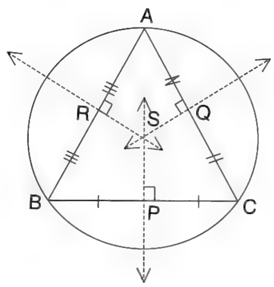


Figure 13.40

The locus of the point equidistant from the end points of the line segment is the perpendicular bisector of the line segment. The three perpendicular bisectors of the three sides of a triangle are concurrent and the point of their concurrence is called the circumcentre of the triangle and is usually denoted by S. The circumcentre is equidistant from all the vertices of the triangle. The circumcentre of the triangle is the locus of the point in the plane of the triangle, equidistant from the vertices of the triangle.

2. Incentre

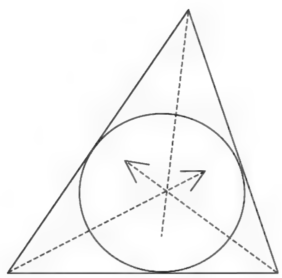


Figure 13.41

The angle bisectors of the triangle are concurrent and the point of concurrence is called the incentre and is usually denoted by I. I is equidistant from the sides of the triangle. The incentre of the triangle is the locus of the point, in the plane of the triangle, equidistant from the sides of the triangle.

3. Orthocentre

The altitudes of the triangle are concurrent and the point of concurrence of the altitudes of a triangle is called orthocentre and is usually denoted by O.

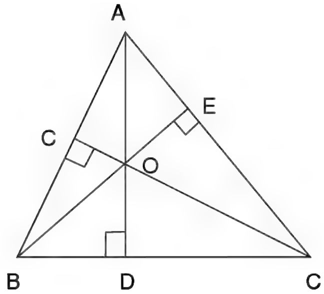


Figure 13.42

4. Centroid

The medians of a triangle are concurrent and the point of concurrence of the medians of a triangle is called the centroid and it is usually denoted by G. The centroid divides each of the medians in the ratio 2 : 1, starting from vertex, i.e., in the figure given below, AG : GD = BG : GE = CG : GF = 2 : 1.

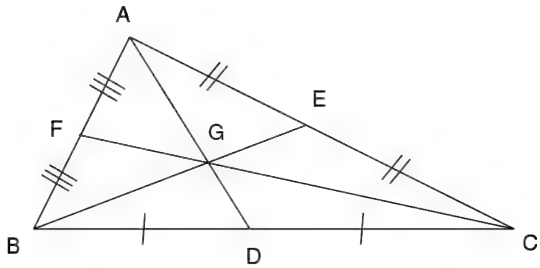


Figure 13.43

Some important points

(i) In an equilateral triangle, the centroid, the orthocentre, the circumcentre and the incentre all coincide.

(ii) In an isosceles triangle, the centroid, the orthocentre, the circumcentre and the incentre all lie on the median to the base.

(iii) In a right-angled triangle the length of the median drawn to the hypotenuse is equal to half of the hypotenuse. The median is also equal to the circumradius. The midpoint of the hypotenuse is the circumcentre.

(iv) In an obtuse-angled triangle, the circumcentre and orthocentre lie outside the triangle and for an acute-angled triangle the circumcentre and the orthocentre lie inside the triangle.

(v) For all triangles, the centroid and the incentre lie inside the triangle.

Circles

A circle is a set of points in a plane which are at a fixed distance from a fixed point.

The fixed point is the centre of the circle and the fixed distance is the radius of the circle.

In the above figure, O is the centre of the circle and OC is a radius of the circle.

AB is a diameter of the circle. OA and OB are also the radii of the circle.

The diameter is twice the radius.

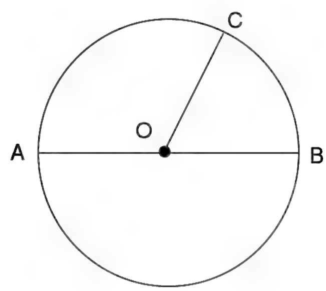


Figure 13.44

The centre of the circle is generally denoted by O, diameter by d and radius by r.
$d = 2r$

The perimeter of the circular line is called the circumference of the circle. The circumference of the circle is π times the diameter.
In the above circle with centre O; A, B and C are three points in the plane in which the circle lies. The points O and A are in the interior of the circle. The point B is located on the circumference of the circle.

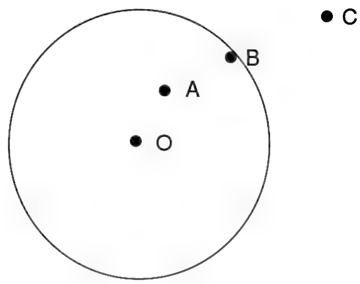


Figure 13.45

Hence, B belongs to the circle.

C is located in the exterior of the circle.

If $OB = r$, B is a point on the circumference of the circle.

As $OA < r$, A is a point in the interior of the circle.

As $OC > r$, C is a point in the exterior of the circle.

Chord

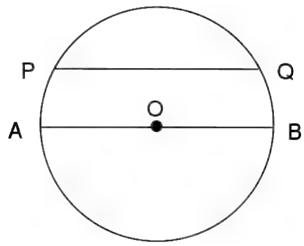


Figure 13.46

The line segment joining any two points on the circumference of a circle is a chord of the circle.
In the above figure, $\overline{PQ}$ and $\overline{AB}$ are the chords.

AB passes through centre O, hence, it is a diameter of the circle. A diameter is the longest chord of the circle. It divides the circle into two equal parts.

Theorem 1

One and only one circle exists through three non-collinear points.

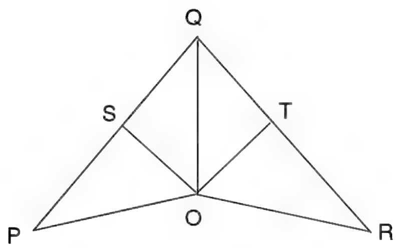


Figure 13.47

Given

P, Q and R are three non-collinear points.

RTP: One and only one circle passes through the points P, Q and R.

Construction

1. Join PQ and RQ and draw the perpendicular bisectors of PQ and RQ.

   Let them meet at the point O.
   Join $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$.
   In triangles OPS and OQS, $PS = SQ$
   $\angle OSP = \angle OSQ$ (right angles)
   $OS = OS$ (common)
   $\therefore$ $\Delta OSP \cong OSQ$. (SAS Congruence Property)
   $\therefore$ $OQ = OP$
   Similarly, it can be proved that $OQ = OR$.
   $\therefore$ $OP = OQ = OR$. Therefore, the circle with O as the centre and passing through P, also passes through Q and R.

2. The perpendicular bisectors of PQ and QR intersect at only one point and that point is O. A circle passing through P, Q and R has to have this point as the centre. Thus, there can only be one circle passing through P, Q and R.

Properties of chords and related theorems

Theorem 2

The perpendicular bisector of a chord of a circle passes through the centre of the circle.

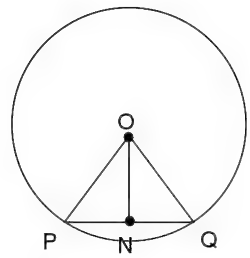


Figure 13.48

Given

PQ is a chord of a circle with centre O. N is the midpoint of chord PQ.

RTP: ON is perpendicular to PQ.

Construction

Join OP and OQ.

Proof

In triangles OPN and OQN,
OP = OQ (radius of the circle)
PN = QN (given)
ON is common.
By SSS Congruence Property,
$\Delta OPN \cong \Delta OQN$.
$\therefore \angle ONP = \angle ONQ$
But,
$\angle ONP + \angle ONQ = 180°$ (straight line)
$\Rightarrow \angle ONP = \angle ONQ = 90°$
$\therefore$ ON is perpendicular to chord PQ.

Note: The converse of the above theorem is also true i.e., the diameter which is perpendicular to a chord of a circle bisects the chord.

Theorem 3

Two equal chords of a circle are equidistant from the centre of the circle.

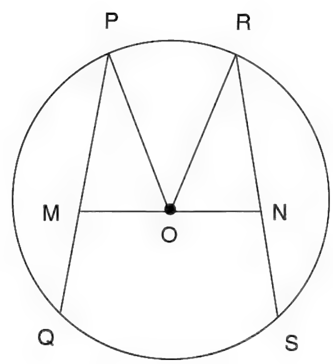


Figure 13.49

Given

In a circle with centre O, chord PQ = chord RS.
$OM \perp PQ$ and $ON \perp RS$.

RTP: OM = ON

Construction

Join OP and OR.

Proof

Since PQ is a chord of the circle and OM is perpendicular to PQ, OM bisects PQ.
(theorem 1).
Similarly, ON bisects RS.

PQ = RS (given)

$\frac{1}{2}$PM = $\frac{1}{2}$ RN

⇒ PM = RN

∠OMP = ∠ONR (Right angles)

OP = OR (radii of the circle)

In triangles OMP and ONR,

OP = OR,

PM = RN and ∠QMP = ∠ONR

∴ ΔOMP ≅ ΔONR (SAS axiom)

∴ OM = ON

Hence proved.

The converse of the above theorem, is also true, i.e., two chords which are equidistant from the centre of a circle are equal in length.

Note: Longer chords are closer to the centre and shorter chords are farther from the centre.

In the given figure, AB and CD are two chords of the circle with centre at O and AB > CD. OP is perpendicular to chord AB and OQ is perpendicular to chord CD. By observation, we can say that OQ > OP.

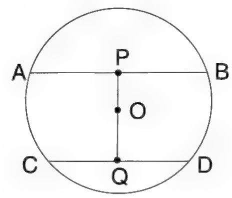


Figure 13.50

Example

AB and CD are two equal and parallel chords of lengths 24 cm each, in a circle of radius 13 cm. What is the distance between the chords?

Solution

In the above circle with centre O, AB and CD are the two chords each of length 24 cm and are equidistant from the centre of the circle. Therefore the distance between AB and CD =5 + 5 = 10 cm.

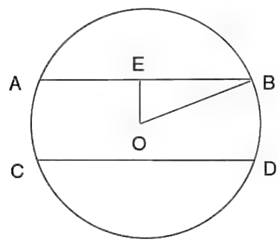


Figure 13.51

Angles subtended by equal chords at the centre

Theorem 4

Equal chords subtend equal angles at the centre of the circle.

Given

AB and CD are equal chords of a circle with centre O. Join OA, OB, OC and OD.

RTP: ∠AOB = ∠COD.

Proof

In triangles ABO and CDO,

OA = OC, (radii of the same circle)

OB = OD

AB = CD (given)

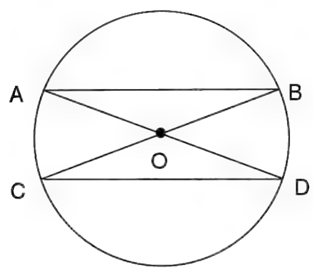


Figure 13.52

By SSS (side-side-side congruence property), triangles ABO and CDO are congruent. Hence, the corresponding angles are equal.

$\angle AOB = \angle COD$

Hence proved.

The converse of the theorem is also true, i.e., chords of a circle subtending equal angles at the centre are equal.

Angles subtended by an arc

Property 1

Angles subtended by an arc at any point on the rest of the circle are equal.

In the above circle, AXB is an arc of the circle. The angles subtended by the arc AXB at C and D are equal, i.e., $\angle ADB = \angle ACB$.

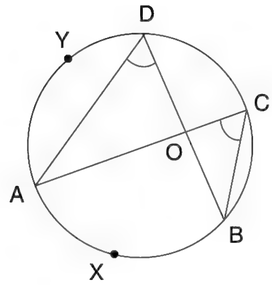


Figure 13.53

Property 2

Angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining part of the circle.

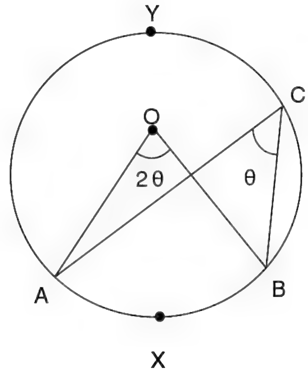


Figure 13.54

In the above figure, AXB is an arc of the circle. $\angle ACB$ is subtended by the arc AXB at the point C (a point on the remaining part of the circle), i.e., arc AYB. If O is the centre of the circle.
$\angle AOB = 2\ \angle ACB$.

Example

In the following figure, AB is an arc of the circle. C and D are the points on the circle.

If $\angle ACB = 30°$, find $\angle ADB$.

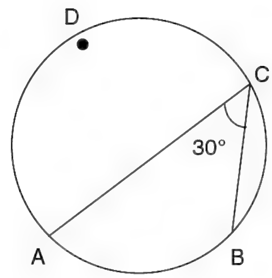


Figure 13.55

Solution

Angles made by an arc in the same segment are equal. Angles made by the arc in the segment ADCB are equal.

$\therefore\ \angle ADB = \angle ACB = 30°$.

☛ *Example*

In the following figure O is the centre of the circle.
AB is an arc of the circle, such that $\angle AOB = 80°$. Find $\angle ACB$.

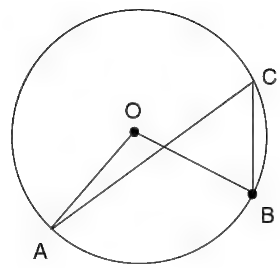


Figure 13.56

Solution

The angle made by an arc at the centre of a circle is twice the angle made by the arc at any point on the remaining part of the circle.

$\angle AOB = 2\,\angle ACB$

$\Rightarrow 2\,\angle ACB = 80°$

$\Rightarrow \angle ACB = 40°$

☛ *Example*

In the following figure, O is the centre of the circle. AB and CD are equal chords.

If $\angle AOB = 100°$, find $\angle CED$.

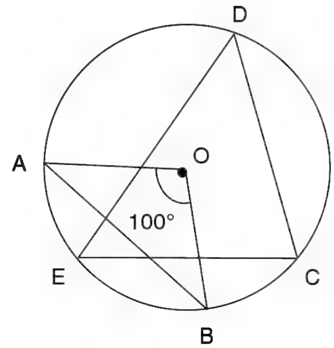


Figure 13.57

Solution

Equal chords subtend equal angles at the centre of the circle.

$\angle AOB = 100°$

$\Rightarrow \angle DOC = 100°$

(Angle subtended by an arc at the centre of the circle is twice the angle subtended by it anywhere in the remaining part of the circle)

$\therefore\ \angle DEC = \frac{1}{2}(100°) = 50°$

Cyclic quadrilateral

If all the four vertices of a quadrilateral lie on a circle, it is called a cyclic quadrilateral.

In the given figure, the four vertices, A, B, C and D of the quadrilateral ABCD lie on the circle. Hence ABCD is a cyclic quadrilateral.

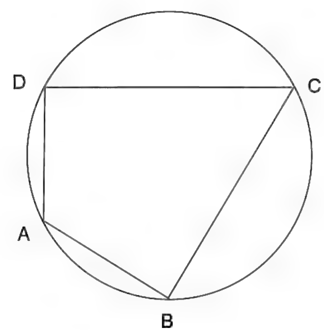


Figure 13.58

Theorem 5

The opposite angles of a cyclic quadrilateral are supplementary.

Given

ABCD is a cyclic quadrilateral.

RTP

$\angle A + \angle C = 180°$ or

$\angle B + \angle D = 180°$

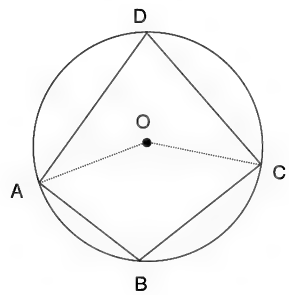


Figure 13.59

Construction

Join OA and OC, where O is the centre of the circle.

$\angle AOC = 2\angle ADC$

Angle subtended by arc ABC is double the angle subtended by arc ABC at any point on the remaining part of the circle and D is such a point.

Similarly,

Reflex $\angle AOC = 2\angle ABC$

$\angle AOC$ + Reflex $\angle AOC = 360°$

$2\angle ADC + 2\angle ABC = 360°$

$\Rightarrow \angle ADC + \angle ABC = 180°$

$\Rightarrow \angle B + \angle D = 180°$.

Similarly, it can be proved that $\angle A + \angle C = 180°$.

Theorem 6

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Given

ABCD is a cyclic quadrilateral.

Construction

Extend BC to X.

RTP: $\angle DCX = \angle BAD$.

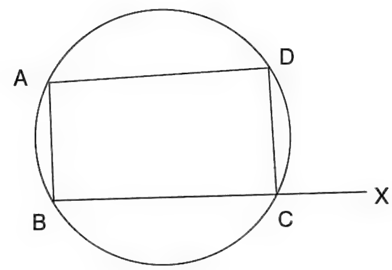


Figure 13.60

Proof

$\angle BAD + \angle BCD = 180°$ ------- (1)

(The opposite angles of a cyclic quadrilateral are supplementary).

$\angle BCD + \angle DCX = 180°$ ------ (2) (Angle of a straight line)

From (1) and (2), we get

$\angle BAD + \angle BCD = \angle BCD + \angle DCX$

$\Rightarrow \angle DCX = \angle BAD$.

Hence proved.

We are already familiar with what a circle is and studied some of its properties. Now, we shall study the properties of tangents and chords.

Tangents

When a line and a circle are drawn in the same plane we have the following cases.

1. The line and the circle may not intersect at all, as shown in the figure (a) below. This means the line and the circle do not meet.
2. The line may intersect the circle at two points as shown in figure (b).
3. The line may touch the circle at only one point as shown in figure (c).

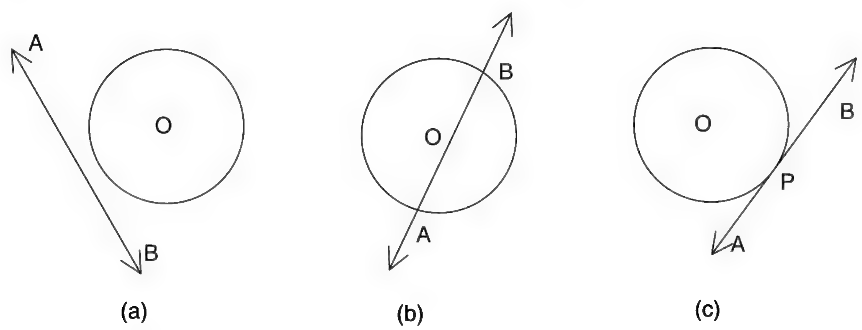


Figure 13.61

(i) If a line meets a circle at two points, then the line is called a secant of the circle.

(ii) When a line touches the circle at only one point, or a line meets a circle at only one point, then the line is called a tangent to the circle at that point and that point is called point of contact or the point of tangency.

(iii) At a point on a circle, only one tangent can be drawn.

(iv) From any given external point, two tangents can be drawn to a circle.

(v) From any point inside a circle, no tangent can be drawn to the circle.

Theorem 7

The tangent at any point on a circle is perpendicular to the radius through the point of contact.

Given

PQ is a tangent to a circle with centre O. Point of contact of the tangent and the circle is A.

RTP: OA is perpendicular to PQ.

Construction

Take another point B on PQ and join OB.

Proof:

Since B is a point other than A, B may lie inside or outside the circle.

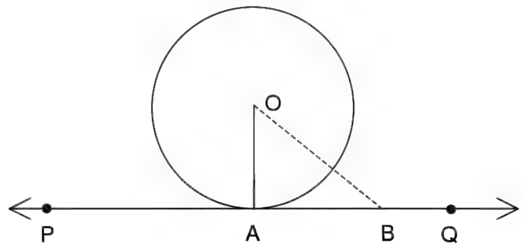


Figure 13.62

Case (i)

If point B lies inside the circle but on the line PQ, then PQ cannot be a tangent to the circle. Hence B does not lie inside the circle.

Case (ii)

If B lies outside the circle,

OB > OA, i.e., among all the segment joining a point on the line to the point O, OA is the shortest.

But, among all the line segments joining point O to a point on PQ, the shortest is the perpendicular line.

$\therefore$ OA is perpendicular to PQ.

Converse of the theorem

A line drawn through the end point of a radius of a circle and perpendicular to it is a tangent to the circle.

Theorem 8

Two tangents drawn to a circle from an external print are equal in length.

Given

PQ and PT are two tangents drawn from point P to circle with centre O.

RTP: PQ = PT

Construction

Join OP, OQ and OT.

Proof

In triangles OPQ and OPT,
OP = OP (Common)

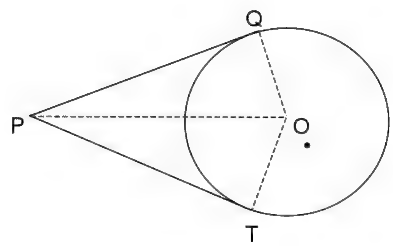


Figure 13.63

$\angle OQP = \angle OTP = 90°$

$OQ = OT$ (Radius)

The tangent is perpendicular to the radius at the point of tangency. By the SAS congruence axiom,

$\Delta OPQ \cong \Delta OPT$

$\Rightarrow PQ = PT$

NOTE: We also note that $\angle OPQ = \angle OPT$, i.e., the centre of a circle lies of the bisector on the angle formed by the two tangents.

Chords

A line joining any two points on a circle is a chord of the circle. If AB is a chord of a circle and P is a point on it, P is said to divide AB internally into two segments AP and PB. Similarly, if Q is a point on the line AB, outside the circle, S is said to divide AB externally into two segments AQ and QB.

Theorem 9

If two chords of a circle intersect each other, then the products of the lengths of their segments are equal.

Case (i)

Let the two chords intersect internally.

Given: AB and CD are two chords intersecting at point P in the circle.

RTP: (PA) (PB) = (PC) (PA)

Construction

Join AC and BD.

Proof

In triangles APC and PDB,

$\angle APC = \angle DPB$ (Vertically opposite angels)

$\angle CAP = \angle CDB$ (Angles made by arc BC in the same segment)

ΔAPC is similar to ΔDPB. (If two angles of one triangle are equal to the corresponding angles of another triangle, the two triangles are similar or the AA Similarity Property)

$$\therefore \frac{PA}{PD} = \frac{PC}{PB}$$

$\Rightarrow$ (PA) (PB) = (PC) (PD)

Case (ii)

Let the two chords intersect externally.

Given: Two chords BA and CD intersect at point P which lies outside the circle.

RTP: (PA)(PB) = (PC)(PD)

Construction: Join AC and BD.

Proof

In triangles PAC and PDB,

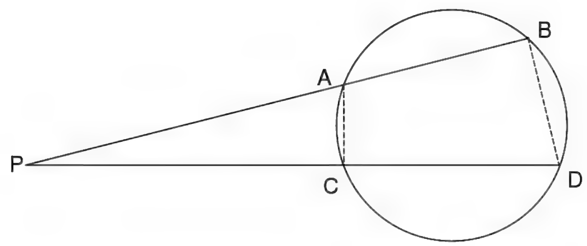


Figure 13.64

$\angle PAC = \angle PDB$ and $\angle PCA = \angle PBD$ (An external angle of a cyclic quadrilateral is equal to the interior angle at the opposite vertex.

$\frac{PA}{PD} = \frac{PC}{PB}$ (The AA Similarity Property)

$\Rightarrow$ (PA) (PB) = (PC) (PD).

NOTE:

1. The converse of the above theorem is also true, i.e., if two line segments AB and CD intersect at P and (PA)(PB) = (PC)(PD), then the four points are concyclic.
2. If one of the secants (say PCD) is rotated around P so that it becomes a tangent, i.e., points A and D say at T, coincide. We get the following result.

If PAB is a secant to a circle intersecting the circle at A and B and PT is the tangent drawn from P to the circle, then $PA \cdot PB = PT^2$

P is any point out side the circle with centre O and PAB is the secant drawn from P and PT is the tangent. Then

(PA) (PB) = PT^2

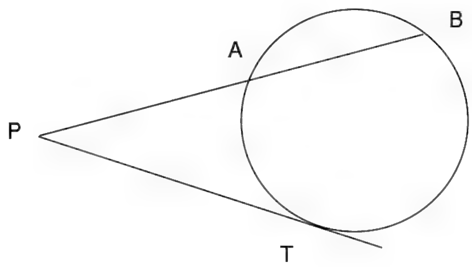


Figure 13.65

Alternate segment and its angles

AB is a chord in a circle with centre O. A tangent is drawn to the circle at A. Chord AB makes two angles with the tangents $\angle BAY$ and $\angle BAX$. Chord AB divides the circle into two segments ACB and ADB. The segments ACB and ADB are called alternate segments to angles $\angle BAY$ and $\angle BAX$ respectively.

Angles made by the tangent with the chord are $\angle BAY$ and $\angle BAX$. ACB and ADB are alternate segments to those angles respectively.

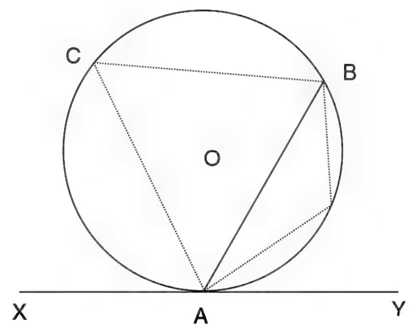


Figure 13.66

Theorem 10

Alternate segment theorem

If a line touches the circle at a point and if a chord is drawn from the point of contact then the angles formed between the chord and the tangent are equal to the angles in the alternate segments.

Given

XY is a tangent to the given circle with centre O at the point A, which lies in between X and Y. AB is a chord. C and D are points on the circle either side of line AB.

RTP: $\angle BAY = \angle ACB$ and $\angle BAX = \angle ADB$.

Construction

Draw the diameter AOP and join PB.

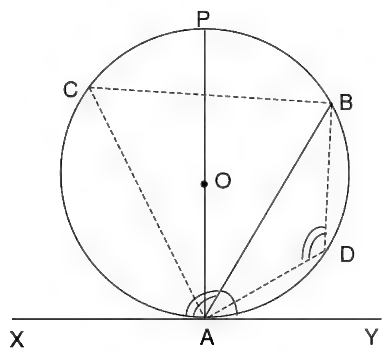


Figure 13.67

Proof

$\angle ACB = \angle APB$ (Angles in the same segment)

$\angle ABP = 90°$ (Angle in a semi-circle)

In the triangle ABP,

$\angle APB + \angle BAP = 90°$ - - - - - (1)

$\angle PAY = 90°$ (the radius makes a right angle with the tangent at the point of tangency).

$\Rightarrow \angle BAP + \angle BAY = 90°$ - - - - - (2)

From the equations (1) and (2),

$\angle APB = \angle BAY$

$\Rightarrow \angle ACB = \angle BAY.(\because \angle APB = \angle ACB)$

Similarly, it can be proved that

$\angle BAX = \angle ADB$

Converse of alternate segment theorem

A line is drawn through the end point of a chord of a circle such that the angle formed between the line and the chord is equal to the angle subtended by the chord in the alternate segment. Then, the line is tangent to the circle at the point.

Common tangents to circles

When two circles are drawn on the same plane with radii r_1 and r_2, with their centres d units apart, then we have the following possibilities.

1. The two circles are concentric, then $d = 0$. The points C_1 and C_2 coincide.

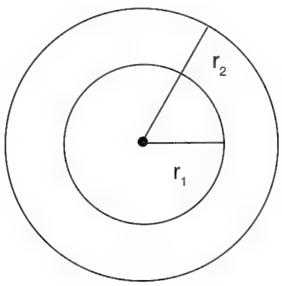


Figure 13.68

2. The two circles are such that one lies in side the other, then $|r_1 - r_2| > d$.

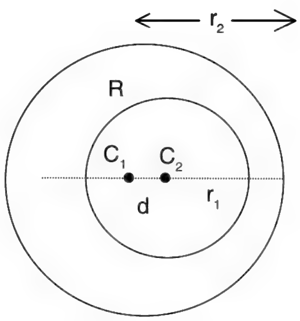


Figure 13.69

3. The two circles may touch each other internally, then $d = |r_1 - r_2|$

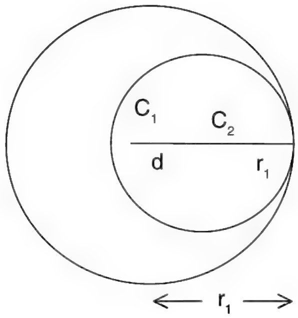


Figure 13.70

4. The two circles intersect at two points, in which case, $|r_1 - r_2| < d < r_1 + r_2$ and d

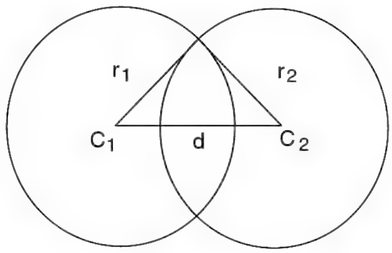


Figure 13.71

5. The two circle may touch each other externally, then $d = r_1 + r_2$

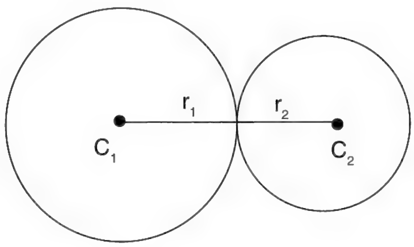


Figure 13.72

6. The two circles do not meet each other, then $d > r_1 + r_2$

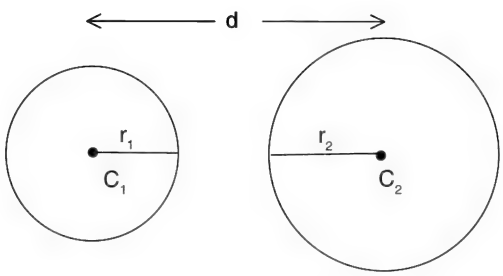


Figure 13.73

Common tangent

If the same line is tangent to two circles drawn on the same plane, then the line is called a common tangent to the circles. The distance between the point of contacts is called the length of the common tangent.

In the figure, PQ is a common tangent to the circles, C_1 and C_2. The length of PQ is the length of the common tangent.

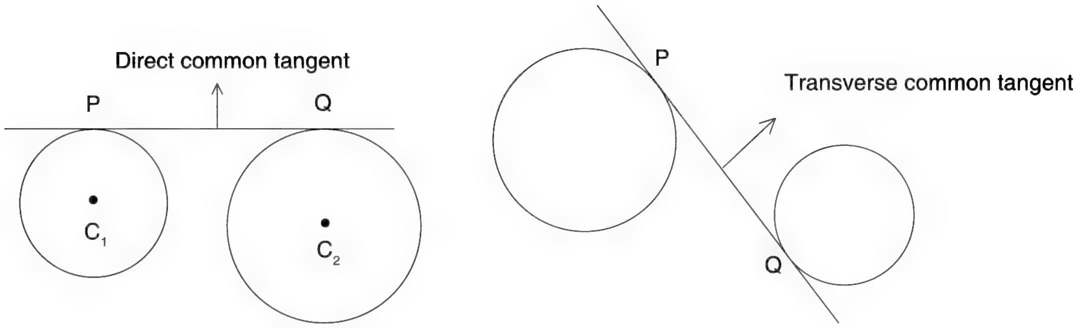


Figure 13.74

In figure (i), we observe that both the circles lie on the same side of PQ. In this case, PQ is a ***direct*** common tangent and in figure (ii), we notice that the two circles lie on either side of PQ. Here PQ is a ***transverse*** common tangent.

1. The number of common tangents to the circles one lying inside the other is zero.

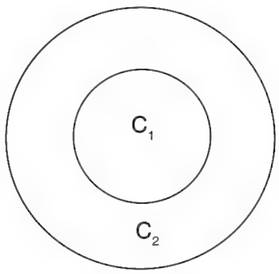


Figure 13.75

2. The number of common tangents to two circles touching internally is one.

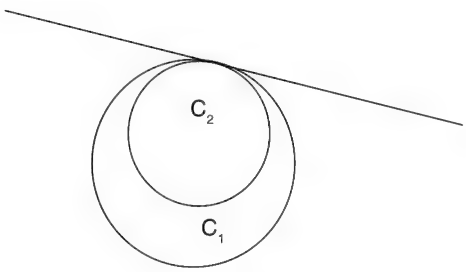


Figure 13.76

3. The number of common tangents to two intersecting circles is two, i.e., two direct common tangents.

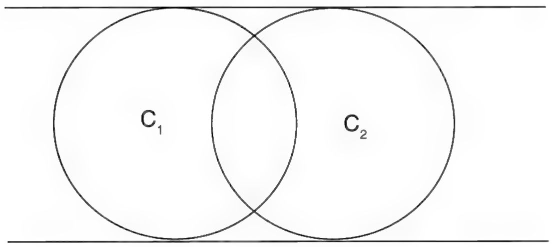


Figure 13.77

4. The number of common tangents to two circles touching externally is three, i.e., two direct tangents and one transverse tangent.

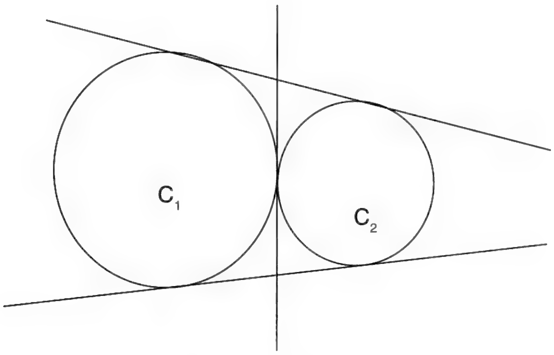


Figure 13.78

5. The number of common tangents to non-intersecting circles is four, i.e. 2 direct tangents and 2 transverse tangents.

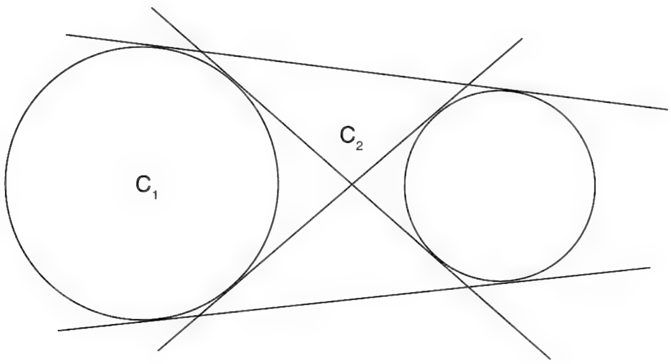


Figure 13.79

Properties of common tangents

1. When two circles touch each other internally or externally, then the line joining the centres is perpendicular to the tangent drawn at the point of contact of the two circles.

Case (i)

Two circles with centres C_1 and C_2 touch each other internally at P. $C_1\ C_2$ P is the line drawn through the centres and XY is the common tangent drawn at P which is common tangent to both the circles.

$\therefore$ $C_1\ C_2$ is perpendicular to XY

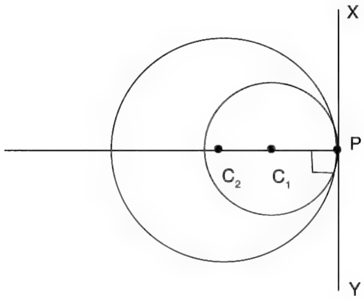


Figure 13.80

Case (ii)

The given two circles with centres C_1 and C_2 touch each other externally at P.

C_1PC_2 is the line joining the centres of the circles and XY is the common tangent to the two circles drawn at P.

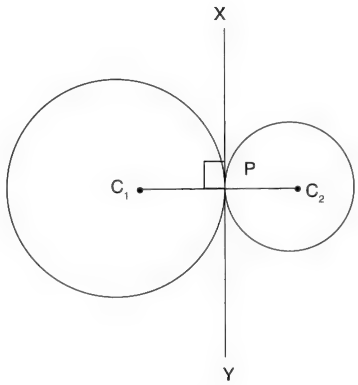


Figure 13.81

$\therefore$ C_1C_2 is perpendicular to XY.

2. The direct common tangents to two circles of equal radii are parallel to each other.

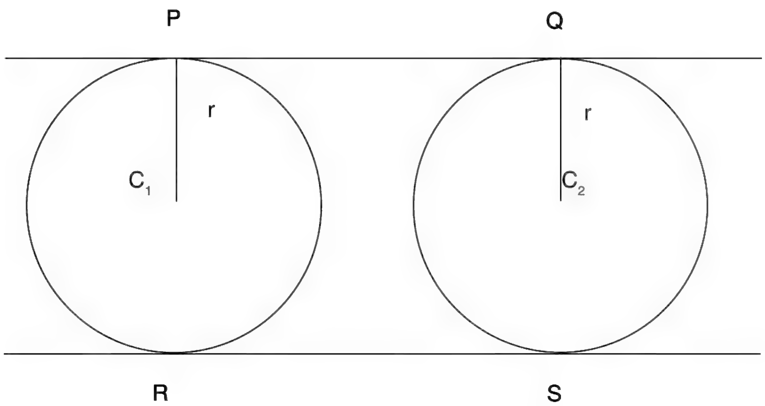


Figure 13.82

Let two circles of equal radii ‘r’ have centres C_1 and C_2 and PQ and RS be the direct common tangents drawn to the circles. Then PQ is parallel to RS.

☛ *Example*

In the figure, find the value of x.

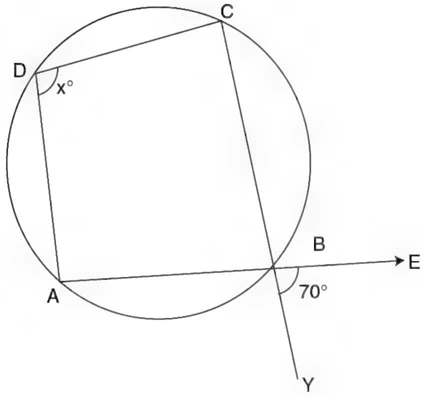


Figure 13.83

Solution

In the figure, $\angle CBE$ is an exterior angle which is equal to the opposite interior angle at the opposite vertex, $\angle ADC$.

$\therefore \angle CBE = \angle ADC \quad \rightarrow \quad (1)$

$\angle CBE + \angle EBY = 180°$ ($\because$ linear pair)

$\therefore \angle CBE = 180° - 70° = 110°$

$x° = \angle ADC = \angle CBE = 110°$.

Constructions related to circles

Construction 1: To construct a segment of a circle, on a given chord and containing a given angle.

☛ *Example*

Construct a segment of a circle on a chord of length 8.5 cm, containing an angle of 65° (θ)

Step 1: Draw a line segment BC of the given length, 8.5 cm.

Step 2: Draw $\overrightarrow{BX}$ and $\overrightarrow{CY}$ such that $\angle CBX = \angle BCY = \dfrac{180 - 2\theta}{2} = 25°$.

Step 3: Mark the intersection of $\overrightarrow{BX}$ and $\overrightarrow{CY}$ as O.

Step 4: Taking O as centre and OB or OC as radius, draw BAC

Step 5: In $\Delta BOC, \angle BOC = 130°$

$\Rightarrow \angle BAC = (1/2)\ \angle BOC = (1/2)\ (130°) = 65°$

$\angle BAC = 65°$.

The segment bounded by BAC and $\overline{BC}$ is the required segment.

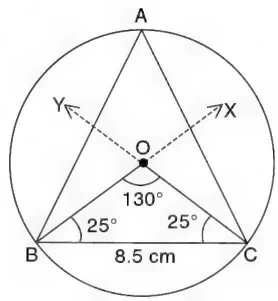


Figure 13.84

Construction 2: Construct an equilateral triangle inscribed in a circle of radius 3.5 cm.

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ such that $\angle AOB = \angle BOC = 120°$. ($\therefore \angle COA = 120°$)

Step 3: Join AB, BC and CA which is the required equilateral ΔABC in the given circle.

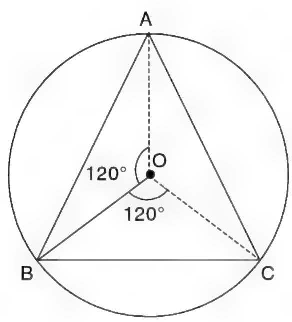


Figure 13.85

Construction 3: Construct an equilateral triangle circumscribed over a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm with centre O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ such that $\angle POR = \angle ROQ = 120°$. $(\therefore \angle QOP = 120°)$

Step 3: At P, Q and R draw perpendiculars to $\overline{OP}$, $\overline{OQ}$ and $\overline{OR}$ respectively to form ΔABC. ΔABC is the required circumscribing equilateral triangle.

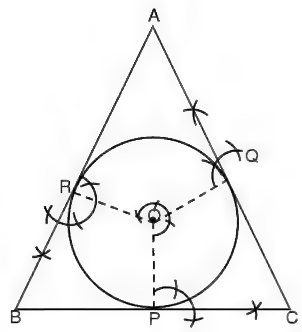


Figure 13.86

Construction 4: Draw the circumcircle of a given triangle.

Step 1: Draw ΔABC with the given measurements.

Step 2: Draw perpendicular bisectors of the two of the sides, say $\overline{AB}$ at S

Step 3: Taking S as centre, and the radius equal to AS or BS or CS, draw a circle. The circle passes through all the vertices A, B and C of the triangle.

$\therefore$ The circle drawn is the required circumcircle.

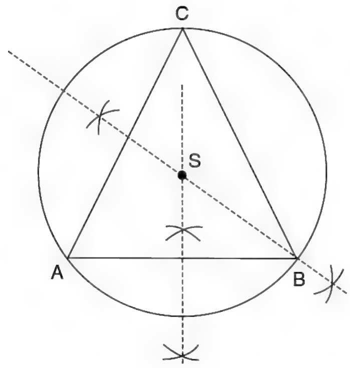


Figure 13.87

Note: The circumcentre is equidistant from the vertices of the triangle.

Construction 5: Construct the incircle of a given triangle ABC.

Step 1: Draw a ΔABC with the given measurements.

Step 2: Draw bisectors of two of the angles, say $\angle B$ and $\angle C$ to intersect at I.

Step 3: Draw perpendicular $\overline{IM}$ from I onto $\overline{BC}$.

Step 4: Taking I as centre and IM as the radius, draw a circle.

This circle touches all the sides of the triangle. This is the incircle of the triangle.

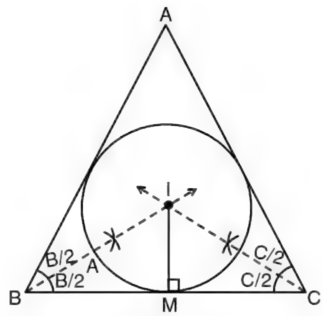


Figure 13.88

Note: The incentre is equidistant from all the sides of the triangle.

Construction 6: Construct a square inscribed in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm and mark centre as 'O'.

Step 2: Draw diameters $\overline{AC}$ and $\overline{BD}$ such that $\overline{AC} \perp \overline{BD}$.

Step 3: Join A, B, C and D. Quadrilateral ABCD is the required square inscribed in the given circle.

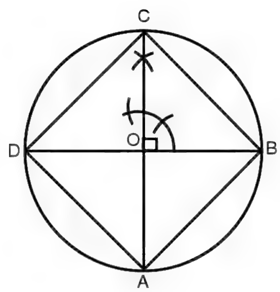


Figure 13.89

Construction 7: Construct a square circumscribed over the given circle of radius 2.5 cm.

Step 1: Draw a circle of radius 2.5 cm.

Step 2: Draw two mutually perpendicular diameters $\overline{PR}$ and $\overline{SQ}$.

Step 3: At P, Q, R and S, draw lines perpendicular to OP, OQ, OR and OS respectively to form square ABCD as shown in the figure.

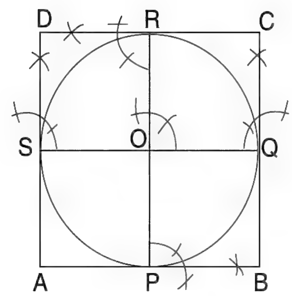


Figure 13.90

Construction 8: Construct a regular pentagon of side 4cm. Circumscribe a circle to it.

Steps

1. Draw a line segment, AB = 4 cm.
2. Draw $\overrightarrow{BX}$ such that $\angle ABX = 108°$.
3. Mark the point C on $\overrightarrow{BX}$ such that BC = 4 cm.
4. Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ and mark their instersecting point as O.
5. With O as centre and OA as radius draw a circle. And it passes through the points B and C.
6. With C as centre and 4cm as radius draw an arc which cuts the circle at the point D.
7. With D as centre and 4cm as radius draw an arc which cuts the circle at the point E.
8. Join CD, DE and AE.
9. ABCDE is the required pentagon.

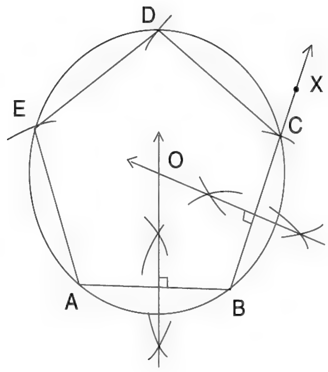


Figure 13.91

Construction 9: Construct a regular pentagon in a circle of radius 4cm.

Steps

1. Construct a circle with radius 4cm.
2. Draw two radii $\overline{OA}$ and $\overline{OB}$ such that $\angle AOB = 72°$
3. Join AB.
4. With A as centre and AB as radius draw an arc which cuts the circle at the point E.
5. With E as centre and AB as radius draw an arc which cuts the circle at the point D.
6. With D as centre and AB as radius draw an arc which cuts the circle at the point C.
7. Join AE, ED, DC and CB.
8. ABCDE is the required pentagon.

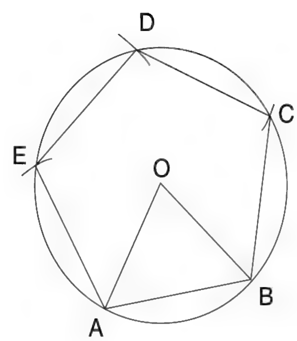


Figure 13.92

Construction 10: Construction regular pentagon about a circle of radius 3.5cm.

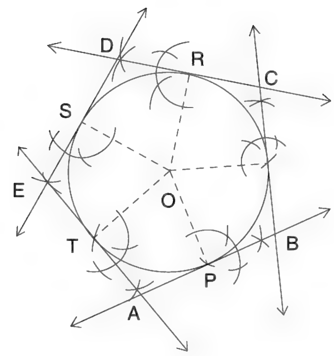


Figure 13.93

Steps

1. Construct a circle with radius 3.5cm.
2. Draw radii OP, OQ, OR, OS and OT such that the angle between any two consecutive radii is 72°.
3. Draw tangents to the circle at the points P, Q, R, S and T.
4. Mark the intersecting points of the tangents as A, B, C, D and E.
5. ABCDE is the required pentagon.

Construction 11: Construct a regular hexagon circumscribed over a circle of radius 3.5 cm.

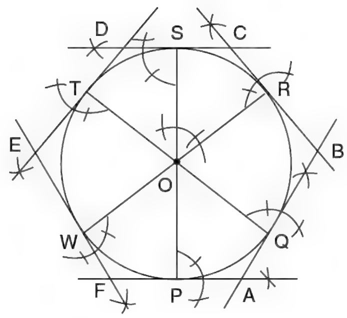


Figure 13.94

Step 1: Draw a circle of radius 3.5 cm and mark its centre as O.

Step 2: Draw radii $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ such that the angle between any two adjacent radii is 60°.

Step 3: Draw lines at P, Q, R, S, T and U perpendicular to $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$ and $\overline{OU}$ respectively, to form the required circumscribed hexagon ABCDEF.

Construction 12: Inscribe a regular hexagon in a circle of radius 3 cm.

Step 1: Draw a circle of radius 3 cm taking the centre as O

Step 2: Draw radius $\overline{OA}$. With radius, equal to OA and starting with A as the centre, mark points B, C, D E and F one after the other.

Step 3: Join A, B, C, D, E and F. Polygon ABCDEF is the required hexagon.

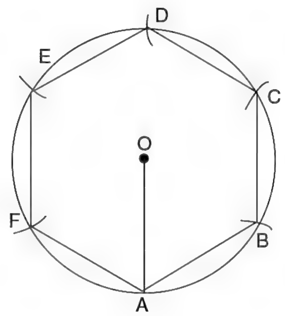


Figure 13.95

Construction 13

1. **To construct a tangent to a circle at a point on it.**
   (a) When the location of the centre is known.

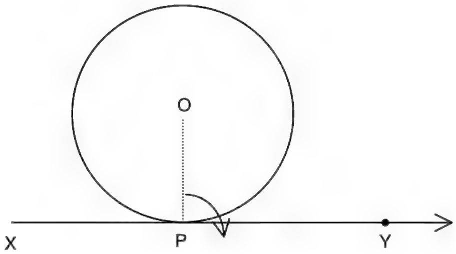


Figure 13.96

Step 1: Draw a circle with centre O with any radius. Let P be a point on the circle.

Step 2: Join OP

Step 3: Construct $\angle OPY = 90°$

Step 4: Produce YP to X and XPY is the required tangent to the given circle at point P. (b) When the location of the centre is not known.

2. **We have a circle, but we do not know where the centre is located.**

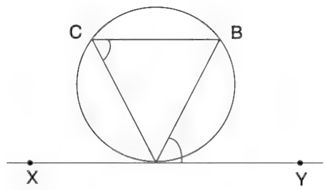


Figure 13.97

Step 1: Draw a chord AB.

Step 2: Mark point C on major arc ACB. Join BC and AC.

Step 3: Draw $\angle BAY = \angle ACB$

Step 4: Produce YA to X as shown in the figure. XAY is the required tangent at A.

Construction 14: To construct the two tangents to a circle with centre at O, from an external point P.
Draw a circle with OP as diameter.

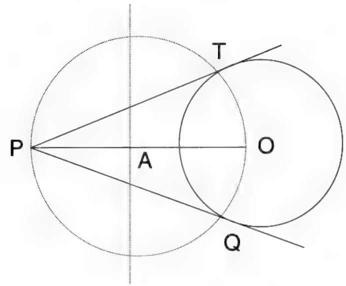


Figure 13.98

(a) When the location of the centre is known.

Step 1: Bisect OP at A. With centre A and radius equal to OA or AP, draw a circle that intersects the given circle at two points T and Q.

Step 2: Join PT and PQ, which are the required tangents from PQ.

(b) When the location of the centre is not known.

Step 1: Draw a circle of radius 2·5 cm. Mark a point P outside the circle.
Step 2: Draw a secant PQR through 'P' to intersect the circle at Q and R.
Step 3: Produce QP to S, such that PS = QP
Step 4: With SR as diameter, construct a semi-circle.
Step 5: Construct a line perpendicular to SR at P to intersect the semi-circle at U.
Step 6: Draw arcs with PU as radius and P as the centre to intersect the given circle at T and T'
Step 7: Join PT and PT'. These are the required tangents from P to the given circles.

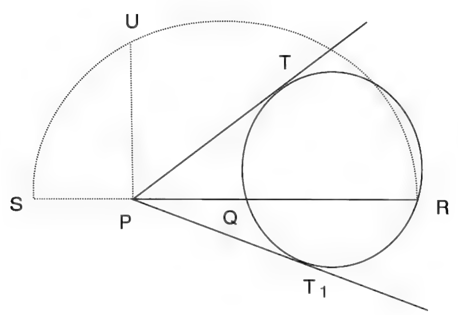


Figure 13.99

Construction 15: To construct the direct common tangents to the circles of radii 3·5 cm and 1·5 cm whose centres are 6·5 cm apart.

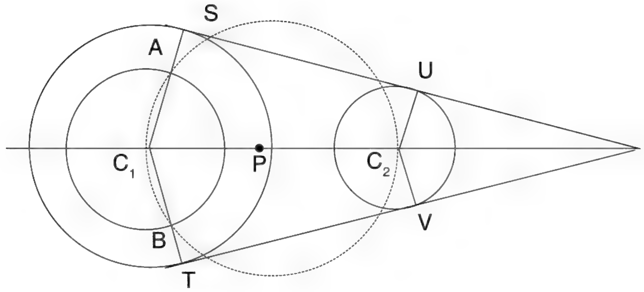


Figure 13.100

Step1: Draw line segment, $C_1C_2 = 6{\cdot}5$ cm

Step 2: Draw circles with radii 3·5 cm and 1·5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2 at P and draw a circle taking P as centre and radius equal to PC_1 or PC_2.

Step 4: Draw another circle with radius equal to $(3{\cdot}5 - 1{\cdot}5) = 2$ cm taking the centre of larger circle C_1 intersecting the circle drawn in step 3 at A and B.

Step 5: Join C_1A and C_1B and produce C_1A and C_1B to meet the outer circle at S and T respectively.

Step 6: Draw line segments C_2U and C_2V, such that $C_2U \parallel C_1S$ and $C_2V \parallel C_1T$, where U,V are points on the circle with centre C_2.

Step 7: Join SU and TV. These are the required direct common tangents.

Note: The length of a direct common tangent to two circles with radii r_1 and r_2 and centres d units apart, is given by $\sqrt{d^2 - (r_1 - r_2)^2}$.

Construction 16: To construct the transverse common tangents to two circles with radii 2.5 cm and 1.5 cm, with centres at a distance of 6 cm from each other.

Step 1: Draw a line segment, $C_1C_2 = 6$ cm.

Step 2: Draw circles of radii 2·5 cm and 1·5 cm respectively with C_1 and C_2 as centres.

Step 3: Bisect C_1C_2. Let M be the mid point of C_1C_2

Step 4: Draw a circle with M as centre and C_1M or C_2M as radius.

Step 5: With C_2 as centre and radius equal to $(2{\cdot}5 + 1{\cdot}5)$ or 4 cm mark off 2 points A and B on the circle in step 4.

Step 6: Join C_2A and C_2B to meet the other circle with centre C_2 at S and T respectively.

Step 7: Draw lines, C_1U and C_1V such that $C_1U \parallel C_2B$ and $C_1V \parallel C_2S$, where U, V are points on the circle with centre at C_1.

Step 8: Join TU and SV. These are the required transverse common tangents.

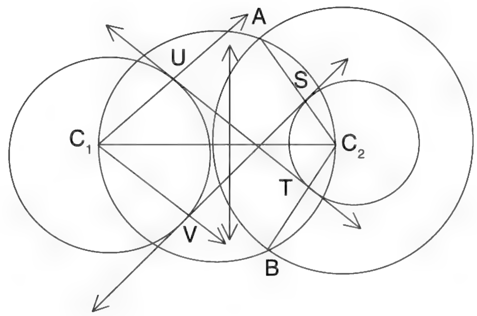


Figure 13.101

Note:

1. The length of the transverse common tangent of two circles with radii r_1 and r_2 and centres at a distance d $(d > r_1 + r_2)$ is $\sqrt{d^2 - (r_1 + r_2)^2}$.

2. Transverse common tangents can be drawn to non intersecting and non touching circles only or to the circles that satisfy the condition, $d > r_1 + r_2$.

Construction 17: To construct a circle of given radius passing through two given points.

Example

Construct a circle of radius 2·5 cm passing through two points A and B 4 cm apart.

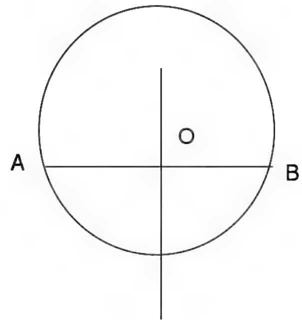


Figure 13.102

Step 1: Draw a line segment, AB = 4 cm.

Step 2: Bisect $\overline{AB}$ by drawing a perpendicular bisector of $\overline{AB}$.

Step 3: Draw an arc with A or B as centre and with radius 2·5 cm which intersect the perpendicular bisector of $\overline{AB}$ at O.

Step 4: Draw a circle that passes through A and B with O as centre and radius 2·5 cm.

∴ This is the required circle with given radius and passing through the given points A and B.

Construction 18: To draw a circle touching a given line AB at a given point P and passing through a given point Q.

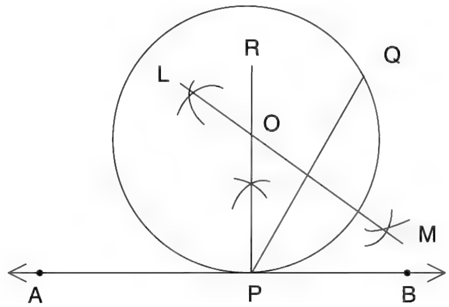


Figure 13.103

Step 1: Draw a line AB and mark point P on it.

Step 2: Draw PR at P such that PR ⊥ AB i.e., ∠APR = 90°.

Step 3: Join PQ and draw the perpendicular bisector of PQ to intersect PR at O

Step 4: Draw a circle. The circle touches line AB at P and passes through the given point Q with O as centre and OP as radius.

∴ The circle drawn is the required circle.

Construction 19: To construct a circle with a given radius and touching the two given intersecting lines.

Example

Construct an angle of 70° and draw a circle of radius 2·5 cm touching the arms of the angle.

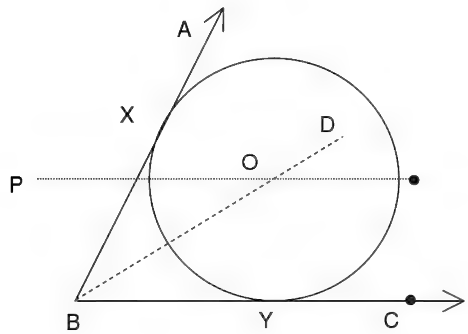


Figure 13.104

Step 1: Draw an angle ∠ABC = 70°.

Step 2: Draw BD, the bisector of the angle ABC.

Step 3: Draw a line $\overline{PD}$ // $\overline{BC}$ at a distance of 2·5 cm (equal to the radius of the circle) to intersect the angle bisector of step 2 at O.

Step 4: Draw a circle with O as centre and radius 2·5 cm

∴ The circle drawn touches the lines at X and Y.

Construction 20: To construct a circle of given radius which touches a given circle and a given line.

Example

Draw a circle of radius 3 cm which touches a given line AB and a given circle of radius 2 cm with centre C.

Step 1: Let AB be the given line.

Step 2: Draw a line PQ parallel to $\overline{AB}$ at a distance 3 cm from $\overline{AB}$.

Step 3: Draw an arc with C as centre and with radius equal to (2 + 3) = 5 cm to cut PQ at point O.

Step 4: Draw a circle which touches the given circle at E and the given line AB at F with O as centre and radius 3 cm.

∴ The circle drawn in step 4 is the required circle.

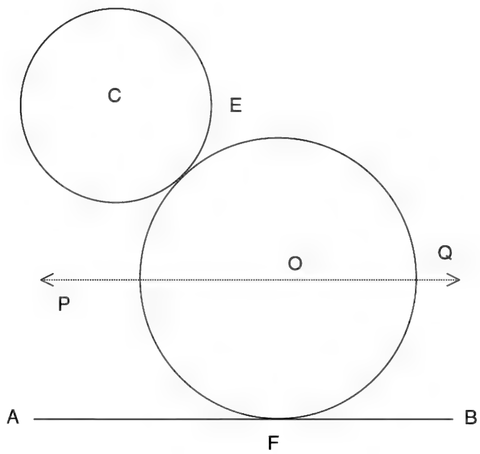


Figure 13.105

Construction 21: Construct a triangle ABC in which BC = 6 cm, ∠A = 60° and altitude through A is 4 cm.

Steps

1. Draw a line segment, BC = 6 cm.
2. Draw $\overrightarrow{BP}$ such that ∠CBL = 60°
3. Draw $\overrightarrow{BM}$ such that ∠MBL = 90°
4. Draw a perpendicular bisector $\left(\overleftrightarrow{XY}\right)$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point P.
5. Mark the intersecting point of $\overleftrightarrow{XY}$ and $\overrightarrow{BM}$ as O.
6. With O as centre and OB as radius draw a circle.
7. Mark the point N on $\overleftrightarrow{XY}$ such that PN = 4cm.
8. Through N, draw a line parallel to $\overrightarrow{BC}$ which intersects the circle at points A and A^1.
9. Join AB, AC and A'B, A'C.
10. Now, ABC or A'BC is required triangle.

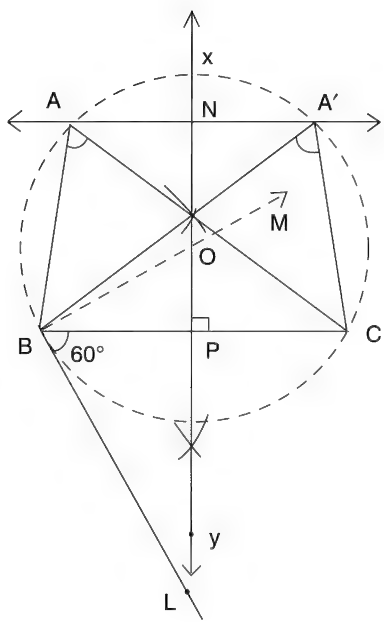


Figure 13.106

Proof

By alternate segment theorem, ∠BAC = ∠CBL = ∠BA'C = 60°

Altitude through A = Altitude through A' = PN = 4 cm

Construction 22: Construct a triangle ABC in which BC = 7 cm, $\angle A = 65°$ and median AT is 5 cm.

Steps

1. Draw a line segment, BC = 7 cm.
2. Draw $\overrightarrow{BP}$ such that $\angle CBP = 65°$.
3. Draw $\overrightarrow{BQ}$ such that $\angle PBQ = 90°$.
4. Draw a perpendicular bisector $\left(\overleftrightarrow{RS}\right)$ of $\overline{BC}$ which intersects $\overline{BC}$ at the point T.
5. Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{BQ}$ as O.
6. With O as centre and OB as radius draw circle.
7. With T as centre 5 cm as radius draw two arcs which intersect the circle at the points A and A'.
8. Join AB and AC and A' and A'C.
9. Now, ABC or A'BC is the required triangle.

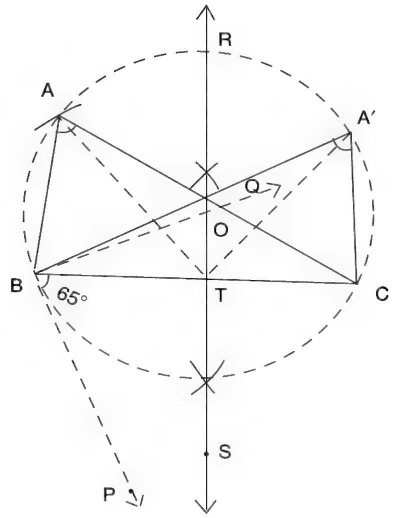


Figure 13.107

Proof

By alternate segment theorem, $\angle BAC = \angle CBP = \angle BA'C = 65°$

Construction 23: Construct a cyclic quadrilateral ABCD in which AB = 3 cm, AD = 4 cm, AC = 6 cm and $\angle D = 70°$.

Rough figure

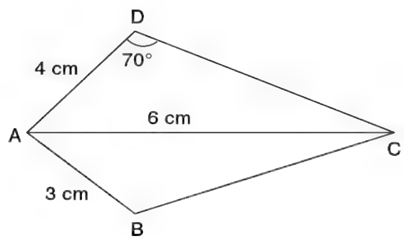


Figure 13.108

Steps

1. Draw a line segment, AC = 6 cm.
2. Draw $\overrightarrow{AP}$ such that $\angle CAP = 70°$.
3. Draw $\overrightarrow{AQ}$ such that $\angle PAQ = 90°$
4. Draw a perpendicular bisectors $\left(\overleftrightarrow{RS}\right)$ of $\overline{AC}$ which intersects $\overline{AC}$ at the point T.
5. Mark the intersecting point of $\overleftrightarrow{RS}$ and $\overrightarrow{AQ}$ as O.
6. With O as centre and OA as radius draw a circle.
7. With A as centre, 4cm as radius draw an arc which intersects the circle at the point D.
8. With A as centre, 3cm as radius draw an arc which intersects the circle at the point B.
9. oin AB, BC, CD and AD.

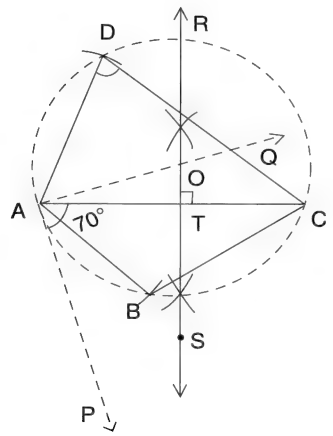


Figure 13.109

Construction 24: Find the mean proportional of the given segments p cm and q cm.

Steps

1. Draw $\overrightarrow{XZ}$
2. Mark the point Y on $\overrightarrow{XZ}$ such that XY = p cm and YZ = q cm.
3. Draw the perpendicular bisector $\left(\overleftrightarrow{AB}\right)$ of $\overline{XZ}$ which meets $\overline{XZ}$ at the point O.
4. With O as centre and OX as radius draw a semi circle. $\overline{XZ}$ is the diameter of the semi circle.
5. Draw $\overline{YM}$ perpendicular to $\overline{XZ}$ which meets semicircle at the point M.
6. Now MY, is the mean proportional of XY and YZ i.e., p and q.

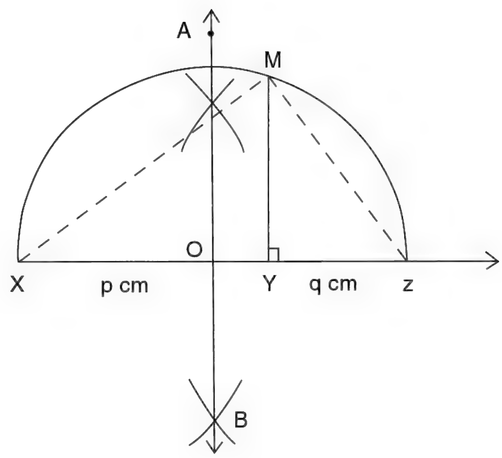


Figure 13.110

Proof

$\Delta XYM \sim \Delta MYZ$

$\Rightarrow \dfrac{XY}{YM} = \dfrac{MY}{YZ}$

$\Rightarrow MY = \sqrt{(XY)\,(YZ)} = \sqrt{pq}$

Uses

By using this construction, square root ($\sqrt{12}$, $\sqrt{15}$ ....) can be found.

Construction 25: Construct a square whose area is equal to the area of the rectangle.

Steps

1. Produce $\overline{AB}$ of rectangle ABCD to the point E such that BE = BC.
2. Construct mean proportional (BF) of AB and BE.
3. Construct a square with side BF.
4. BFGH is the required square.

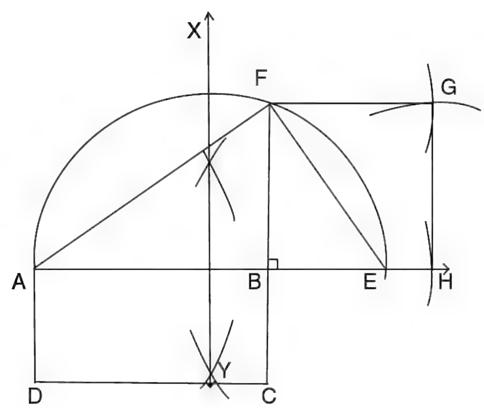


Figure 13.111

Proof

BF is the mean proportional of AB and BE.

$\Rightarrow (BF)^2 = AB \times BE$

$\Rightarrow (BF)^2 = AB \times BC \;(\because BC = BE)$

$\Rightarrow$ Area of square BFGH = Area of rectangle ABCD.

Construction 26: Construct a triangle similar to a given triangle ABC with its sides equal to $\left(\frac{5}{9}\right)$th of the corresponding sides of ΔABC.

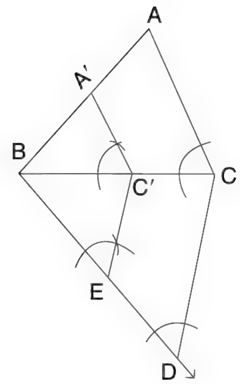


Figure 13.112

Steps

1. ABC is the given triangle.
2. Draw $\overrightarrow{BD}$ which makes non-zero angle with $\overline{BC}$ and D is 9 cm away from the point B.
3. Mark the point E on $\overrightarrow{BD}$ such that BE = 5 cm.
4. Join $\overline{CD}$.
5. Draw $\overline{EC'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.
6. Draw $\overline{C'A'}$ parallel to $\overline{CA}$ which meets $\overline{BA}$ at the point A^1.
7. A'B'C' is the required triangle.

Proof

In ΔBCD, $\frac{BE}{BD} = \frac{5}{9}$ and $\overline{DC} \,||\, \overline{EC'}$

$\therefore$ ΔBEC' ~ ΔBDC

$\therefore \frac{BC'}{BC} = \frac{5}{9}$

In ΔBCA, $\frac{BC'}{BC} = \frac{5}{9}$ and $\overline{C'A} \,||\, \overline{CA}$

$\therefore$ ΔBCA ~ ΔBC'A'

$\therefore \frac{BC'}{BC} = \frac{BA'}{BA} = \frac{A'C'}{AC} = \frac{5}{9}$

Construction 27: Construct a quadrilateral similar to a given quadrilateral ABCD with its sides equal to $\left(\frac{6}{10}\right)$th of the corresponding sides of ABCD.

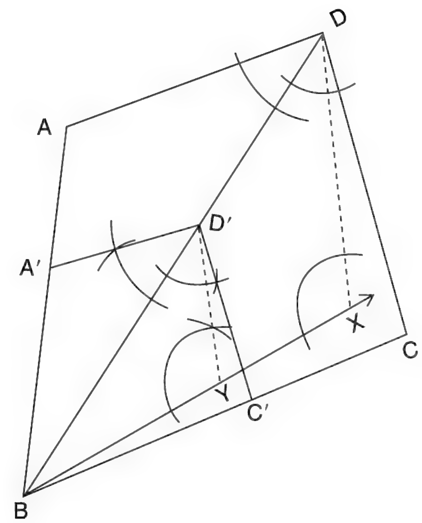


Figure 13.113

Steps

1. ABCD is the given quadrilateral.
2. Join BD.
3. Draw $\overrightarrow{BX}$ which makes non zero angle with $\overline{BD}$ and X is 10 cm away from the point B.
4. Mark the point Y on $\overrightarrow{BX}$ such that BY = 6 cm.

5. Join XD.
6. Draw $\overline{YD'}$ parallel to $\overline{XD}$ which meets BD at the point D'.
7. Draw $\overline{D'A'}$ parallel to $\overline{DA}$ which meets $\overline{AB}$ at the point A'.
8. Draw $\overline{D'C'}$ parallel to $\overline{DC}$ which meets $\overline{BC}$ at the point C'.
9. A'BC'D' is the required quadrilateral.

Construction 28: Construction of a pentagon similar to the given pentagon on a side of given length.

Steps

1. ABCDE is a given pentagon and $\overline{XY}$ is the given line segment.
2. Mark the point B' on $\overline{AB}$ such that AB'= XY
3. Join AC and AD.
4. Draw $\overline{B'C'}$ parallel to $\overline{BC}$ which meets $\overline{AC}$ at the point C'.
5. Draw $\overline{C'D'}$ parallel to $\overline{CD}$ which meets $\overline{AD}$ at the point D'.
6. Draw $\overline{D'E'}$ parallel to $\overline{DE}$ which meets AE at the point E'.
7. AB'C'D'E' is the required pentagon.

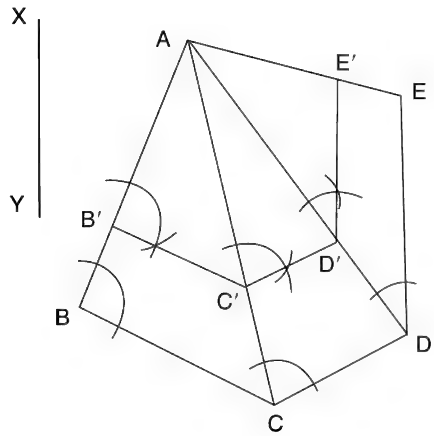


Figure 13.114

Locus

Mark a fixed point O on a sheet of paper. Now, start marking points P_1, P_2, P_3, P_4,....... on the sheet of paper such that $OP_1 = OP_2 = OP_3 = \ldots\ldots\ldots = 4$ cm. What do we observe on joining these points by a smooth curve? We observe a pattern, which is circular in shape such that any point on the circle obtained is at a distance of 4 cm from the point O.

It can be said that whenever a set of points satisfying a certain condition are plotted, a pattern is formed. This pattern formed by all possible points satisfying the given condition is called the locus of points. In the above given example, we have a locus of points which are equidistant (4 cm) from the given point O.

The collection (set) of all points and only those points which satisfy certain given geometrical conditions is called **locus of a point.**

Alternatively, locus can be defined as the path or curve traced by a point in a plane when subjected to some geometrical conditions.

Consider the following examples.

1. The locus of the point in a plane which is at a constant distance 'r' from a fixed point 'O' is a circle with centre O and radius r units.
2. The locus of the point in a plane which is at a constant distance from a fixed straight line is a pair of lines, parallel to the fixed line. Let the fixed line be ℓ. The lines *m* and *n* form the set of all points which are at a constant distance from ℓ.

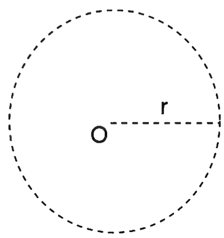


Figure 13.115

3. The locus of a point in a plane, which is equidistant from a given pair of parallel lines is a straight line, parallel to the two given lines and lying midway between them.

In the given above, m and n are the given lines and line ℓ is the locus.

To prove that a given path or curve is the desired locus, it is necessary to prove that

(i) every point lying on the path satisfies the given geometrical conditions.

(ii) every point that satisfies the given conditions lies on the path.

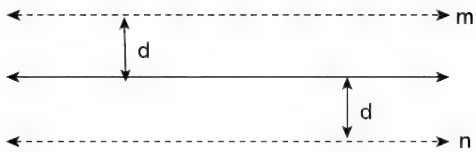


Figure 13.116

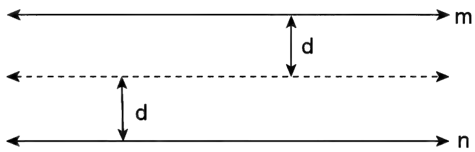


Figure 13.117

Worked out examples

Examples

1. Show that the locus of a point equidistant from the end points of a line segment is the perpendicular bisector of the segment.

Solution

The proof will be taken up in two steps.

Step I: We, initially prove that the any point equidistant from the end points of a line segment lies on the perpendicular bisector of the line segment.

Given: M and N are two points on a plane. A is a point in the same plane such thatAM = AN.

RTP: A lies on the perpendicular bisector of MN.

Proof: Let M and N be the two fixed points in a plane.

Let A be a point such that AM = AN and L be the mid-point of $\overline{MN}$.

If A coincides with L, then A lies on the bisector of MN.

Suppose A is different from L.

Then, in ΔMLA and ΔNLA,

ML = NL, AM = AN and MP is a common side.

∴ By SSS congruence property, ΔMLA ≅ ΔNLA.

⇒ ∠MLA = ∠NLA (∵ corresponding elements of congruent triangles are equal) --- (1)

But ∠MLA + ∠NLA = 180° (∵ They form a straight angle)

⇒ 2 ∠MLA = 180° (using (1))

∴ ∠MLA = ∠NLA = 90°

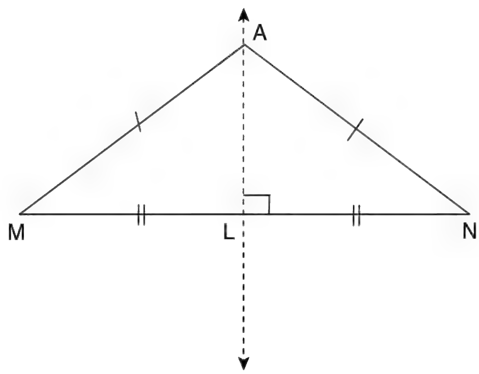


Figure 13.118

So, $\overline{AL} \perp \overline{MN}$ and hence $\overline{AL}$ is the perpendicular bisector of $\overline{MN}$.

∴ A lies on the perpendicular bisector of $\overline{MN}$.

Step II: Now, we prove that any point on the perpendicular bisector of the line segment is equidistant from the end points of the line segment.

Given: MN is a line segment and P is a point on the perpendicular bisector. L is the mid-point of MN.

RTP: MP = NP.

Proof: If P coincides with L, then MP = NP.

Suppose P is different from L. Then, in ΔMLP and NLP,

ML = LN

LP is the common side and

∠MLP = ∠NLP = 90°

∴ By the SAS congruence property, ΔMLP ≅ ΔNLP

So, MP = PN (∵ The corresponding elements of congruent triangles are equal)

∴ Any point on the perpendicular bisector of $\overline{MN}$ is equidistant from the points M and N.

Hence, from the steps I and II of the proof it can be said that the locus of the point equidistant from two fixed points is the perpendicular bisector of the line segment joining the two points.

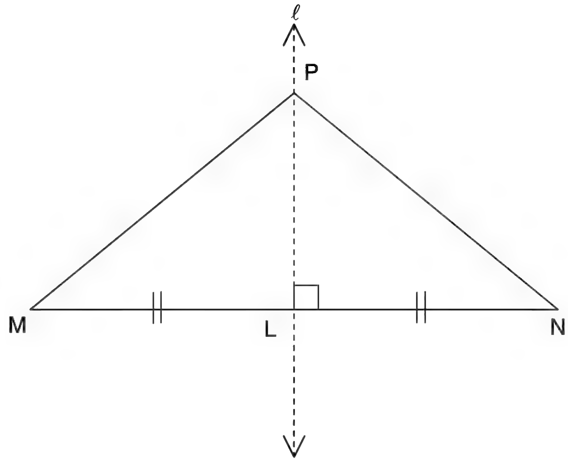


Figure 13.119

2. Show that the locus of a point equidistant from two intersecting lines in the plane determined by the lines is the union of the pair of lines bisecting the angles formed by given lines.

Solution

Step I: We initially prove that any point equidistant from two given intersecting lines lies on one of the lines bisecting the angles formed by given lines.

Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines intersecting at O. P is the point on the plane such that PM = PN. Line ℓ is the bisector of ∠BOD and ∠AOC.

Line m is the bisector of ∠BOC and ∠AOD.

RTP: P lies on either on line ℓ or line m.

Proof: In ΔPOM and ΔPON,

PM = PN,

OP is a common side and

∠PMO = ∠PNO = 90°

∴ By RHS congruence property, ΔPOM ≅ ΔPON.

So, ∠POM = ∠PON, i.e., P lies on the angle bisector of ∠BOD.

As ℓ is the bisector of ∠BOD and ∠AOC, P lies on the line ℓ.

Similarly if P lies in any of the regions of ∠BOC, ∠AOC or ∠AOD, such that it is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, then we can conclude that P lies on the angle bisector ℓ or on the angle bisector m.

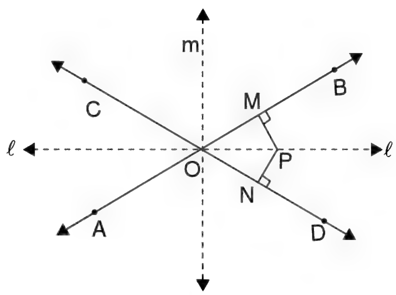


Figure 13.120

Step II: To prove that any point on the bisector of one of angles formed by two intersecting lines is equidistant from the lines.

Given: Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O. Lines ℓ and m are the angle bisectors.

Proof: Let ℓ be the angle bisector of $\angle BOD$ and $\angle AOC$, and m be the angle bisector of $\angle BOC$ and $\angle AOD$.

Let P be a point on the angle bisector ℓ, as shown in the figure.

If P coincides with O, then P is equidistant from the line $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Suppose P is different from O.

Draw the perpendiculars $\overline{PM}$ and $\overline{PN}$ from the point P onto the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ respectively.

Then in ΔPOM and ΔPON,

$\angle POM = \angle PON$, $\angle PNO = \angle PMO = 90°$ and OP is a common side.

$\therefore$ By the AAS congruence property

$\Delta POM \cong \Delta PON$

So, PN = PM ($\because$ corresponding sides)

i.e., P is equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

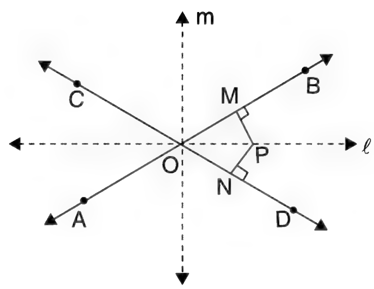


Figure 13.121

Hence, from the steps I and II of the proof it can be said that the locus of the point which is equidistant from the two intersecting lines is the pair of the angle bisectors of the two pairs of vertically opposite angles formed by the lines.

Equation of a locus

We know a locus is the set of points that satisfy a given geometrical condition. When we express the geometrical condition in the form of an algebraic equation that equation is called the equation of the locus.

Steps to find the equation of a locus

1. Consider any point (x_1, y_1) on the locus.
2. Express the given geometrical condition in the form of an equation using x, and y.
3. Simplify the equation obtained in step 2.
4. Replace (x_1, y_1) by (x, y) in the simplified equation obtained in step 3, which gives the required equation of the locus.

The following formulae will be helpful in finding the equation of a locus.

1. Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Area of the triangle formed by joining the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$, where the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3. Equation of the circle with centre (a, b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$

4. The perpendicular distance from a point $P(x_1, y_1)$ to a given line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

☛ *Example*

Find the equation of the locus of a point that forms a triangle of area 5 units with the points A(2, 3) and B(–1, 4).

Solution

Let $P(x_1, y_1)$ be point on the locus, $(x_2, y_2) = (2, 3)$ and $(x_3, y_3) = (-1, 4)$

Given area of ΔPAB = 5 sq.units.

$$\therefore \frac{1}{2}\begin{vmatrix} x_1 - 2 & 2-(-1) \\ y_1 - 3 & 3-4 \end{vmatrix} = 5$$

$$\begin{vmatrix} x_1 - 2 & 3 \\ y_1 - 3 & -11 \end{vmatrix} = 10$$

$$-(x_1 - 2) - 3(y_1 - 3) = \pm 10$$

$$x_1 + 3y_1 + 11 = \pm 10$$

Required equation is $x + 3y = 10 - 11$ or $x + 3y = -10 - 11$

$x + 3y + 1 = 0$ or $x + 3y + 21 = 0$

test your concepts

Very short answer type questions

1. There are no congruent figures which are similar. (True/False).

2. Two triangles with measures 3 cm, 4 cm, 5 cm and 0.60 cm, 0.8 cm, 1 cm are similar. (Agree/disagree).

3. A triangle is formed by joining the mid-points of the sides of a given triangle. This process is continued indefinitely. All such triangles formed are similar to one another. (True/False).

4. All the similar figures are congruent if their areas are equal. (Yes/No).

5. The ratio of corresponding sides of two similar triangles is 2 : 3, then the ratio of the perimeters of two triangles is 4 : 9. (True/False).

6. Which of the following is/are true?

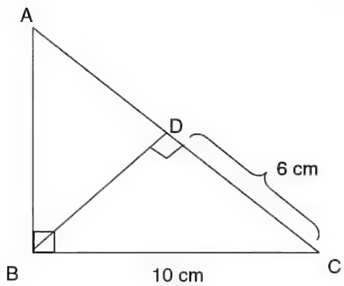
(1) All triangles are similar.
(2) All circles are similar.
(3) All squares are similar.

7. Two equal chords of a circle are always parallel. (True/False).

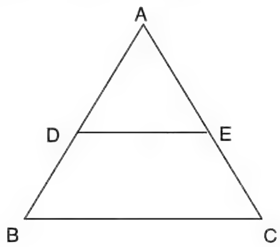
8. In a circle, chord PQ subtends an angle of 80° at the centre and chord RS subtends 100°, then which chord is longer?

9. Number of circles that pass through three collinear points is _______.

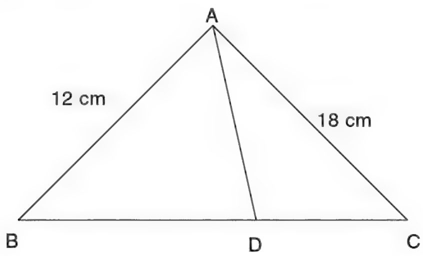
10. In the following figure, $\angle ABC = 90°$, $BC = 10$ cm, $CD = 6$ cm, then $AD =$ ______.



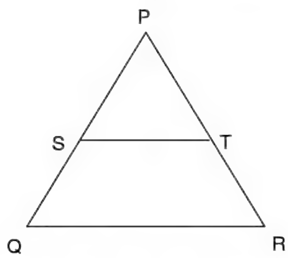
11. In the following figure, AD = DB and DE || BC, then AE = EC. (True/False)



12. In the following figure (not to scale), if BC = 20 cm and $\angle BAD = \angle CAD$, then BD = ______.



13. In the following figure, ST || QR, then $\Delta PST \sim \Delta PQR$. (Yes/No).

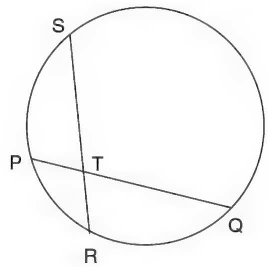


14. In the above figure, if ST | | QR and PS : PQ = 2 : 5 and TR = 15 cm, then PT = ______.

15. The number of tangents drawn from an external point to a circle is ______.

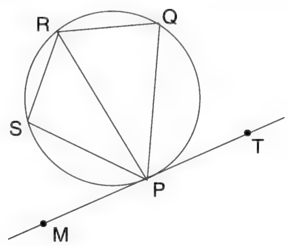
16. If two circles intersect at two distinct points, then the number of common tangents is _____.

17. In the following figure, PT = 4 cm, TQ = 6 cm and RT = 3 cm, then TS = ______.



18. If two circles touch each other externally, then the number of transverse common tangents is ______.

19. In the following figure, to find ∠PQR, ______ must be given.(∠PRQ/∠QPT /∠RPT)



20. From a point P which is at a distance of 13 cm from the centre of the circle of radius 5 cm, a tangent is drawn to the circle. The length of the tangent is ______.

21. A line drawn from the centre of a circle to a chord always bisects it. (True/False).

22. Distance between two circles with radii R and r is d units. If $d^2 = R^2 + r^2$ then the two circles ______ (intersect at one point/do not intersect/intersect at two distinct points).

23. Two circles with radii r_1 and r_2 touch externally. The length of their direct common tangent is ______.

24. In a circle, angle made by an arc in the major segment is 60°. Then the angle made by it in the minor segment is ______.

25. If a trapezium is cyclic, then its ______ are equal. (parallel sides/oblique sides)

26. In a circle, two chords PQ and RS bisect each other. Then PRQS is ______.

27. Line joining the centers of two intersecting circles always bisect their common chord. (True/False).

28. The locus of the tip of a seconds hand of a watch is a ______.

29. A parallelogram has no line of symmetry (True/False).

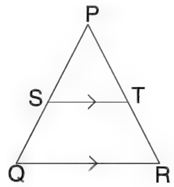
30. If the length of an enlarged rectangle is 12 cm and the scale factor is $\frac{3}{2}$, then the length of the original rectangle is _____.

Short answer type questions

31. If the altitude of an equilateral triangle is 6 units, then the radius of its incircle is _________.

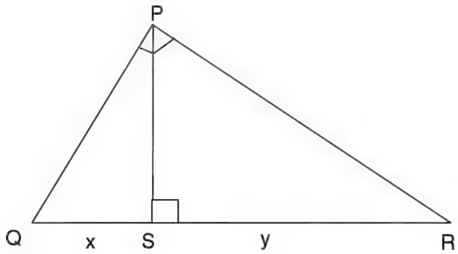
32. In a rhombus PQRS, PR and QS are the diagonals of the rhombus. If PQ = 10 cm, then find the value of $PR^2 + QS^2$.

33. In a triangle PQR, ST is parallel to QR. Show that RT(PQ + PS) = SQ(PR + PT)



34. In a right angled triangle PQR, angle Q = 90° and QD is the altitude. Find DR, if PD = 17 cm and QD = 21 cm.

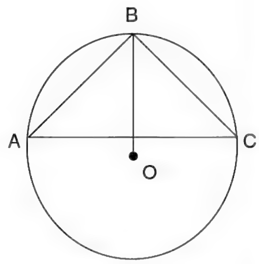
35. In the following figure, QS = x, SR = y, $\angle QPR = 90°$ and $\angle PSR = 90°$, then find $(PQ)^2 - (PR)^2$ in terms of x and y.



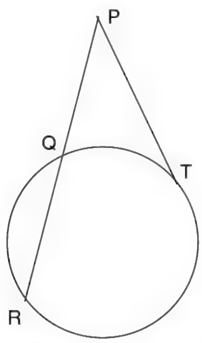
36. In the following figure, AB and BC are equidistant from the centre 'O' of the circle. Show that

(1) ABC is an isosceles triangle

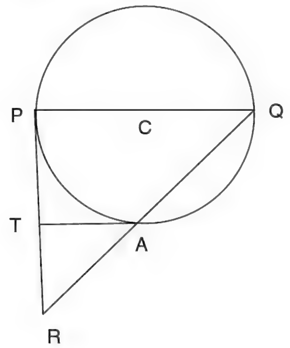
(2) OB bisects angle ABC.



37. In the following figure, PR is a secant and PT is a tangent to the circle. If PT = 6 cm and QR = 5 cm, then PQ = ______ cm.



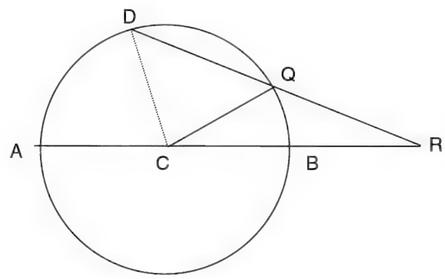
38. In the following figure, PQ is the diameter of the circle with radius 5 cm. If AT is the tangent and equal to the radius of the circle, then find the length of AR.



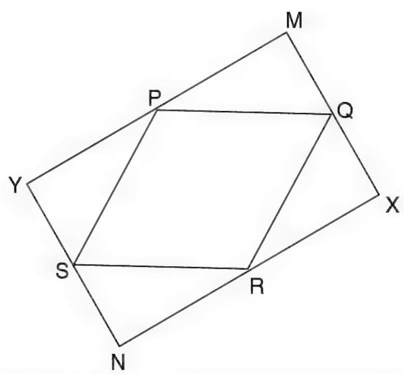
39. The distance between two buildings is 24 m. The height of the buildings are 12 m and 22 m. Find the distance between the tops.

40. Find the distance between the centres of the two circles, if their radii are 11 cm and 7 cm, and the length of the transverse common tangent is $\sqrt{301}$ cm.

41. In the figure given along side, AB is the diameter of the circle, C is the centre of the circle and CQR is an isosceles triangle, such that CQ = QR. Prove that $\angle DCA = 3 \cdot \angle QCR$.



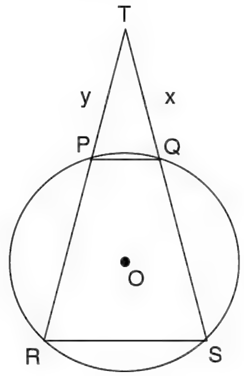
42. In the following figure, PQRS is a rhombus formed by joining the mid-points of a quadrilateral YMXN, show that $3PQ^2 = SN^2 + NR^2 + QX^2 + XR^2 + PY^2 + YS^2$.



43. Find the locus of a point which is at a distance of 5 units from (–1, –2).

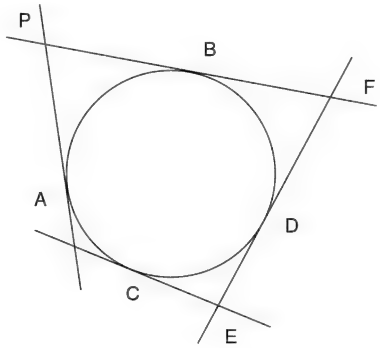
44. In the following figure, PR and SQ are chords, of the circle with centre O, intersecting at T and TQ = x; TP = y.

Show that (TS + TR) : (TS – TR) = (x + y) : (y – x)



45. In the given figure

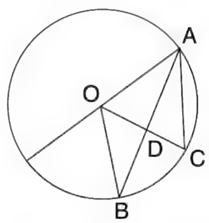
PA, PB, EC and ED are tangents to the circle. If PA = 13 cm, CE = 4·5 cm and FE = 9 cm, then find PF.



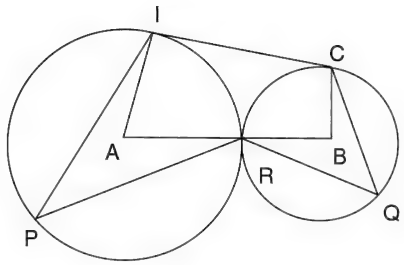
Essay type questions

46. EF and EH are the two chords of a circle with centre O intersecting at E. The diameter ED bisects the angle HEF. Show that the triangle FEH is an isosceles triangle.

47. In the above figure, O is the centre of the circle, AC are parallel lines. If ∠ACO = 80°, then find ∠ADO.

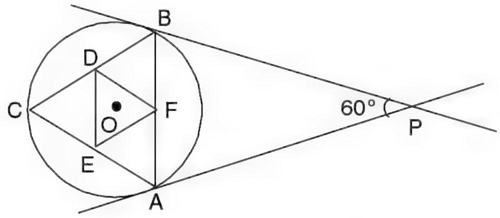


48.



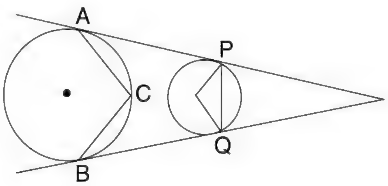
In the above figure, A and B are the centres of two circles, DC being the common tangent. If $\angle DPR = 35°$, then $\angle RQC =$

49.



In the diagram given above, D, E and F are mid-points of BC, CA and AB. If the angle between the tangents drawn at A and B is 60°, find $\angle EFD$.

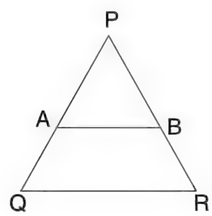
50.



In the diagram above, A, B, P and Q are points of contacts of direct common tangents of the two circles. If $\angle ACB$ is 120°, then find the angle between the two tangents and angle made by PQ at the centre of same circle.

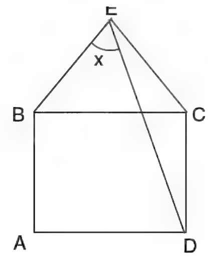
CONCEPT APPLICATION

Concept Application Level—1



1. In the triangle PQR, AB is parallel to QR. The ratio of the areas of two similar triangles PAB and PQR is 1 : 2. Then PQ : AQ = _____.
   (1) $\sqrt{2} : 1$ (2) $1 : \sqrt{2} - 1$
   (3) $1 : (\sqrt{2} + 1)$ (4) $\sqrt{2} : \sqrt{2} - 1$

2. In the figure given below, equilateral triangle ECB surmounts square ABCD. Find the angle BED represented by x.



(1) 15°

(2) 30°

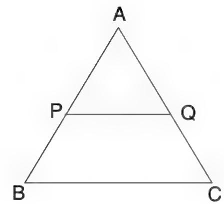
(3) 45°

(4) 60°

3. In two triangles ABC and DEF, ∠A = ∠D. The sum of the angles A and B is equal to the sum of the angles D and E. If BC = 6 cm and EF = 8 cm, find the ratio of the areas of the triangles, ABC and DEF.

(1) 3 : 4 (2) 4 : 3 (3) 9 : 16 (4) 16 : 9

4. In the shown figure, PQ is parallel to BC and PQ : BC = 1 : 3. If the area of the triangle ABC is 144 cm², then what is the area of the triangle APQ?



(1) 48 cm²

(2) 36 cm²

(3) 16 cm²

(4) 9 cm²

5. In triangle ABC, sides AB and AC are extended to D and E respectively, such that AB = BD and AC = CE. Find DE, if BC = 6 cm.

(1) 3 cm (2) 6 cm

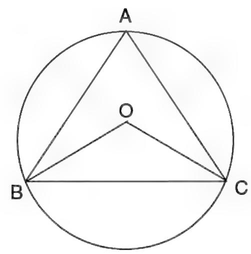
(3) 9 cm (4) 12 cm

6. A man travels on a bicycle, 10 km east from the starting point A to reach point B, then he cycles 15 km south to reach point C. Find the shortest distance between A and C.

(1) 25 km (2) 5 km

(3) $25\sqrt{13}$ km (4) $5\sqrt{13}$ km

7. In the following figure, O is the centre of the circle. If ∠BAC = 60°, then ∠OBC =



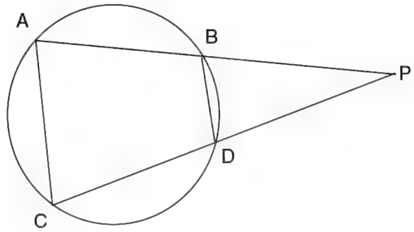
(1) 120°

(2) 30°

(3) 40°

(4) 60°

8. In the shown figure (not to scale), AB = CD and $\overline{AB}$ and $\overline{CD}$ are produced to meet at the point P. If ∠BAC = 70°, then find ∠P.



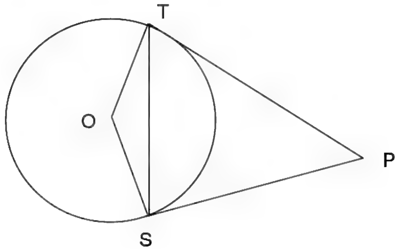
(1) 30°

(2) 40°

(3) 45°

(4) 50°

9.

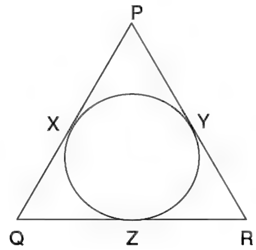


PT and PS are the tangents to the circle with centre O. If $\angle TPS = 65°$, then $\angle OTS =$

(1) $32°$ (2) $45°$ (3) $57\frac{1}{2}°$ (4) $32\frac{1}{2}°$

10. In the following figure X, Y and Z are the points at which the incircle touches the sides of the triangle as shown below.

If PX = 4 cm, QZ = 7cm and YR = 9 cm, then the perimeter of triangle PQR is



(1) 20 cm
(2) 46 cm
(3) 40 cm
(4) 80 cm

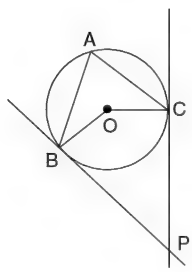
11. The locus of the point P which is at a constant distance of 2 units from the origin and which lies in the first or the second quadrants is

(1) $y = -\sqrt{4-x^2}$ (2) $y = \sqrt{4-x^2}$ (3) $x = \sqrt{4-y^2}$ (4) $x = -\sqrt{4-y^2}$

12. If PAB is a triangle in which $\angle B = 90°$ and A(1, 1) and B(0, 1), then the locus of P is ___________.

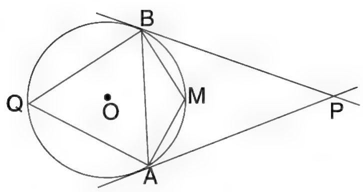
(1) $y = 0$ (2) $xy = 0$ (3) $x = y$ (4) $x = 0$

13. In the shown diagram, if the angle between two chords AB and AC is 65°, then the angle between two tangents which are drawn at B and C is _______



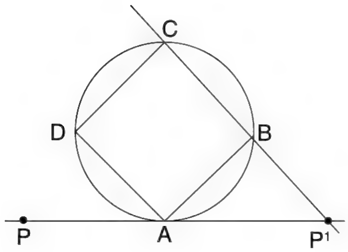
(1) 50°
(2) 30°
(3) 60°
(4) 40°

14. In the shown diagram, O is the centre of the circle and $\angle AMB = 120°$, Find the angle between the two tangents AP and BP.



(1) 30°
(2) 45°
(3) 70°
(4) 60°

15.

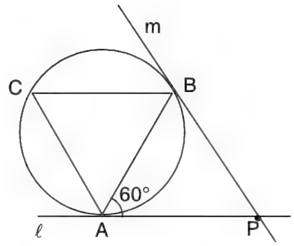


If ABCD is a square inscribed in a circle and PA is a tangent, then the angle between the lines P^1A and P^1B is

(1) 30°   (2) 20°

(3) 40°   (4) 45°

16. In the diagram shown, if ℓ and m are two tangents and AB is a chord making an angle of 60° with the tangent ℓ, then the angle between ℓ and m is



(1) 45°

(2) 30°

(3) 60°

(4) 90°

17. Find the length of a transverse common tangent of the two circles whose radii are 3·5 cm, 4·5 cm and the distance between their centres is 10 cm.

(1) 36 cm   (2) 6 cm

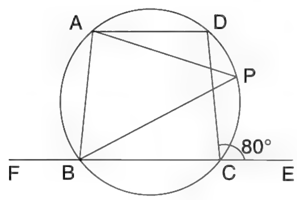
(3) 64 cm   (4) 8 cm

18. If ABCD is a trapezium, AC and BD are the diagonals intersecting each other at point O. Then AC : BD =

(1) AB : CD   (2) AB + AD : DC + BC

(3) $AO^2 : OB^2$ (4) AO – OC : OB – OD

19. In the shown figure (not to scale), $\overline{PA}$ and $\overline{PB}$ are equal chords and ABCD is a cyclic quadrilateral. If $\angle DCE = 80°$, $\angle DAP = 30°$ then find $\angle APB$.



(1) 40°

(2) 80°

(3) 90°

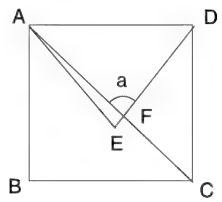
(4) 160°

20. In trapezium KLMN, KL and MN are parallel sides. A line is drawn, from the point A on KN, parallel to MN meeting LM at B. KN : LM is equal to

(1) KL : NM   (2) (KL + KA) : (NM + BM)

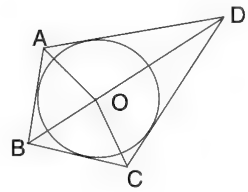
(3) (KA – AN) : (LB – BM)   (4) $KL^2 : MN^2$

21. In the following figure, ABCD is a square and AED is an equilateral triangle. Find the value of a.



(1) 30°
(2) 45°
(3) 60°
(4) 75°

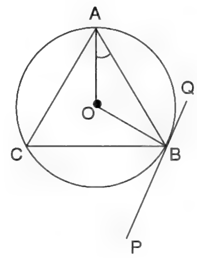
22. A circle with centre O is inscribed in a quadrilateral ABCD as shown in the figure. Which of the following statements is/are true?



(a) ∠AOD + ∠BOC = 180°
(b) ∠AOB and ∠COD are complementary
(c) OA, OB, OC and OD are the angle bisectors of ∠A, ∠B, ∠C and ∠D respectively.

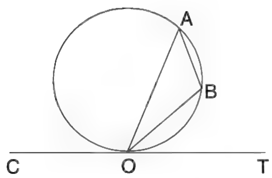
(1) Both (a) and (b) (2) Both (b) and (c)
(3) Both (a) and (c) (4) All the three

23. In the diagram, O is the centre of the circle and if ∠OAB = 30°, then the acute angle between AB and the tangent PQ at B is



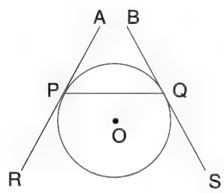
(1) 30°
(2) 60°
(3) 45°
(4) 90°

24. In the shown figure, AB = OB and CT is the tangent to the circle at O. If ∠COA = 125°, then ∠OAB is



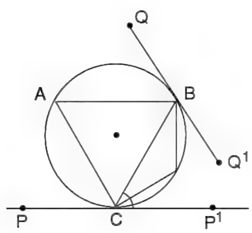
(1) 55°
(2) 27 ½°
(3) 82 ½ °
(4) 45°

25. AR and BS are the tangents to the circle, with centre O, touching at P and Q respectively and PQ is the chord. If ∠OQP = 25°, then ∠RPQ = ________.



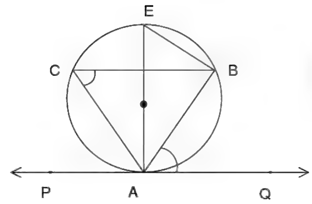
(1) 100°
(2) 115°
(3) 150°
(4) 90°

26. In the diagram, if $\angle BCP^1 = \angle ABQ = 60°$, then the triangle ABC is ______.



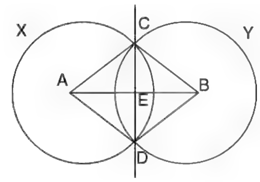
(1) scalene
(2) equilateral
(3) right angled
(4) acute angled

27. In the following figure, AQ is a tangent to the circle at A. If $\angle ACB = 60°$, then $\angle BAQ =$



(1) 30°
(2) 60°
(3) 120°
(4) 45°

28. In the diagram, two circles X and Y with centres A and B respectively intersect at C and D. The radii AC and AD of circle X are tangents to the circle Y. Radii BC and BD of circle Y are tangents to the circle X. Find $\angle AEC$.

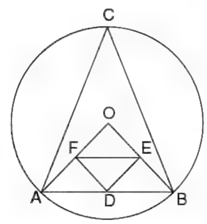


(1) 45° (2) 60°
(3) 90° (4) Cannot be determined

29. The tangent AB touches a circle, with centre O, at the point P. If the radius of the circle is 5 cm, OB = 10 cm and OB = AB, then find AP.

(1) $5\sqrt{5}$ cm (2) $10\sqrt{5}$ cm (3) $(10-5\sqrt{3})$ cm (4) $\left(10-\frac{5}{\sqrt{3}}\right)$ cm

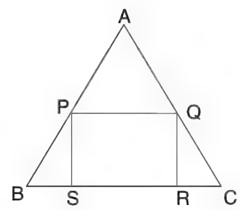
30. In the diagram above, O is the centre of the circle and D, E and F are mid points of AB, BO and OA respectively. If $\angle DEF = 30°$, then find $\angle ACB$.



(1) 30°
(2) 60°
(3) 90°
(4) 120°

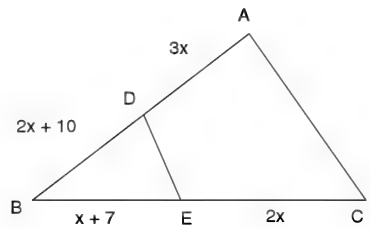
Concept Application Level—2

31. In the figure given below, ABC is an equilateral triangle and PQRS is a square of side 6 cm. By how many cm^2 is the area of the triangle more than that of the square?



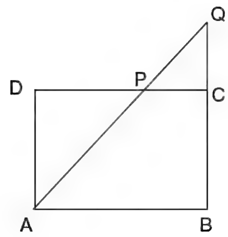
(1) $\frac{21}{\sqrt{3}}$
(2) 21
(3) $21\sqrt{3}$
(4) 63

32. In the given figure, $\overline{DE} \parallel \overline{AC}$. Find the value of x.



(1) 1
(2) 2
(3) 3
(4) 4

33 In the shown figure (not to scale) ABCD is a rectangle, BC = 24 cm, DP = 10 cm and CD = 15 cm. Then, AQ and CQ respectively are

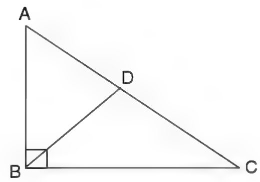


(1) 39 cm, 13 cm
(2 13 cm, 12 cm
(3) 25 cm, 13 cm
(4) 39 cm, 12 cm

34. At a particular time, the shadow cast by a tower is 6 m long. If the distance from top of the tower to the end of the shadow is 10 m long, determine the height of the tower.

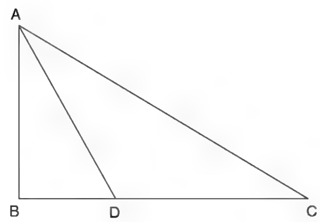
(1) 4 m
(2) 8 m
(3) 16 m
(4) 12 m

35. In the figure above, $\angle ABC = 90°$, AD = 15 and DC = 20. If BD is the bisector of $\angle ABC$, what is the perimeter of the triangle ABC?



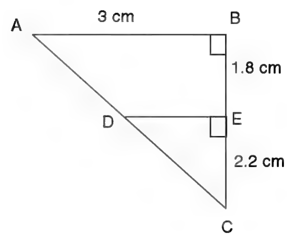
(1) 74
(2) 84
(3) 91
(4) 105

36. In the triangle ABC, $\angle ABC$ or $\angle B = 90°$. AB : BD : DC = 3 : 1 : 3. If AC = 20 cm, then what is the length of AD (in cm)?



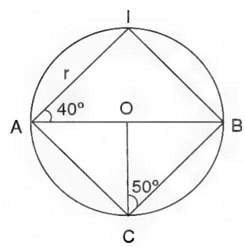
(1) $5\sqrt{2}$
(2) $6\sqrt{3}$
(3) $4\sqrt{5}$
(4) $4\sqrt{10}$

37. In the given figure, find AD.



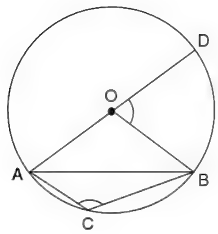
(1) 1.8 cm
(2) 2.25 cm
(3) 2.2 cm
(4) 1.85 cm

38. In the diagram, AB is a diameter, O is the centre of the circle and $\angle OCB = 50°$, then find $\angle DBC$.



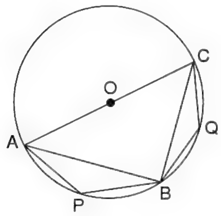
(1) 80°
(2) 100°
(3) 120°
(4) 140°

39. In the above diagram, O is the centre of the circle and AD is the diameter. If $\angle ACB = 135°$, then find $\angle DOB$.



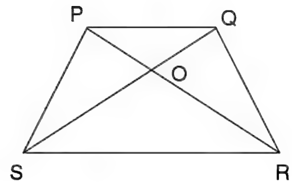
(1) 135°
(2) 60°
(3) 90°
(4) 45°

40. In the diagram, O is the centre of the circle, AC is the diameter and if $\angle APB = 120°$, then find $\angle BQC$.



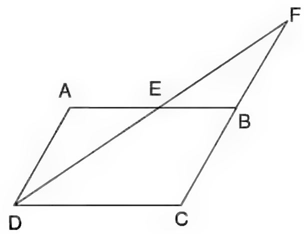
(1) 30°
(2) 150°
(3) 90°
(4) 120°

41. In the trapezium PQRS, PQ is parallel to RS and the ratio of the areas of the triangle POQ to triangle ROS is 225 : 900. Then SR =?



(1) 30 PQ
(2) 25 PQ
(3) 2 PQ
(4) PQ

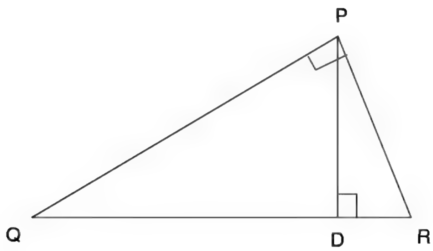
42. In the following figure, ABCD is a parallelogram, CB is extended to F and the line joining D and F intersect AB at E. Then,



(1) $\frac{AD}{AE} = \frac{BF}{BE}$ (2) $\frac{AD}{AE} = \frac{CF}{CD}$

(3) $\frac{BF}{BE} = \frac{CF}{CD}$ (4) All of them are true

43.

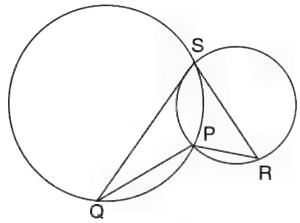


PQR is a right angled triangle, where $\angle P = 90°$. $\overline{PD}$ is perpendicular to $\overline{QR}$. PQ : PR =

(1) QD : DR (2) $\sqrt{QD} : \sqrt{DR}$

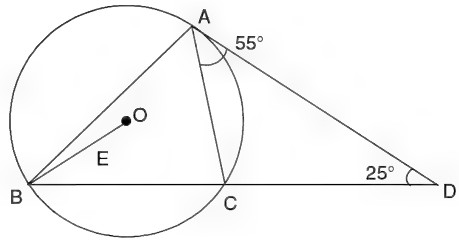
(3) $QD^2 : RD^2$ (4) None of these

44. Two circles intersect at two points P and S. QR is a tangent to the two circles at Q and R. If $\angle QSR = 72°$, then $\angle QPR$ = __________.



(1) 84°
(2) 96°
(3) 102°
(4) 108°

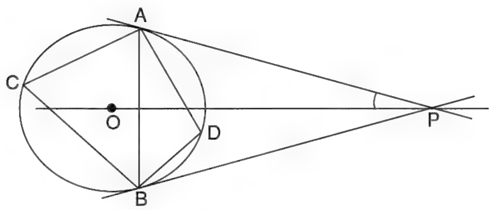
45. In the shown figure, O is the centre of the circle and AD is a tangent to the circle at A. If $\angle CAD = 55°$ and $\angle ADC = 25°$, then $\angle ABO$ =



(1) 10°
(2) 15°
(3) 20°
(4) 25°

Concept Application Level—3

46. In the diagram, O is the centre of the circle and $\angle OPA = 30°$. Find $\angle ACB$ and $\angle ADB$ respectively.



(1) 120°, 60°
(2) 60°, 120°
(3) 75°, 105°
(4) 35°, 145°

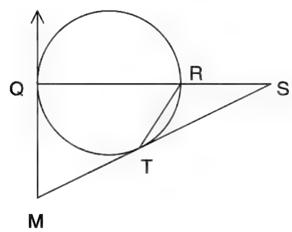
47. Side of a square PQRS is 4 cm long. $\overline{PR}$ is produced to the point M such that PR = 2RM. Find SM.

(1) $\sqrt{10}$ cm (2) $\sqrt{5}$ cm (3) $2\sqrt{5}$ cm (4) $2\sqrt{10}$ cm

48. ABC is an equilateral triangle of side 6 cm. If a circle of radius 1 cm is moving inside and along the sides of the triangle, then locus of the centre of the circle is an equilateral triangle of side __________.

(1) 5 cm (2) 4 cm (3) $(6 - 2\sqrt{3})$ cm (4) $(3 + \sqrt{3})$ cm

49. In the shown figure (not to scale), STM and MQ are tangents to the circle at T and Q respectively. SRQ is a straight line. SR = TR and $\angle TSR = 25°$. Find $\angle QMT$.



(1) 55°
(2) 60°
(3) 75°
(4) 80°

50. PQ is the direct common tangent of two circles (S, 9 cm) and (R, 4 cm) which touch each other externally. Find the area of the quadrilateral PQRS. (in cm^2)

(1) 72 (2) 65 (3) 78 (4) 69

KEY

Very short answer type questions

1. False
2. Agree
3. True
4. Yes
5. False
6. 2 and 3.
7. False
8. RS
9. zero
10. $\frac{32}{3}$ cm
11. True
12. 8 cm
13. Yes
14. 10 cm
15. 2
16. 2
17. 8 cm
18. 1
19. $\angle RPT$
20. 12 cm
21. False
22. intersect at two distinct points
23. $2\sqrt{r_1 r_2}$
24. 120°
25. Oblique sides
26. rectangle
27. True
28. Circle
29. true
30. 8 cm

Short answer type questions

31. 2 units
32. 400 cm^2.
34. 26 cm (approximately).
35. $x^2 - y^2$
37. 4 cm
38. $5\sqrt{2}$
39. 26 m.
40. 25 cm
43. $x^2 + y^2 + 2x + 4y = 0$.
45. 17.5 cm

Essay type questions

47. 120°
48. 55°
49. 60°
50. 60°, 120°

key points for selected questions

32. (i) In a rhombus, diagonals bisect each other perpendicularly.
(ii) $PR^2 + QS^2 = 4$(side of the rhombus)

33. (i) Apply Basic proportionality theorem.
(ii) Apply componendo rule.

34. QD is the mean proportional of PD and DR.

35. Apply Pythagoras theorem in triangles PSQ and PSR.

36. In a circle, chords equidistant from the centre are equal.

37. $PQ.PR = PT^2$

38. (i) Apply concept of similarity.
(ii) $AT : PQ = 1 : 2$, find $PT : TR$.

39. Apply Pythagoras theorem.

40. Length of the transverse common tangent

$= \sqrt{d^2 - (R + r)^2}$

41. (i) Let $\angle QCR$ be x
(iii) Then find $\angle DCQ$, where triangle DCQ is an isosceles triangle.

42. Apply Pythagoras theorem.
All the sides of a rhombus are of equal length.

43. (i) Distance between (x_1, y_1) and (x_2, y_2) is

$\sqrt{(x_1 - x_2 + (y_1 - y_2)^2}$

(ii) Given distance is 5 units.

44. (i) $\Delta TPQ \sim \Delta TSR$
(ii) Apply componendo-dividendo rule.

45. (i) Tangents drawn from an external point to a circle are equal.
(ii) PF = PB + FB
(iii) PB = PA and FB = FD
(iv) FD = FE-DE

Essay type questions

46. (i) ED is perpendicular bisector of FH.
(ii) Apply the concept of congruency.

47. $\angle BAC = \frac{1}{2}\angle BOC$ and proceed.

48. (i) Radius is perpendicular to the tangent, at the point of tangency.
(ii) $\angle DAR = 2\angle DPR$
(iii) In the quadrilateral ABCD, find $\angle RBC$.

49. (i) Join OB and OA.
(ii) Find $\angle BOA$ and then $\angle BCA$.
(iii) Triangles ABC and DEF are similar.

50. (i) Let S be the point of intersection of the tangents AP and BQ and O be the centre of the bigger circle.
(ii) Find reflex $\angle AOB$ and then $\angle AOB$, in the quadrilateral AOBS,
(iii) $\angle PSQ + \angle AOB = 180°$.

KEY

Concept Application Level 1—3

1. 4	2. 3
3. 3	4. 3
5. 4	6. 4
7. 2	8. 2
9. 4	10. 3
11. 2	12. 4
13. 1	14. 4
15. 4	16. 3
17. 2	18. 4
19. 2	20. 3
21. 4	22. 3
23. 2	24. 2
25. 2	26. 2
27. 2	28. 3
29. 3	30. 2
31. 3	32. 1
33. 4	34. 2
35. 2	36. 4
37. 2	38. 2
39. 3	40. 2
41. 3	42. 4
43. 2	44. 4
45. 1	46. 2
47. 4	48. 3
49. 4	50. 3

Concept Application Level 1—3

Key points for select questions

1. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
2. BC = CD = EC and proceed using $\angle ECD = 90° + 60° = 150°$
3. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
4. The ratio of areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.
5. Apply Basic proportionality theorem.
6. Apply Pythagoras theorem.
7. $\angle BOC = 2\angle BAC$ and proceed.

8. Exterior angle of a cyclic quadrilateral is equal to its interior opposite angle.

   $\angle BAC = \angle DCA$ and proceed.

9. Radius of a circle is perpendicular to the tangent at the point of contact and tangents drawn to a circle from an external point are equal.

10. Tangents drawn to a circle from an external point are equal.

11. Recall the definition of locus.

    In the first or the second quadrant, y is positive.

12. Apply Pythagoras theorem.

13. Radius is perpendicular to the tangent at the point of contact.

    In the quadrilateral BOCP, $\angle BOC + \angle BPC = 180^\circ$

14. Recall the properties of cyclic quadrilateral and also of tangents.
    $\angle BOA + \angle BPA = 180^\circ$ and $\angle BOA = 2\angle BQA$

15. Apply 'Alternate segment theorem'.

16. Tangents drawn to a circle from an external point are equal.

17. Length of the transverse common tangent $= \sqrt{d^2 - (R + r)^2}$

18. Diagonals of a trapezium divide each other proportionally.

19. Recall the properties of cyclic quadrilateral.
    $\angle PAB = \angle PBA$ and $\angle DAB = \angle DCE$.

20. Apply BPT and use componondo–dividendo after drawing the complete figure.

21. Diagonal of a square bisects the angle at the vertices.
    $\angle FDC = 30^\circ$ and $\angle FCD = 45^\circ$.

22. Evaluate the solution from the options.

23. Apply 'Alternate segment theorem'.

24. Recall 'Alternate segment theorem'.

25. Radius is perpendicular to the tangent at the point of contact.

26. Apply 'Alternate segment theorem'.

27. Apply 'Alternate segment theorem'.

28. Recall the properties of a kite.

29. Apply Pythagoras theorem.

30. (i) ADEF is a parallelogram.
    (ii) $\angle FAD = 30^\circ$ and
    $\angle OAD = \angle OBA$
    (angles opposite to equal sides)

31. (i) In triangle PBS, $\angle B = 60^\circ$.
    $\therefore\ \angle P = 30^\circ$ and $\angle S = 90^\circ$.
    (ii) The sides of the triangle PBS, i.e., BS, SP and PB are in the ratio $1 : \sqrt{3} : 2$.
    (iii) Given PS = 6 cm

32. (i) Apply 'Basic proportionality theorem'.
    (ii) $\dfrac{BD}{DA} = \dfrac{BE}{EC}$
    (iii) Substitute the values of BD, AD, BE and EC in the above equation.

33. (i) Use Pythagoras theorem to find AP.
    (ii) Triangle QAB and triangle QPC are similar.
    $\therefore \dfrac{QP}{PA} = \dfrac{QC}{CB}$

34. (i) Apply 'Pythagoras theorem'.
    (ii) The tower is perpendicular to the surface of the ground.

35. (i) Apply 'Angle bisector theorem'.
    (ii) AB : BC = AD : DC
    (iii) Apply Pythagoras theorem, and find AB and BC.

36. (i) Apply 'Pythagoras theorem' for triangle ABC.
    (ii) Let AB = 3x, BD = x and CD = 3x
    (iii) First find AB and BC using $AB^2 + BC^2 = AC^2$.
    (iv) Then find AD by using $AD^2 = AB^2 + BD^2$

37. (i) Apply 'Basic proportionality theorem'.
    (ii) Apply Pythagoras theorem to find AC.
    (iii) Then apply basic proportionality theorem. i.e., CE/EB = CD/DA

38. (i) In triangle OBC, angles opposite to equal sides are equal.
(ii) $\angle ADB = \angle ACB = 90°$ (Since angle in a semi circle is 90°)
(iii) $\angle OCB = \angle OBC$ and $\angle OAC = \angle OCA$
(Angles opposite to equal sides)

39. (i) ACBD is a cyclic quadrilateral.
(ii) $\angle ACB$ and $\angle ADB$ are supplementary angles.
(iii) $\angle AOB = 2\angle ADB$

40. (i) APBC is a cyclic quadrilateral.
(ii) $\angle ABC$ is an angle in a semi circle.
(iii) ABQC is a cyclic quadrilateral.

41. (i) POQ and ROS are similar triangles.
(ii) SR and PQ are proportional to the square roots of the areas of similar triangles SOR and POQ.

42. (i) Triangles FEB and FDC are similar.
(ii) Triangles AED and EFB are similar.

43. (i) Triangle PDR, QDP and QPR are similar.
(ii) Corresponding sides of similar triangles are proportional.
(iii) $\Delta PDQ \sim \Delta RDP$.

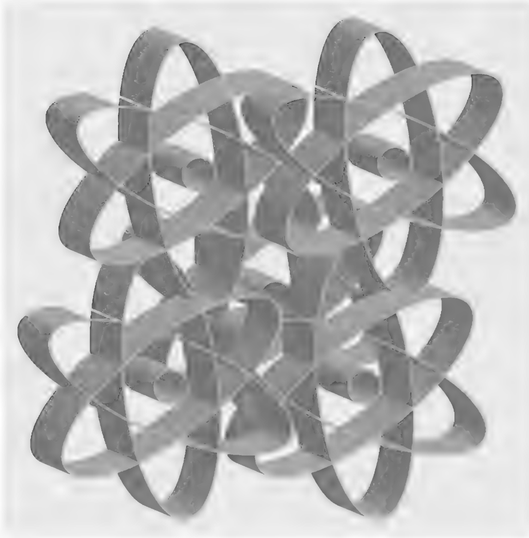
44. (i) Apply 'Alternate segment theorem'.
(ii) Join QR and join PS.
$\angle PQR = \angle PSQ$ and
$\angle PRQ = \angle PSR$ (By alternate segment theorem)

45. (i) Join OA and OC.
(ii) Join OA
(iii) $\angle ACD = 180° - 80° = 100°$
(iv) $\angle ACB$ and $\angle ACD$ are supplementary.
(v) $\angle AOB = 2\angle ACB$.

46. (i) Tangents drawn from an external point to the same circle are equal i.e., PA = PB
(ii) $\angle APO = \angle OPB$ and $\angle AOB + \angle APB = 180°$
(iii) $\angle ACB = \frac{1}{2} \angle AOB$
(iv) ACBD is a cyclic quadrilateral.

CHAPTER 14

Mensuration



INTRODUCTION

Mensuration is a branch of mathematics that deals with the computation of geometric magnitudes, such as the length of a line, the area of a surface and the volume of a solid. In this chapter we shall deal with the areas and volumes of three dimensional figures like prisms, pyramids, cones, spheres, hemispheres etc. However, some problems on plane figures like circles, sectors, segments etc as an exercise of revision.

I. Circle and semi-circle

1. Area of circle = πr^2 sq.units
2. Area of the semi circle = $\dfrac{\pi r^2}{2}$ sq. units.
3. Circumference of the circle = $2\pi r$ units = πd units
4. Circumference of the semicircle = $(\pi + 2)r$ units $= \dfrac{36r}{7}$ units

(Where r is radius and d is diameter)

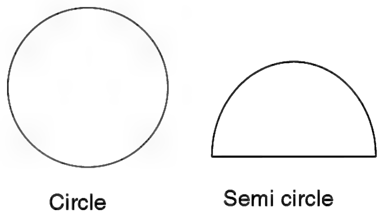


Figure 14.1

II. Circular ring

1. Area of the ring = $\pi(R^2 - r^2) = \pi (R + r) (R - r)$

   (Where R and r are outer radius and inner radius of a ring and $(R - r)$ is the width of the ring)

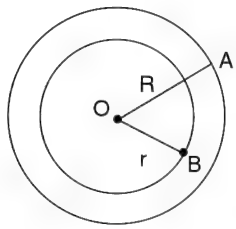


Figure 14.2

III. Sectors and segements

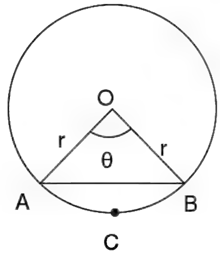


Figure 14.3

1. Length of the arc ACB

   $1 = \left(\dfrac{\theta}{360°}\right) 2\pi r$ units

2. Area of the sector AOBC

   $A = \left(\dfrac{\theta}{360°}\right) \pi r^2$ sq.units

3. Perimeter of the sector = 1 + 2r) units

4. Area of the segment ACB = $\left(A - \dfrac{r^2}{2}\right)$ sq. units.

5. Perimeter of the segment ACB = (length of arc ACB + AB) units.

   (Where r is the radius of the circle and θ is sector angle)

Rotations made by a wheel

1. Distance covered by a wheel in one revolution = Circumference of the wheel.

2. Number of rotations made by a wheel in unit time = $\dfrac{\text{Distance covered by it in unit time}}{\text{Circumference of the wheel}}$

3. Angle made by minute hand in one minute = $\dfrac{360°}{60} = 6°$.

4. Angle made by hour hand in one minute = $\dfrac{30°}{60} = \left(\dfrac{1}{2}\right)^{\circ}$.

Equilateral triangle

1. Circumference of an equilateral triangle = 3a units.

2. Area of the equilateral triangle = $\dfrac{\sqrt{3}a^2}{4}$ sq. units.

3. Height of the equilateral triangle = $\dfrac{\sqrt{3}a}{2}$ units.

4. Radius of incircle of equilateral triangle = $\dfrac{1}{3}\left(\dfrac{\sqrt{3}a}{2}\right) = \dfrac{a}{2\sqrt{3}}$ units.

5. Circumradius of equilateral triangle = $\dfrac{2}{3}\left(\dfrac{\sqrt{3}a}{2}\right) = \dfrac{a}{\sqrt{3}}$ units.

   (Where a is side of the triangle)

Prisms

Prism is a solid in which two congruent and parallel polygons form the top and the bottom faces.The lateral faces are parallelograms.

The line joining the centres of the two parallel polygons is called the axis of the prism and the length of the axis is referred to as the height of prism.

If two parallel and congruent polygons are regular and if the axis is perpendicular to the base, then the prism is called a right prism. The lateral surfaces of a right prism are rectangles.

Consider two congruent and parallel triangular planes ABC and PQR. If we join the corresponding vertices of both the planes, i.e., A to P, B to Q and C to R, then the resultant solid formed is a triangular prism. A right prism, the base of which is a rectangle is called a cuboid and the one, the base of which is a pentagon is called a pentagonal prism. If all the faces of the solid are congruent, it is a cube. In case of a cube or a cuboid, any face may be the base of the prism. A prism whose base and top faces are squares but the lateral faces are rectangular is called a square prism.

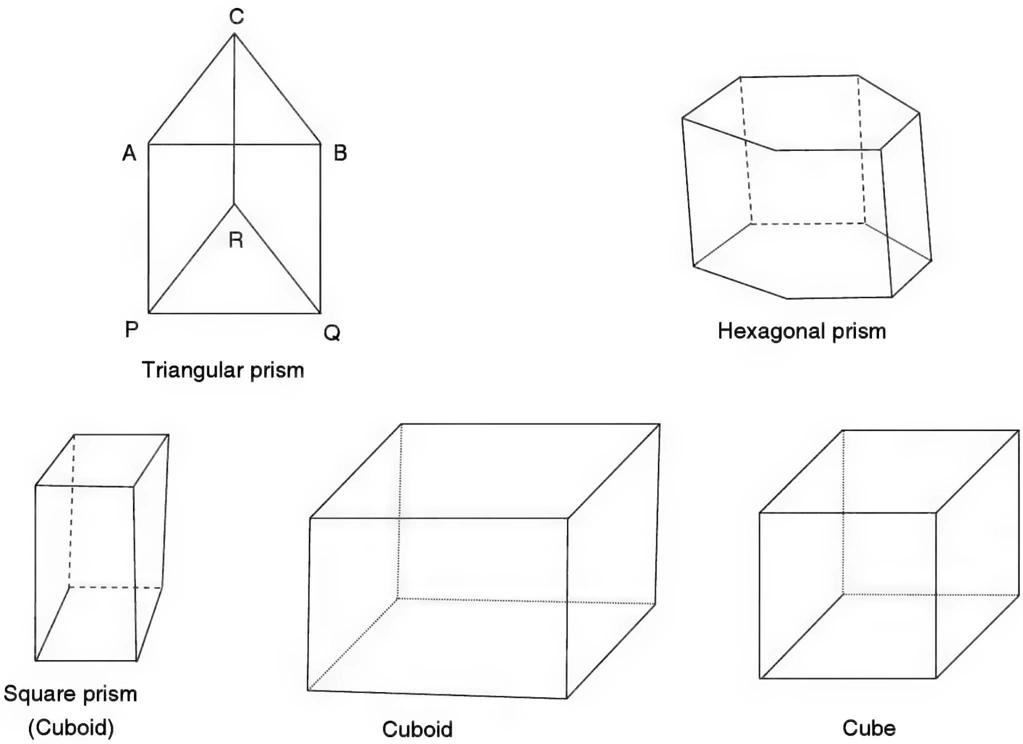


Figure 14.4

NOTE: The following points hold good for all prisms.

1. The number of lateral faces = the number of sides of the base.
2. The number of edges of a prism = number of sides of the base × 3.
3. The sum of the lengths of the edges = 2(perimeter of base) + number of sides × height.

Lateral surface area (LSA) of a prism

L.S.A = Perimeter of base × height = ph

Total surface area (TSA) of a prism

T.S.A = L.S.A + 2(area of base)

Volume of a prism

Volume = Area of base × height = Ah.

Note: The volume of water flowing in a canal = The cross section of the canal × The speed of water.

Example

1. The base of a right prism is a right angled triangle. The measure of the base of the right angled triangle is 3 m and its height 4 m. If the height of the prism is 7 m, then find
   (i) the number of edges of the prism.
   (ii) the volume of the prism.
   (iii) the total surface area of the prism.

Solution

(i) The number of the edges = The number of sides of the base × 3 = 3 × 3 = 9

(ii) The volume of the prism = Area of the base × Height of the prism = $\frac{1}{2}(3 \times 4) \times 7 = 42 \text{ m}^3$

(iii) TSA = LSA + 2(area of base)

= ph + 2(area of base)

where, p = perimeter of the base = sum of lengths of the sides of the given triangle.

As, hypotenuse of the triangle $\sqrt{3^2+4^2} = \sqrt{25} = 5$ m

∴ Perimeter of the base = 3 + 4 + 5 = 12 m

⇒ LSA = ph = 12 × 7 = 84 m^2

TSA = LSA + 2(area of base) = $84 + 2\left(\frac{1}{2}\times 3\times 4\right) = 84 + 12 = 96 \text{ m}^2$

Cubes and cuboids

Cuboid

In a right prism, if the base is a rectangle, then it is called a cuboid. A match box, a brick, a room etc., are in the shape of a cuboid.

The three dimensions of the cuboid, its length (l), breadth (b) and height (h) are generally denoted by l × *b* × *h*.

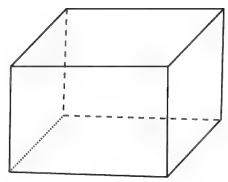


Figure 14.5 Cuboid

1. The lateral surface area of a cuboid = ph = 2(l + b)h sq.units, where p is the perimeter of the base.
2. The total surface area of a cuboid = LSA + 2(base area) = 2(l + b)h + 2lb
   = 2(lb + bh + lh) sq.units.
3. The volume of a cuboid = Ah = (lb)h = lbh cubic units, where A is the area of the base.
4. Diagonal of cuboid = $\sqrt{\ell^2+b^2+h^2}$ units

Note: If a box made of wood of thickness t has inner dimensions of l, b and h, then

the outer length = l + 2t,
the outer breadth = b + 2t and
the outer height = h + 2t.

Cube

In a cuboid, if all the dimensions, i.e., its length, breadth and height are equal, then the solid is called a cube.

All the edges of a cube are equal in length and each edge is called the side of the cube.

Thus, the size of a cube is completely determined by its side.

If the side of cube is 'a' units, then

1. The lateral surface area of a cube = $4a^2$ sq.units
2. The total surface area of a cube = LSA + 2 (area of base) = $4a^2 + 2a^2 = 6a^2$ sq units.
3. The volume of a cube = a^3 cubic units.
4. The diagonal of a cube = $\sqrt{a^2+a^2+a^2} = \sqrt{3}$ a units.

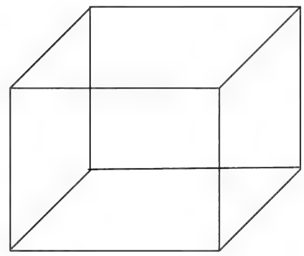


Figure 14.6 Cube

Note: If the inner edge of a cube made of wood of thickness 't' is 'a' units then the outer edge of the cube is given by (a + 2t) units.

Examples

1. The dimensions of a room are 12m × 7m × 5m. Find
   (i) the diagonal of the room.
   (ii) the cost of flooring at the rate of Rs 2 per m^2.
   (iii) the cost of whitewashing the room excluding the floor at the rate of Rs 3 per m^2.

Solution

(i) The diagonal of the room = $\sqrt{\ell^2+b^2+h^2} = \sqrt{12^2+7^2+5^2} = \sqrt{144+49+25} = \sqrt{218}$

(ii) To find the cost of the flooring, we should know the area of the base.

Base area = lb = 12 × 7 = 84 m^2

∴ The cost of flooring = 84 × 2 = Rs 168.

(iii) The total area that is to be whitewashed

= LSA + Area of roof = 2(l + b)h + lb

= 2(12 + 7)5 + 12 × 7 = 2(19) (5) + 84

= 190 + 84

= 274 m^2

∴ The cost of whitewashing = 274 × 3 = Rs 822

2. A box is in the form of a cube. Its edge is 5 m long. Find
   (a) the total length of the edges.
   (b) the cost of painting the outside of the box, on all the surfaces, at the rate of Rs 5 per m^2.
   (c) the volume of liquid which the box can hold.

Solution

(a) Length of edges = Number of edges × 3 × Length of each edge = 4 × 3 × 5 = 60 m

(b) To find the cost of painting the box, we need to find the total surface area.

TSA = $6a^2 = 6 \times 5^2 = 6 \times 25 = 150$ m^2

$\therefore$ Cost of painting = 150×5 = Rs 750

(c) Volume = $a^3 = 5^3 = 125$ m^3.

Right circular cylinder

A cylinder has two congruent and parallel circular planes which are connected by a curved surface. Each of the circular planes is called the base of the cylinder. A road roller, water pipe, power cables, round pillars are some of the objects which are in the shape of a cylinder.

In the above figure, a right circular cylinder is shown. Let A be the centre of the top face and A^1 be the centre of the base. The line joining the centres (i.e., AA^1) is called the axis of cylinder. The length AA^1 is called the height of the cylinder. If the axis is perpendicular to the base, then it is a right circular cylinder. The radius r of the base of the cylinder and the height h, completely describe the cylinder.

Lateral (curved) surface area = Perimeter of base × Height = $2\pi rh$ sq.units

The total surface area = LSA + 2(Base area) = $2\pi rh + 2(\pi r^2) = 2\pi r(h + r)$ sq.units.

Volume = Area of base × Height = $\pi r^2 h$ cubic units.

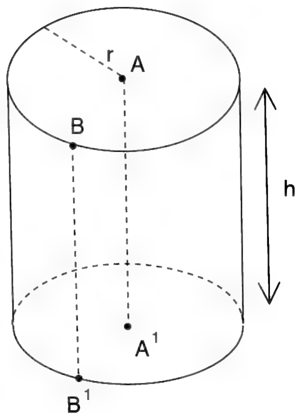


Figure 14.7

Hollow cylinder

The part of a cylinder from which a smaller cylinder of the same axis is cut out is a hollow cylinder.
Let R and r be the external and internal radii of the hollow cylinder and h be the height.

Volume of the material used = $\pi R^2 h - \pi r^2 h = \pi h(R + r)(R - r)$ cubic units.

Curved surface area = $2\pi Rh + 2\pi rh = 2\pi h(R + r)$ sq. units.

Total surface area = Curved surface area + Area of the two ends.

$= 2\pi h(R + r) + 2\pi(R^2 - r^2) = 2\pi(R + r)(R - r + h)$ sq.units.

Note: If a plastic pipe of length ℓ is such that its outer radius is R and the inner radius is r, then the volume of the plastic content of the pipe = $\ell\pi(R^2 - r^2)$ cubic units.

Examples

1. A closed cylindrical container, the radius of which is 7 cm and height 10 cm is to be made out of a metal sheet. Find
   (i) the area of metal sheet required.
   (ii) the volume of the cylinder made.
   (iii) the cost of painting the lateral surface of the cylinder at the rate of Rs 4 per cm^2.

Solution

(i) The area of the metal sheet required = The total surface area of the cylinder = $2\pi r(r + h)$

$$= 2 \times \frac{22}{7} \times 7(7 + 10) = 44(17) = 748 \text{ cm}^2.$$

(ii) Volume = $\pi r^2 h = \frac{22}{7} \times 7 \times 7 \times 10 = 22 \times 70 = 1540$ cm^3

(iii) To find the cost of painting the lateral surface, we need to find the curved (lateral) surface area.

$\therefore$ LSA $= 2\pi rh = 2 \times \frac{22}{7} \times 7 \times 10 = 440$ cm^2

Cost of painting $= 440 \times 4 =$ Rs 1760.

Pyramid

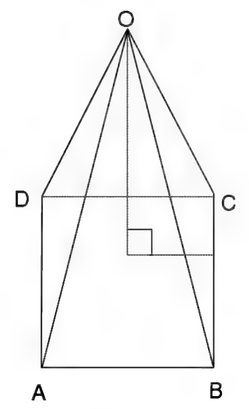


Figure 14.8

A pyramid is a solid obtained by joining the vertices of a polygon to a point in the space by straight lines. The base of the solid obtained is the polygon and lateral faces are triangles. The fixed point in space where all the triangles (i.e., lateral faces) meet is called its vertex.

In the above figure, the base ABCD is a quadrilateral. All the vertices of the base are joined to a fixed point O in space, by straight lines. The resultant solid obtained is called a pyramid.

The straight line joining the vertex and the centre of the base is called the axis of the pyramid. If the axis is not perpendicular to the base, it is an oblique pyramid.

Right pyramid

If the base of a pyramid is a regular polygon and if the line joining the vertex to the centre of the base is perpendicular to the base, then the pyramid is called a right pyramid.

The length of the line segment joining the vertex to the centre of the base of a right pyramid is called the height of the pyramid and is represented by 'h'.

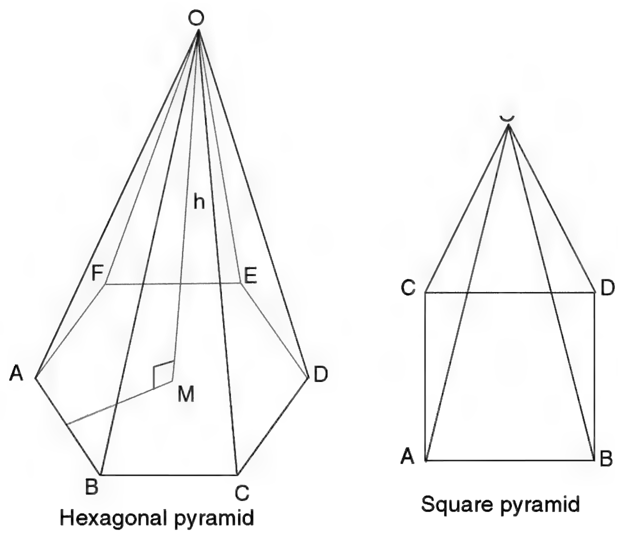


Figure 14.9

The perpendicular distance between the vertex and the mid point of any of the sides of the base (i.e., regular polygon) of a right pyramid is called its slant height and is represented by 'ℓ'.

For a right pyramid with perimeter of base = p, height = h and slant height = ℓ,

(i) Lateral surface area $= \frac{1}{2}$(Perimeter of base) $\times$ (Slant height) $= \frac{1}{2}p\ell$

(ii) Total surface area = Lateral surface area + Area of base.

(iii) Volume of a pyramid $= \frac{1}{3} \times$ Area of base $\times$ Height.

Examples

1. An hexagonal pyramid is 20 m high. Side of the base is 5 m. Find the volume and the slant height of the pyramid.

Solution

Given h = 20 m,
Side of base = a = 5 m

$\therefore$ Area of base $= \dfrac{\sqrt{3}}{4} \times a^2 \times 6 = \dfrac{6\sqrt{3}}{4} \times 5^2 = \dfrac{3\sqrt{3}}{2} \times 25 \text{ m}^2$

Volume $= \dfrac{1}{3}$Ah, where A = area of the base and h = height $= \dfrac{1}{3} \times \dfrac{3\sqrt{3}}{2}(25) \times 20$

$= \sqrt{3} \times 250 = 250\sqrt{3} \text{ m}^3$

To find slant height, refer to the figure shown. In the figure, O is the vertex of the pyramid and G is the centre of the hexagonal base. H is the mid-point of AB.
OG is the axis of the pyramid.
OH is the slant height of the pyramid.
ΔOGH is a right angled triangle.

$\therefore$ $OH^2 = GH^2 + OG^2$

GH = altitude of ΔAGB $= \dfrac{\sqrt{3}}{2}a = \dfrac{\sqrt{3}}{2} \times 5 = \dfrac{5\sqrt{3}}{2}$ m

$\therefore$ $OH^2 = \left(\dfrac{5\sqrt{3}}{2}\right)^2 + (20)^2 = \dfrac{25\times 3}{4} + 400 = \dfrac{75+1600}{4} = \dfrac{1675}{4}$

$\Rightarrow$ OH $= \dfrac{\sqrt{1675}}{2}$ m

$\therefore$ Slant height $= \dfrac{\sqrt{1675}}{2}$ m

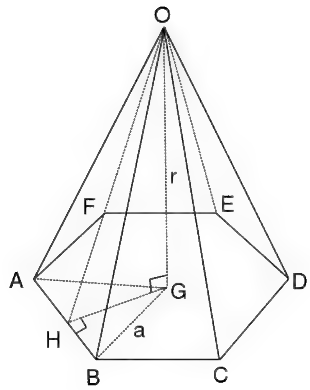


Figure 14.10

Cone

A cone is a solid pointed figure with a circular base. A cone is a kind of pyramid whose base is a circle.

A cone has one vertex, one plane surface (i.e., the base) and a curved surface. (i.e., the lateral surface).

The line joining the vertex to the centre of base (i.e., AO) is called the axis of the cone. The length of the line segment AO is called the height or perpendicular height of the cone. An ice cream cone and a conical tent are some of the examples of conical objects.

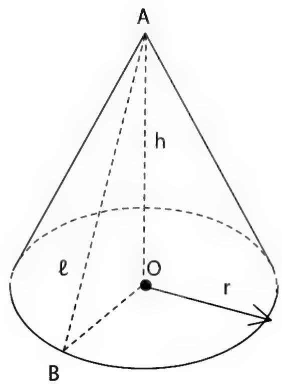


Figure 14.11 Cone

Right circular cone

In a cone, if the line joining the vertex and the centre of the base of the cone is perpendicular to the base, then it is a right circular cone. In other words, if the axis of the cone is perpendicular to the base of the cone, then it is a right circular cone. We generally deal with problems on right circular cones.

A cone is generally defined as a solid obtained by the revolution of a right angled triangle about one of its two perpendicular sides.

If we consider any point B on the periphery of the base of the cone, then the line joining B and the vertex A is called the slant height of the cone and is denoted by 'ℓ'.

From the figure it is clear that ΔAOB is right angled.

$\therefore\ \ell = \sqrt{r^2 + h^2}$

Hollow cone

In earlier classes we have studied about sector. We may recall that sector is an area bounded by an arc of a circle and its two radii. (as shown in figure given below)

Now consider the sector AOB. If we roll the sector up and bring (join) together the radii OA and OB such that they coincide, then the figure formed is called a hollow cone. The radius of the circle becomes the slant height of the cone and the length of the arc of the sector becomes the perimeter of the base of the cone.

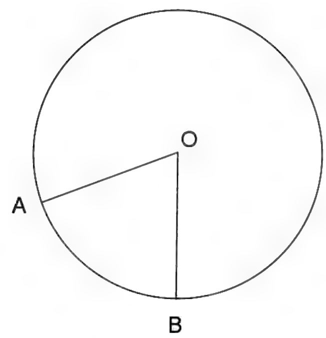


Figure 14.12

For a cone of radius r, height h and slant height ℓ,

1. Curved surface area of a cone = $\pi r\ell$ sq.units
2. Total surface area of a cone = curved surface area + area of base = $\pi r\ell + \pi r^2 = \pi r(r + \ell)$ sq.units.
3. Volume of a cone = $\frac{1}{3}\pi r^2 h$ cubic units.

Cone frustum (or a conical bucket)

If a right circular cone is cut by a plane perpendicular to its axis (i.e., a plane parallel to the base), then the solid portion containing the base of the cone is called the frustum of the cone.

From the figure above, we observe that a frustum is in the shape of a bucket.

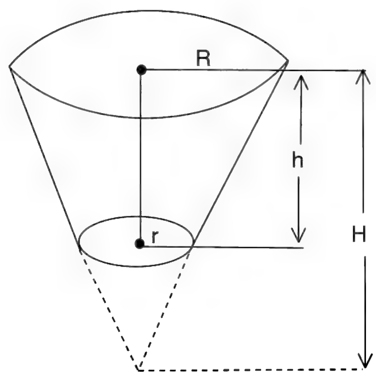


Figure 14.13

Let,

Radius of upper base be R,

Radius of lower base = r,

Height of frustum = h,

Slant height of frustum = ℓ

1. Curved surface area of a frustrum = $\pi\ell(R + r)$ sq.units
2. Total surface area of a frustrum = curved surface area + area of upper base + area of lower base = $\pi\ell(R + r) + \pi r^2 + \pi R^2$ sq.units

3. Volume of a frustrum = $\frac{1}{3}\pi h(R^2 + Rr + r^2)$ cubic units

4. Slant height (ℓ) of a frustrum = $\sqrt{(R-r)^2 + h^2}$ units

Examples

1. A joker's cap is in the form of a cone of radius 7 cm and height 24 cm. Find the area of the cardboard required to make the cap.

Solution

Area of the cardboard required = curved surface area of the cap (or cone) = $\pi r\ell$

Now, $\ell = \sqrt{r^2 + h^2} = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$ cm

$\Rightarrow$ Curved surface area = $\frac{22}{7} \times 7 \times 25 = 550$ cm^2

$\therefore$ Area of the cardboard required = 550 cm^2

2. The diameter of an ice-cream cone is 7 cm and its height is 12 cm. Find the volume of ice cream that the cone can contain.

Solution

Volume of ice cream = $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times \left(\frac{7}{2}\right)^2 \times 12 = 22 \times 7 = 154$ cm^3

3. The diameters of top and bottom portions of a milk can are 56 cm and 14 cm respectively. The height of the can is 72 cm. Find the
   (i) area of metal sheet required to make the can (with out lid).
   (ii) volume of milk which the container can hold.

Solution

The milk can is in the shape of a frustum with R = 28 cm, r = 7 cm and h = 72 cm.

(i) Area of metal sheet required = curved surface area + area of bottom base

$= \pi\ell(R + r) + \pi r^2$

Now, $\ell = \sqrt{(R - r^2) + h^2} = \sqrt{(28-7)^2 + 72^2} = \sqrt{21^2 + 72^2} = \sqrt{9(7^2 + 24^2)}$

$= 3\sqrt{49+576} = 3 \times \sqrt{625} = 3 \times 25 = 75$ cm

$\therefore$ Area of metal sheet = $\frac{22}{7} \times 75(28 + 7) + \frac{22}{7} \times 7^2 = 22 \times 75 \times 5 + 22 \times 7$

$= 22(375 + 7)$

$= 22(382) = 8404$ cm^2

(ii) Amount of milk which the container can hold = $\frac{1}{3}\pi h\ (R^2 + Rr + r^2)$

$= \frac{1}{3} \times \frac{22}{7} \times 72\ (28^2 + 7 \times 28 + 7^2)$

$$= \frac{22}{7} \times 24\,(7 \times 4 \times 28 + 7 \times 28 + 7 \times 7)$$

$$= \frac{22}{7} \times 24 \times 7\,(112 + 28 + 7)$$

$$= 22 \times 24 \times (147) = 77616 \text{ cm}^3$$

4. From a circular canvas of diameter 56 m, a sector of 270° was cut out and a conical tent was formed by joining the straight ends of this piece. Find the radius and the height of the tent.

   As shown in the figure, when the free ends of the torn canvas are joined to form a cone, the radius of sector becomes slant height.

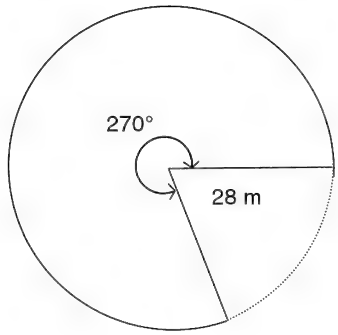


Figure 14.14

$$\therefore\ \ell = \frac{56}{2} = 28 \text{ m}$$

The length of the arc of the sector becomes the circumference of the base of the cone.

Let the radius of the base of the cone = r.

$$\Rightarrow 2\pi r = 2 \times \frac{22}{7} \times \frac{56}{2} \times \left(\frac{270}{360}\right)$$

$$\Rightarrow 2\pi r = 2 \times \frac{22}{7} \times 28 \times \frac{3}{4} \quad \Rightarrow r = 21 \text{ m}$$

$$\therefore \text{ height } h = \sqrt{\ell^2 - r^2} = \sqrt{28^2 - 21^2} = \sqrt{7^2(4^2 - 3^2)} = 7\sqrt{16-9} = 7\sqrt{7} \text{ m}$$

$$\therefore\ h = 7\sqrt{7} \text{ m and } r = 21 \text{ m.}$$

Sphere

Sphere is a set of points in the space which are equidistant from a fixed point. The fixed point is called the centre of the sphere, and the distance is called the radius of the sphere. A lemon, a foot ball, the moon, globe, the Earth, small lead balls used in cycle bearings are some objects which are spherical in shape.

A line joining any two points on the surface of sphere and passing through the centre of the sphere is called its diameter.

The size of sphere can be completely determined by knowing its radius or diameter.

Solid sphere

A solid sphere is the region in space bounded by a sphere. The centre of a sphere is also a part of solid sphere whereas the centre is not a part of hollow sphere. Marbles, lead shots, etc. are the examples of solid spheres while a tennis ball is a hollow sphere.

Hollow sphere

From a solid sphere a smaller sphere having the same centre of the solid sphere, is cut off, then we obtain a hollow sphere. This can also be called a spherical shell.

Hemisphere

If a sphere is cut into two halves by a plane passing through the centre of sphere, then each of the halves is called a hemisphere.

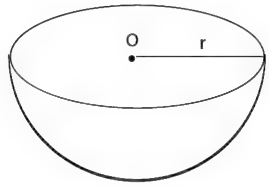


Figure 14.15 Hemisphere

Hemispherical shell

A hemispherical shell is shown in the figure given below.

Figure 14.16 Hemispherical shell

Formulae to memorize

Sphere

1. Surface area of a sphere = $4\pi r^2$ sq.units.
2. Volume of a sphere = $\frac{4}{3}\pi r^3$ sq.units.

Spherical shell/hollow sphere

1. Thickness = R − r, where R = outer radius, r = inner radius.
2. Volume = $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$ cubic units.
3. Total surface area of a hemi-spherical shell = $\frac{1}{2}$(surface area of outer hemisphere + surface area of inner hemisphere + area of ring)

Hemisphere

1. Curved surface area of a hemisphere = $2\pi r^2$ sq.units
2. Total surface area of a hemisphere = $3\pi r^2$ sq.units
3. Volume of a hemisphere = $\frac{2}{3}\pi r^3$ cubic units.

Examples

1. The cost of painting a solid sphere at the rate of 50 paise per square metre is Rs 1232. Find the volume of steel required to make the sphere.

Solution

Cost of painting = Surface area × Rate of painting.

$$\therefore \text{ Surface area} = \frac{\text{Cost of painting}}{\text{Rate of painting}} = \frac{1232}{0.5} = 2464 \text{ m}^2$$

$$\Rightarrow 4\pi r^2 = 2464 \Rightarrow r^2 = \frac{2464}{4\pi} = \frac{616}{\left(\frac{22}{7}\right)} = \frac{616\times7}{22} = 28 \times 7$$

$\Rightarrow r = 7 \times 2 = 14$ m

$$\therefore \text{Volume of steel required} = \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times 14\times 14\times 14 = \frac{34496}{3}\text{m}^3$$

2. A hollow hemispherical bowl of thickness 1 cm has an inner radius of 6 cm. Find the volume of metal required to make the bowl.

Solution

Inner radius, r = 6 cm
thickness, t = 1 cm
$\therefore$ outer radius, R = r + t = 6 + 1 = 7 cm

Figure 14.17

$$\therefore \text{Volume of steel required} = \frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\times\frac{22}{7}\times 7^3 - \frac{2}{3}\times\frac{22}{7}\times 6^3 = \frac{44}{21}(7^3-6^3)$$

$$= \frac{44}{21}(343-216) = \frac{44}{21}\times 127 = \frac{5588}{21}\text{ cm}^3$$

3. A thin hollow hemispherical sailing vessel is made of metal covered by a conical canvas tent. The radius of the hemisphere is 14 m and total height of vessel (including the height of tent) is 28 m. Find area of metal sheet and the canvas required.

Solution

The vessel (with the conical tent) is shown in figure.

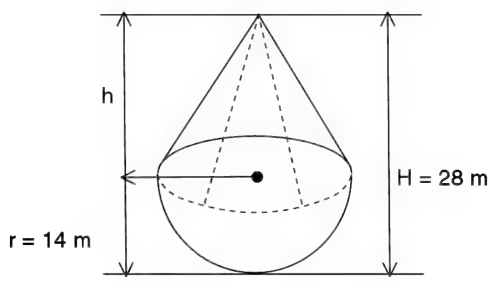


Figure 14.18

Total height, H = 28 m
Radius of hemisphere = r = 14 m
$\therefore$ height of conical tent = h = H – r = 28 – 14 = 14 m.
We can observe that radius of base of cone = radius of the hemisphere = 14 m.

$$\therefore \text{Area of canvas required} = \pi rl = \frac{22}{7}\times 14\times\sqrt{14^2+14^2} = 44\times 14\sqrt{2} = 616\sqrt{2}\text{ m}^2$$

$$\text{Area of metal sheet required} = \text{surface area of hemisphere} = 2\pi r^2 = 2\times\frac{22}{7}\times 14\times 14 = 1232\text{ m}^2$$

test your concepts

Very short answer type questions

1. If the length of the side of an equilateral triangle is 12 cm, then what is its inradius?
2. The radius of a circle is 8 cm and O is its centre. If $\angle AOB = 60°$ and AB is a chord, then what is the length of the chord AB?
3. The circum radius of an equilateral triangle is x cm. What is the perimeter of the triangle in terms of x?
4. If the difference between the outer radius and the inner radius of a ring is 14 cm, then what is the difference between its outer circumference and inner circumference?
5. The area of a ring is 22 cm^2. What is the difference of the square of the outer radius and the square of the inner radius?
6. A cone is formed by joining together the two straight edges of a sector, so that they coincide with each other. The length of the arc of the sector becomes the __________ of the circular base and radius of sector becomes the __________ of the cone.
7. The volume of a cube with diagonal d is __________.
8. If the total surface area of a cube is $\frac{50}{3}$ m^2, then find its side.
9. Find the maximum number of soaps of size 2 cm × 3 cm × 5 cm that can be kept in a cuboidal box of dimensions 6 cm × 3 cm × 15 cm.
10. Total number of faces in a prism which has 12 edges is ________.
11. W, P, H and A are whole surface area, perimeter of base, height and area of the base of a prism respectively. The relation between W, P, H and A is __________.
12. If s is the perimeter of the base of a prism, n is the number of sides of the base, S is the total length of the edges and h is the height, then S = __________.
13. If the number of lateral surfaces of a right prism is equal to n, then the number of edges of the base of the prism is __________.
14. If f, e and v represent the number of rectangular faces, number of edges and number of vertices respectively of a cuboid, then the values of f, e, and v respectively are __________.
15. Find the number of vertices of a pyramid, whose base is a pentagon.
16. A and B are the volumes of a pyramid and a right prism respectively. If the pyramid and the prism have the same base area and the same height, then what is the relation between A and B?
17. If the ratio of the base radii of two cones having the same curved surface areas is 6 : 7, then the ratio of their slant heights is __________.
18. The heights of two cones are equal and the radii of their bases are R and r. The ratio of their volumes is __________.

19. If the heights of two cylinders are equal and their radii are in the ratio of 7 : 5, then the ratio of their volumes is ____________.

20. Volumes of two cylinders of radii R, r and heights H, h respectively are equal. Then R^2H = ____________.

21. The volumes of two cylinders of radii R, r and heights H, h respectively are equal. If R : r = 2 : 3, then H : h = ____________.

22. A sector of a circle of radius 6 cm and central angle 30° is folded into a cone such that the radius of the sector becomes the slant height of the cone. What is the radius of the base of the cone thus formed?

23. If R and r are the external and the internal radii of a hemispherical bowl, then what is the area of the ring, which forms the edge of the bowl (in sq.units)?

24. What is the volume of a hollow cylinder with R, r and h as outer radius, inner radius and height respectively?

25. The side of a cube is equal to the radius of the sphere. Find the ratio of their volumes.

26. A sphere and the base of a cylinder have equal radii. The diameter of the sphere is equal to the height of the cylinder. The ratio of the curved surface area of the cylinder and surface area of the sphere is ___________.

27. A road roller of length 3ℓ metres and radius $\frac{\ell}{3}$ metres can cover a field in 100 revolutions, moving once over. The area of the field in terms of ℓ is ______ m^3.

28. What is the volume of sand to be spread uniformly over a ground of dimensions. 10x m × 8x m upto a height of 0.1x m?

29. The outer radius and the inner radius of a hollow cylinder are (2 + x) cm and (2 – x) cm. What is its thickness?

30. The slant height, outer radius and inner radius of a cone frustum are 2a cm, (a + b) cm and (a – b) cm. What is its curved surface area?

Short answer type questions

31. A circle is inscribed in an equilateral triangle. If the inradius is 21 cm, what is the area of the triangle?

32. Three cubes each of side 3.2 cm are joined end to end. Find the total surface area of the resulting cuboid.

33. A square is drawn with the length of side equal to the diagonal of a cube. If the area of the square is 72075 cm^2, then find the side of the cube.

34. What is the area of a ground that can be levelled by a cylindrical roller of radius 3·5 m and 4 m long by making 10 rounds?

35. A square of side 28 cm is folded into a cylinder by joining its two sides. Find the base area of the cylinder thus formed.

36. Find the number of cubes of side 2 m to be dropped in a cylindrical vessel of radius 14 m in order to increase the water level by 5 m.

37. Find the capacity of a closed cuboidal cistern whose length is 3 m, breadth is 2 m and height is 6 m. Also find the area of iron sheet required to make the cistern.

38. An open metallic conical tank is 6 m deep and its circular top has diameter of 16 m. Find the cost of tin plating its inner surface at the rate of Rs 0.8 per 100 cm^2. (Take $\pi = 3.14$)

39. The total surface area of a hemisphere is 3768 cm^2. Find the radius of the hemisphere. (Take $\pi = 3.14$)

40. The base radius of a conical tent is 120 cm and its slant height is 750 cm. Find the area of the canvas required to make 10 such tents (in m^2). (Take $\pi = 3.14$)

41. From a cylindrical wooden log of length 30 cm and base radius $7\sqrt{2}$ cm, biggest cuboid of square base is made. Find the volume of wood wasted.

42. A right circular cone is such that the angle at its vertex is 90° and its base radius is 49 cm, then find the curved surface area of the cone.

43. The base of a right pyramid is an equilateral triangle, each side of which is $6\sqrt{3}$ cm long and its height is 4 cm. Find the total surface area of the pyramid in cm^2.

44. If the thickness of a hemispherical bowl is 12 cm and its outer diameter is 10·24 m, then find the inner surface area of the hemisphere. (Take $\pi = 3.14$)

45. A spherical piece of metal of diameter 6 cm is drawn into a wire of 4 mm in diameter. Find the length of the wire.

Essay type questions

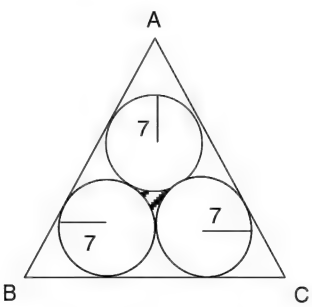
46. The cost of the canvas required to make a conical tent of base radius 8 m at the rate of Rs 40 per m^2 is Rs 10048. Find the height of the tent. (Take $\pi = 3.14$)

47. A hollow sphere which has internal and external diameter as 16 cm and 14 cm respectively is melted into a cone with a height of 16 cm. Find the diameter of the base of the cone.

48. A drum in the shape of a frustum of a cone with radii 24 ft and 15 ft and height 5 ft is full of water. The drum is emptied into a rectangular tank of base 99 ft × 43 ft. Find the rise in the height of the water level in the tank.

49. A cylindrical tank of radius 7 m, has water to some level. If 110 cubes of side 7 dm are completely immersed in it, then find the rise in the water level in the tank. (in metres)

50. Find the area of the shaded portion in the figure given below, where ABC is an equilateral triangle and the radius of each circle is 7 cm.



CONCEPT APPLICATION

Concept Application Level—1

1. The area of a sector whose perimeter is four times its radius (r units) is

(1) $\sqrt{r}$ sq.units. (2) r^4 sq.units. (3) r^2 sq.units. (4) $\frac{r^2}{2}$ sq.units.

2. A chord of a circle of radius 28 cm makes an angle of 90° at the centre. Find the area of the major segment.

(1) 1456 cm^2 (2) 1848 cm^2 (3) 392 cm^2 (4) 2240 cm^2

3. The area of a circle inscribed in an equilateral triangle is 48π square units. What is the perimeter of the triangle?

(1) $17\sqrt{3}$ units (2) 36 units (3) 72 units (4) $48\sqrt{3}$ units

4. Two circles touch each other externally. The distance between the centres of the circles is 14 cm and the sum of their areas is 308 cm^2. Find the difference between radii of the circles. (in cm)

(1) 1 (2) 2 (3) 0 (4) 0.5

5. If the outer and the inner radii of a circular track are 7 m and 3.5 m respectively, then the area of the track is

(1) 100 m^2 (2) 178 m^2 (3) 115.5 m^2 (4) 135.5 m^2

6. The base of a right pyramid is an equilateral triangle of perimeter 8 dm and the height of the pyramid is $30\sqrt{3}$ cm. Find the volume of the pyramid.

(1) 16000 cm^3 (2) 1600 cm^3 (3) $\frac{16000}{3}$ cm^3 (4) $\frac{5}{4}$ cm^3

7. The volume of a cuboid is $20\sqrt{42}\,m^3$. Its length is $5\sqrt{2}$ m, breadth and height are in the ratio $\sqrt{3} : \sqrt{7}$. Find its height.

(1) $\sqrt{7}$ m (2) $3\sqrt{7}$ m (3) $4\sqrt{7}$ m (4) $2\sqrt{7}$ m

8. A metal cube of edge $\frac{3\sqrt{2}}{\sqrt{5}}$ m is melted and formed into three smaller cubes. If the edges of the two smaller cubes are $\frac{3}{\sqrt{10}}$ m and $\frac{\sqrt{5}}{\sqrt{2}}$ m, find the edge of the third smaller cube.

(1) $\frac{3}{\sqrt{7}}$ m (2) $\frac{6}{\sqrt{15}}$ m (3) $\frac{5}{\sqrt{11}}$ m (4) $\frac{4}{\sqrt{10}}$ m

9. Find the volume of the space covered by rotating a rectangular sheet of dimensions 16.1 cm × 7.5 cm along its length.

(1) 2846.25 cm^3 (2) 2664 cm^3 (3) 2864.25 cm^3 (4) 2684 cm^3

10. The base of a right prism is an equilateral triangle of edge 12 m. If the volume of the prism is $288\sqrt{3}$ m^3, then its height is

(1) 6 m (2) 8 m (3) 10 m (4) 12 m

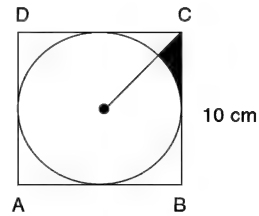
11. A roller levelled an area of 165000 sq m in 125 revolutions, whose length is 28 m. Find the radius of the roller.

(1) 7.5 m (2) 8.5 m (3) 6.5 m (4) 7 m

12. A large sphere of radius 3.5 cm is carved from a cubical solid. Find the difference between their surface areas.

(1) 122 cm^2 (2) 80.5 cm^2 (3) 144.5 cm^2 (4) 140 cm^2

13. In the figure given below, ABCD is a square of side 10 cm and a circle is inscribed in it. Find the area of the shaded part as shown in the figure.



(1) $\left(\dfrac{100-36\pi}{41}\right)$ cm^2 (2) $\left(\dfrac{100-25\pi}{8}\right)$ cm^2

(3) $\left(\dfrac{100+25\pi}{8}\right)$ cm^2 (4) None of these

14. The outer curved surface area of a cylindrical metal pipe is 1100 m^2 and the length of the pipe is 25 m. The outer radius of the pipe is

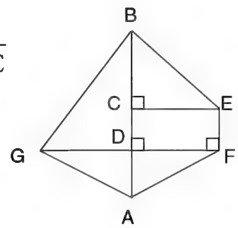
(1) 8 m (2) 9 m (3) 7 m (4) 6 m

15. The volume of a hemisphere is 2.25π cm^3. What is the total surface area of the hemisphere.

(1) 2.25π cm^2 (2) 5π cm^2 (3) 6.75π cm^2 (4) 4.5π cm^2

16. Find the area of the figure given below, in which

AB = 100 m, CE = 30 m, C is mid-point of $\overline{AB}$ and D is mid-point of $\overline{AC}$ and $\overline{GF}$.



(1) 5250 m^2 (2) 3750 m^2

(3) 3375 m^2 (4) 3175 m^2

17. Find the volume of the greatest right circular cone, which can be cut from a cube of a side 4 cm. (in cm^3)

(1) $\dfrac{12\pi}{5}$ (2) $\dfrac{20\pi}{3}$ (3) $\dfrac{18\pi}{5}$ (4) $\dfrac{16\pi}{3}$

18. The area of the base of a right equilateral triangular prism is $16\sqrt{3}$ cm^2. If the height of the prism is 12 cm, then the lateral surface area and the total surface area of the prism respectively are

(1) 288 cm^2, $(288 + 32\sqrt{3})$ cm^2 (2) 388 cm^2, $(388 + 32\sqrt{3})$ cm^2

(3) 288 cm^2, $(288 + 24\sqrt{3})$ cm^2 (4) 388 cm^2, $(388 + 24\sqrt{3})$ cm^2

19. A metallic cone of diameter 32 cm and height 9 cm is melted and made into identical spheres, each of radius 2 cm. How many such spheres can be made?

(1) 72 (2) 64 (3) 52 (4) 48

20. A cylindrical vessel open at the top has a base radius of 28 cm. If the total cost of painting the outer part of the vessel is Rs 357 at the rate of Re 0.2 per 100 cm^2, then find the height of the vessel. (approximately)

(1) 10 m (2) 9 m (3) 8 m (4) 4 m

21. The radii of the ends of a bucket 16 cm high are 20 cm and 8 cm. Find the curved surface area of the bucket.

(1) 1760 cm^2 (2) 2240 cm^2 (3) 880 cm^2 (4) 3120 cm^2

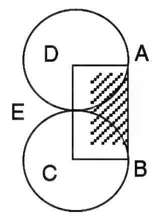
22. A cylindrical vessel of radius 8 cm contains water. A solid sphere of radius 6 cm is lowered into the water until it is completely immersed. What is the rise in the water level in the vessel?

(1) 3 cm (2) 3.5 cm (3) 4 cm (4) 4.5 cm

23. What is the difference in the areas of the regular hexagon circumscribing a circle of radius 10 cm and the regular hexagon inscribed in the circle?

(1) 50 cm^2 (2) $50\sqrt{3}$ cm^2 (3) $100\sqrt{3}$ cm^2 (4) $100\sqrt{3}$ cm^2

24. In the shown figure, two circles of radii of 7 cm each, are shown. ABCD is rectangle and AD and BC are the radii. Find the area of the shaded region (in cm^2).



(1) 20
(2) 21
(3) 19
(4) 18

25. There is a closed rectangular shed of dimensions 10 m × 4 m inside a field. A cow is tied at one corner of outside of the shed with a 6 m long rope. What is the area that the cow can graze in the field?

(1) 66 m^2 (2) 88 m^2 (3) 0.8π m^2 (4) 27π m^2

26. The base of a right prism is a square of perimeter 20 cm and its height is 30 cm. What is the volume of the prism?

(1) 700 cm^3 (2) 750 cm^3 (3) 800 cm^3 (4) 850 cm^3

27. A cylindrical tank with radius 60 cm is being filled by a circular pipe with internal diameter of 4 cm at the rate of 11 m/sec. Find the height of the water column in 18 minutes.

(1) 66 m (2) 12.2 m (3) 13.2 m (4) 6.1 m

28. A conical cup when filled with ice cream forms a hemispherical shape on its open end. Find the volume of ice cream (approximately), if radius of the base of the cone is 3.5 cm, the vertical height of cone is 7 cm and width of the cone is negligible.

(1) 120 cm^3 (2) 150 cm^3 (3) 180 cm^3 (4) 210 cm^3

29. A hemispherical bowl of internal diameter 24 cm contains water. This water is to be filled in cylindrical bottles, each of radius 6 cm and height 8 cm. How many such bottles are required to empty the bowl?

(1) 3 (2) 4 (3) 5 (4) 6

30. A dome of a building is in the form of a hemisphere. The total cost of white washing it from inside, was Rs 1330.56. The rate at which it was white washed is Rs 3 per square metre. Find the volume of the dome approximately.

(1) $1150.53\ m^3$ (2) $1050\ m^3$ (3) $1241.9\ m^3$ (4) $1500\ m^3$

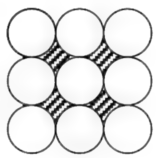
Concept Application Level—2

31. A circular garden of radius 15 m is surrounded by a circular path of width 7 m. If the path is to be covered with tiles at a rate of Rs 10 per m^2, then find the total cost of the work. (in Rs)

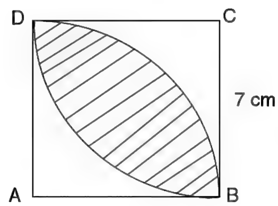
(1) 8410 (2) 7140 (3) 8140 (4) 7410

32. Find the area of the shaded region, given that the radius of each circle is equal to 5 cm.

(1) $(400 - 100\pi)\ cm^2$
(2) $(360 - 100\pi)\ cm^2$
(3) $231\ cm^2$
(4) $(400 - 50\pi)\ cm^2$



33. In the figure given below, ABCD is a square of side 7 cm. BD is an arc of a circle of radius AB. What is the area of the shaded region?



(1) $14\ cm^2$
(2) $21\ cm^2$
(3) $28\ cm^2$
(4) $35\ cm^2$

34. The volume of a right prism, whose base is an equilateral triangle, is $1500\sqrt{3}\ cm^3$ and the height of the prism is 125 cm. Find the side of the base of the prism.

(1) $8\sqrt{3}$ cm (2) $4\sqrt{3}$ cm (3) $16\sqrt{3}$ cm (4) $24\sqrt{3}$ cm

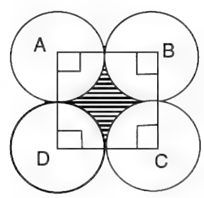
35. A right circular cylinder of volume $1386\ cm^3$ is cut from a right circular cylinder of radius 4 cm and height 49 cm, such that a hollow cylinder of uniform thickness, with a height of 49 cm and an outer raidus of 4 cm is left behind. Find the thickness of the hollow cylinder left behind.

(1) 0.5 cm (2) 2 cm (3) 1.5 cm (4) 1 cm

36. The volume of a hemisphere is $18\pi\ cm^3$. What is the total surface area of the hemisphere?

(1) $18\pi\ cm^2$ (2) $27\pi\ cm^2$ (3) $21\pi\ cm^2$ (3) $24\pi\ cm^2$

37. The diagram shown above has four circles of 7 cm radius with centres at A, B, C and D. If the quadrilateral ABCD represents a square, then find the area of the shaded region.



(1) $42\ cm^2$ (2) $21\ cm^2$
(3) $63\ cm^2$ (4) $84\ cm^2$

38. Find the total surface area of a hollow metallic hemisphere whose internal radius is 14 cm and the thickness of the metal is 7 cm.

(1) $4774\ cm^2$ (2) $4477\ cm^2$ (3) $4747\ cm^2$ (4) $7744\ cm^2$

39. A metal cube of edge $\frac{3}{10}$ m is melted and formed into three smaller cubes. If the edges of the two smaller cubes are $\frac{1}{5}$ m and $\frac{1}{4}$ m, find the edge of the third smaller cube.

(1) $\frac{7}{20}$ m (2) $\frac{1}{20}$ m (3) $\frac{3}{20}$ m (4) None of these

40. The hour hand of a clock is 6 cm long. Find the area swept by it between 11:20 am and 11:55 am (in cm^2)

(1) 2.75 (2) 5.5 (3) 11 (4) None of these

41. Two hemispherical vessels can hold 10.8 litres and 50 litres of liquid respectively. The ratio of their inner curved surface areas is

(1) 16 : 25 (2) 25 : 9 (3) 9 : 25 (4) 4 : 3

42. A cylindrical drum 1.5 m in diameter and 3 m in height is full of water. The water is emptied into another cylindrical tank in which water rises by 2 m. Find the diameter of the second cylinder up to 2 decimal places.

(1) 1.74 m (2) 1.94 m (3) 1.64 m (4) 1.84 m

43. Curved surface area of a conical cup is $154\sqrt{2}$ cm^2 and base radius is 7 cm. Find the angle at the vertex of the conical cup.

(1) 90° (2) 60° (3) 45° (4) 30°

44. The sum of the length, breadth and the height of a cuboid is $5\sqrt{3}$ cm and length of its diagonal is $3\sqrt{5}$ cm. Find the total surface area of the cuboid.

(1) 30 cm^2 (2) 20 cm^2 (3) 15 cm^2 (4) 18 cm^2

45. An equilateral triangle has a circle inscribed in it and is circumscribed by a circle. There is another equilateral triangle inscribed in the inner circle. Find the ratio of the areas of the outer circle and the inner equilateral triangle.

(1) $\frac{16\pi}{3\sqrt{3}}$ (2) $\frac{8\pi}{2\sqrt{3}}$ (3) $\frac{24\pi}{3\sqrt{3}}$ (4) None of these

Concept Application Level—3

46. An ink pen, with a cylindrical barrel of diameter 2 cm and height 10.5 cm, and completely filled with ink, can be used to write 4950 words. How many words can be written using 400 ml of ink? (Take 1 litre = 1000 cm^3)

(1) 40000 (2) 60000 (3) 45000 (4) 80000

47. Each of height and side of the base of a regular hexagonal pyramid is equal to x cm. Find its lateral surface area in terms of x (in cm^2).

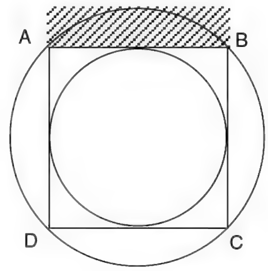
(1) $\frac{9\sqrt{7}}{2}x^2$ (2) $\frac{7\sqrt{7}}{2}x^2$ (3) $\frac{5\sqrt{7}}{2}x^2$ (4) $\frac{3\sqrt{7}}{2}x^2$

48. The diameters of the top and the bottom portions of a bucket are 42 cm and 28 cm. If the height of the bucket is 24 cm, then find the cost of painting its outer surface at the rate of 5 paise/cm^2.

(1) Rs 158.25 (2) Rs 172.45

(3) Rs 168.30 (4) Rs 164.20

49. In the following figure, a circle is inscribed in square ABCD and the square is circumscribed by a circle. If the radius of the smaller circle is r cm, then find the area of the shaded region (in cm^2).

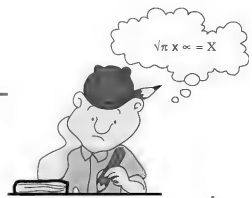


(1) $\left(\frac{\pi - 2}{4}\right)r^2$ (2) $\left(\frac{3\pi - 4}{2}\right)r^2$

(3) $\left(\frac{\pi + 2}{4}\right)r^2$ (4) $\left(\frac{\pi - 2}{2}\right)r^2$

50. ABCD is a square of side 4 cm. If E is a point in the interior of the square such that ΔCED is equilateral, then find the area of ΔACE (in cm^2).

(1) $2(\sqrt{3} - 1)$ (2) $4(\sqrt{3} - 1)$ (3) $6(\sqrt{3} - 1)$ (4) $8(\sqrt{3} - 1)$



KEY

Very short answer type questions

1. $2\sqrt{3}$ cm **2.** 8 cm

3. $3\sqrt{3}$ x cm **4.** 88 cm

5. 7 cm. **6.** 0.5 cm

7. $\frac{d^3}{3\sqrt{3}}$ cu units. **8.** $\frac{5}{3}$ m

9. 9 **10.** 6

11. W = P × H + 2 A **12.** nh + 2s

13. n **14.** 6, 12 and 8

15. 6 **16.** $A = \frac{B}{3}$

17. 7 : 6 **18.** $R^2 : r^2$

19. 49 : 25 **20.** r^2 h

21. 9 : 4

22. circumference, slant height

23. $\pi(R^2 - r^2)$

24. $\pi (R^2 - r^2)h$

25. 21 : 88 **26.** 1 : 1.

27. $200\pi \ell^2$ cm^2 **28.** $8x^3$ m^3

29. 2x cm **30.** $4 \pi a^2$ cm^2

Short answer type questions

31. $1323\sqrt{3}$ sq. units **32.** 143.36 cm^2

33. 155 cm **34.** 880 m^2

35. $\frac{686}{11}$ cm^2 **36.** 385

37. 72 m^2 **38.** Rs 20096

39. 157 m^2 **40.** 282.6 m^2

41. 3360 cm^3 **42.** $7546\sqrt{2}$ cm^2

43. $72\sqrt{3}$ cm^2 **44.** 20 cm

45. 900 cm

48. $1\frac{3}{7}$ ft

Essay type questions

46. 251.2 m^2, 6 m **47.** 13 cm

49. 0.245

50. 7.87 cm^2

key points for selected questions

Short answer type questions

31. In radius of a circle = 1/3 of the median

32. (i) Length of newly formed cuboid is thrice the length of the side of any cube.
(ii) Its breadth and height are equal to the length of the side of the cube.

33. Diagonal of a cube of edge a units is $\sqrt{3}$ a = side of a square

34. The area that can be levelled by a roller for one round = CSA of cylinder.

35. (i) Calculate the radius, equating circumference of the base of the cylinder to the side of the square.
(ii) Now find the area of base (i.e., circle).

36. (i) Let the number of cubes be x.
(ii) Now, equate the volume of x cubes to the volume of cylinder (height 5 m) and find x.

37. (i) Capacity of a cistern is its volume.
(ii) Area of the iron sheet is the total surface area of the cuboid.

38. Cost of tin plating = C.S.A of cone × cost per cm^2

39. Equate the given area to $3\pi r^2$ and solve for r.

40. (i) Calculate the CSA of one tent, and multiply with 10.

41. (i) Equate the base diameter of the cylinder to the diagonal of the square base.
(ii) Then, find the volumes of the cuboid so formed and also of the cylinder.
(iii) The required value is the difference of their volumes.

42. (i) The height, radius and slant height are in the ratio of $1 : 1 : \sqrt{2}$.
(ii) Evaluate the slant height by using this ratio and radius given.
(iii) Then find CSA of the cone.

43. (i) Find the slant height (ℓ) of the pyramid.
(ii) Then, TSA = LSA + area of base, where LSA = $\frac{1}{2}p\ell$, where p is the perimeter of the base.

44. (i) Calculate inner radius i.e., outer radius – thickness.
(ii) Find the inner surface area i.e., CSA of hemisphere.

45. (i) Assume the length of the wire as ℓ cm.
(ii) Calculate the volume of the sphere and the volume of the wire (i.e., cylinder)
(iii) Equate the volumes obtained and find ℓ.

Essay type questions

46. C.S.A of a conical tent × cost per m^2 = Total cost

47. Volume of the hollow sphere = Volume of the cone formed

48. Rise in water level = $\dfrac{\text{volume of water filled}}{\text{base area of the tank}}$

$= \dfrac{\text{volume of the frustum}}{\text{base area of the cuboid}}$.

49. (i) One decimetre = 0.1 metre.

(ii) Volume of water risen in the cylinder = Volume of the total number of cubes.

50. (i) Find the area of the equilateral triangle formed by joining the centres of the circles.

(ii) Area of shaded portion = Area of new equilateral triangle – Area of 3 sectors.

KEY

Concept Application Level—1,2,3

1. 3	**2.** 3
3. 4	**4.** 4
5. 1	**6.** 3
7. 3	**8.** 3
9. 4	**10.** 2
11. 1	**12.** 4
13. 2	**14.** 3
15. 3	**16.** 3
17. 4	**18.** 1
19. 1	**20.** 1
21. 1	**22.** 4
23. 2	**24.** 2
25. 2	**26.** 2
27. 3	**28.** 3
29. 2	**30.** 3
31. 3	**32.** 1
33. 3	**34.** 2
35. 4	**36.** 2
37. 1	**38.** 1
39. 3	**40.** 2
41. 3	**42.** 4
43. 1	**44.** 1
45. 1	**46.** 2
47. 4	**48.** 3
49. 4	**50.** 2

Concept Application Level—1,2,3

Key points for select questions

1. Perimeter of a sector = $\ell + 2r = 4r$

2. Find the area of triangle and the area of sector formed by the chord.

3. Radius of the circle = 1/3 of the median.

4. Use $(a + b)^2 = a^2 + b^2 + 2ab$ and $a - b = \sqrt{(a+b)^2 - 4ab}$.

5. Area of the track = Area of outer circle – Area of inner circle

6. Volume of the pyramid = 1/3 area of the base × height

7. Breadth = $\sqrt{3}$ x m and Height = $\sqrt{7}$ x m.

8. Volume of large cube is equal to the sum of volumes of three small cubes.

9. Find the volume of cylinder. Whose radius is breadth of rectangle and height is equal to length of the rectangle.

10. Volume of the prism = Area of the base × height

11. Area levelled by the roller in one revolution = CSA of the cylinder.

12. Diameter of the sphere = length of the edge of a cube.

13. Draw $\overline{OP}$ from the centre to the midpoint of $\overline{BC}$

14. C.S.A of a cylinder = $2\pi Rh$

15. Find 'r' by using volume formula

16. Find individual areas of different parts of the figure.

17. Diameter of base of cone is edge of cube.

18. Find the edge of the prism.

19. Volume of the cone = Volume of all the spheres formed

20. TSA of outer part = $357 \times \frac{100}{0.2} cm^2$

21. C.S.A of bucket = $\pi\ell(R+r)$

22. Equate the two volumes.

23. Find the area of two hexagon.

24. Required area is the difference of areas of rectangle and sum of areas of two sectors.

25. Draw the diagram and proceed.

26. Volume of the prism = Area of the base × height

27. Volume of water in the tank = Area of the cross sections of the pipe × rate × time

28. Required volume = Volume of cone + volume of sphere.

29. Volume of bowl = Volume of a bottle × number of bottles.

30. Volume of a hemisphere = $\frac{2}{3}\pi r^3$ cubic units.

31. (i) Area of circular path = Area of ring.
 (ii) Area of path = $\pi(R^2 - r^2)$
 (iii) Total cost = Area × cost/m^2.

32. (i) Find the area of the square formed by joining the centers of all outer circles.
 (ii) The required area = Area of the square $- 16(\frac{1}{4}$ area of circle)

33. Area of shaded region = 2(Area of sector $\overline{BAD}$ – Area of ΔABD)

34. (i) Volume of prism = $\frac{\sqrt{3}}{4} a^2 \times$ height
 (ii) Use the above formula and get the value of a.

35. Find the radius of the cylinder which is cut. (i.e., $\pi r^2 h = 1386$)

36. Radius of the hemisphere = Radius of the sphere

37. Area of shaded region = Area of square ABCD $- 4(\frac{1}{4}$ area of circle)

38. T.S.A. = $3\pi R^2 + \pi r^2$

39. Volume of big cube = Sum of the volumes of smaller cubes.

40. Angle made by the hours hand of the clock in 35 minutes is 17.5° (angle of sector)

41. Ratio of their C.S.A's is $r_1^2 : r_2^2$

42. Volume of water in the cylindrical drum = Volume of the second cylinder up to the water risen.

43. Use, $\tan\left(\frac{\theta}{2}\right) = \frac{r}{h}$ and find θ.

44. (i) Use suitable algebraic identity to find the L.S.A of the cuboid.
 (ii) $\ell + b + h = 5\sqrt{3}$ and $\ell^2 + b^2 + h^2 = 3\sqrt{5}$
 (iii) Square the first equation and evaluate $2(\ell b + bh + h\ell)$

45. (i) Circum radius = $\frac{a}{\sqrt{3}}$ and inradius $= \frac{a}{2\sqrt{3}}$, where a is the side of the outer equilateral triangle.

(ii) For an equilateral triangle of side a, if an incircle and circum circle are drawn whose radii are r and R then $r = \frac{a}{2\sqrt{3}}$ and $R = \frac{a}{\sqrt{3}}$

46. (i) 1 ml = 1 cm^3

(ii) Find the number of words per 1 ml of ink

47. C.S.A of a pyramid = ½ perimeter of the base × slant height.

48. C.S.A of a bucket = $\pi\ell\,(R+r)$

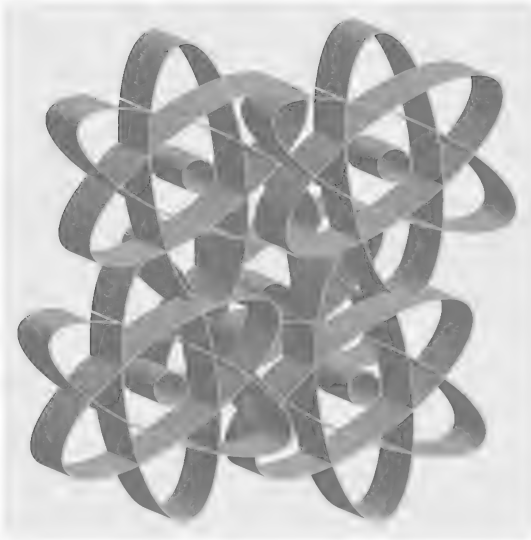
49. Diagonals of the square meet at the centre of the circles.

50. (i) Draw the figure according to the data and draw $\overline{EF} \parallel \overline{CD}$

(ii) Area of ΔACE = Area of the triangle ABC − {Area of Δ ECD + Area of ABED}

CHAPTER 15

Coordinate Geometry



INTRODUCTION

Let X^1OX and YOY^1 be two mutually perpendicular lines intersecting at the point O in a plane.

These two lines are called reference lines or coordinate axes. The horizontal reference line X^1OX is called X-axis and the vertical reference line YOY^1 is called Y-axis.

The point of intersection of these two axes i.e., O is called the origin. The plane containing the coordinate axes is called coordinate plane or XY– plane.

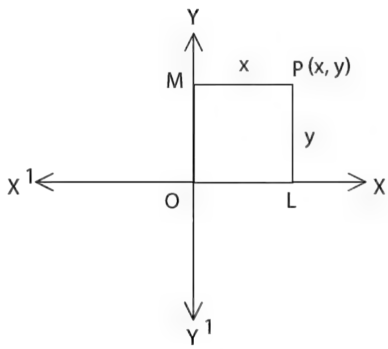


Figure 15.1

Coordinates of a point

Let P be a point in the XY–plane. Draw perpendiculars PL and PM to X-axis and Y-axis respectively.

Let PL = y and PM = x. Then, the point P is taken as (x, y). Here x and y are called the rectangular cartesian coordinates or simply coordinates of the point P. x is called x-coordinate or abscissa and y is called y-coordinate or ordinate of the point P. P (x, y) is x units away from Y-axis and y units away from X-axis.

Convention of signs

(i) Towards the right side of the Y-axis, x-coordinate of any point on the graph paper is taken positive and towards the left side of the Y-axis, x-coordinate is taken negative.

(ii) Above the X-axis, the y-coordinate of any point on the graph paper is taken positive and below the X-axis, y-coordinate is taken negative.

If (x, y) is a point in the plane and Q_1, Q_2, Q_3, Q_4 are the four quadrants of rectangular coordinate system, then

1. If $x > 0$ and $y > 0$, then $(x, y) \in Q_1$.
2. If $x < 0$ and $y > 0$, then $(x, y) \in Q_2$.

3. If $x < 0$ and $y < 0$, then $(x, y) \in Q_3$.
4. If $x > 0$ and $y < 0$, then $(x, y) \in Q_4$.

Example

If $x < 0$ and $y > 0$, then $(-x, y)$ lies in which quadrant?

Solution

$x < 0 \Rightarrow -x > 0$

$\therefore$ The point $(-x, y)$ lies in the first quadrant i.e., Q_1.

Example

If $(a, b) \in Q_3$, then $(-a, -b)$ belongs to which quadrant?

Solution

Given $(a, b) \in Q_3$, $\Rightarrow a < 0$, $b < 0$ then $-a$ is positive and $-b$ is also positive therefore $(-a, -b) \in Q_1$

Example

Plot the points A(2, 3), B(−1, 2), C(−3, −2), and D(4, −2) in the XY plane.

Solution

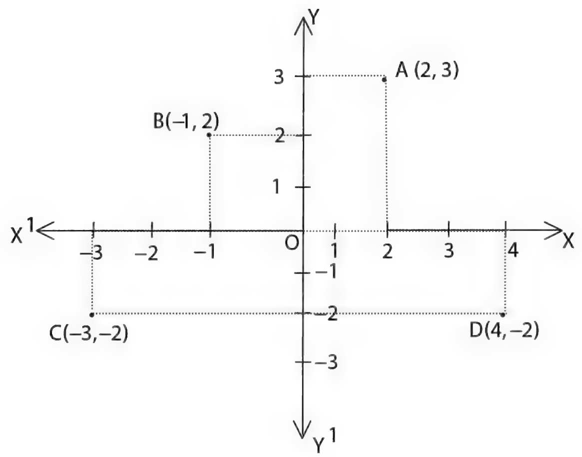


Figure 15.2

Points on the plane

Point on X-axis and Y-axis

Let P be a point on X-axis, so that its distance from X-axis is zero. Hence, the point P can be taken as (x, 0).

Let P^1 be a point on Y-axis, so that its distance from Y-axis is zero. Hence, the point P^1 can be taken as (0, y).

Distance between two points

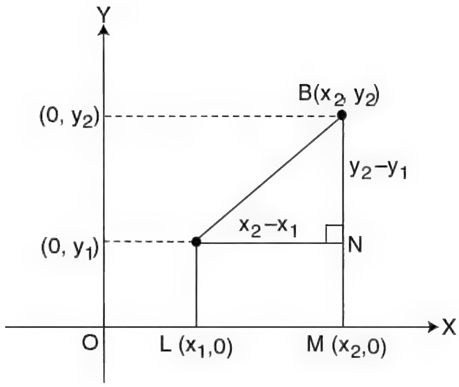


Figure 15.3

Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Draw perpendiculars AL and BM from A and B to X-axis, AN is the perpendicular drawn from A on to BM.

From right triangle ABN, $AB = \sqrt{AN^2 + BN^2}$

Here, $AN = x_2 - x_1$, and $BN = y_2 - y_1$

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Hence, the distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ units.

NOTE: The distance of a point $A(x_1, y_1)$ from origin $O(0, 0)$ is $OA = \sqrt{x_1^2 + y_1^2}$

Example

Find the distance between the points (3, – 5) and (5, – 1).

Solution

Let the given points be $A(3, -5)$ and $B(5, -1)$.

$AB = \sqrt{(5-3)^2 + (-1-(-5))^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$ units.

Example

Find a if the distance between the points $P(11, -2)$ and $Q(a, 1)$ is 5 units.

Solution

Given, PQ = 5

$\Rightarrow \sqrt{(a - 11)^2 + (1 - (-2))^2} = 5$

Taking square on both sides, we get

$(a - 11)^2 = 25 - 9 = 16$

$a - 11 = \sqrt{16}$

$a - 11 = \pm 4$

$a = 15$ or 7

Example

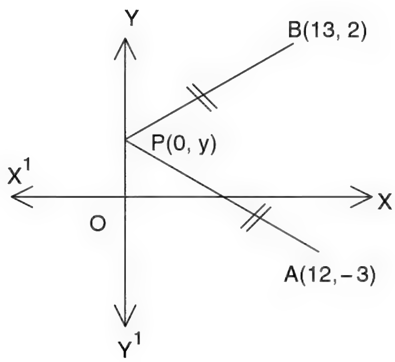


Figure 15.4

Find the coordinates of a point on Y-axis which is equidistant from the points (13, 2) and (12, – 3).

Solution

Let $P(0, y)$ be the required point and the given points be $A(12, -3)$ and $B(13, 2)$.

Then PA = PB (given)

$$\sqrt{(12-0)^2+(-3-y)^2}=\sqrt{(13-0)^2+(2-y)^2} \;=\; \Rightarrow \sqrt{144+(y+3)^2}=\sqrt{169+(2-y)^2}$$

Taking square on both sides, we get

$144 + 9 + y^2 + 6y = 169 + 4 + y^2 - 4y$

$\Rightarrow 10y = 20 \Rightarrow y = 2$

$\therefore$ The required point on Y-axis is (0, 2).

Applications of distance formula

I. Collinearity of three points

Let A, B and C be three given points. The distances AB, BC and CA can be calculated using distance formula. If the sum of any two of these distances is found to be equal to the third distance, then the points A, B and C will be collinear.

NOTE:

1. If AB + BC = AC, then the points A, B and C are collinear.

2. If AC + CB = AB, then the points A, C and B are collinear.

3. BA + AC = BC, then the points B, A and C are collinear.

By Note (1), (2) and (3), we can find the position of points in collinearity.

☛ *Example*

Show that the points A(2, 3), B(3, 4) and C(4, 5) are collinear.

Solution

Given, A = (2, 3), B = (3, 4) and C = (4, 5)

$$AB = \sqrt{(3-2)^2+(4-3)^2} = \sqrt{2} \text{ units.}$$

$$BC = \sqrt{(4-3)^2+(5-4)^2} = \sqrt{2} \text{ units.}$$

$$AC = \sqrt{(4-2)^2+(5-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units.}$$

Now $AB + BC = \sqrt{2} + \sqrt{2} = 2\sqrt{2} = AC$

i.e., AB + BC = AC

Hence, the points A, B and C are collinear.

☛ *Example*

Show that the points (2, 4), (6, 8) and (2, 8) form an isosceles right angled triangle when joined.

Solution

Let A(2, 4), B(6, 8) and C(2, 8) be the given points.

$AB = \sqrt{(6-2)^2 + (8-4)^2} = \sqrt{32}$ units.

$BC = \sqrt{(2-6)^2 + (8-8)^2} = \sqrt{16} = 4$ units.

$AC = \sqrt{(2-2)^2 + (8-4)^2} = \sqrt{16} = 4$ units.

Clearly, $BC^2 + AC^2 = AB^2$.

$\Rightarrow \angle C = 90°$

Also, BC = AC

Hence, the given points form the vertices of a right angled isosceles triangle.

Example

Show that the points (1, –1), (–1, 1) and $(\sqrt{3}, \sqrt{3})$ when joined, form an equilateral triangle.

Solution

Let A(1, –1), B(–1, 1) and $C(\sqrt{3}, \sqrt{3})$ be the given points.

Then, $AB = \sqrt{(-1-1)^2 + (1+1)^2} = \sqrt{4+4} = \sqrt{8}$ units

$BC = \sqrt{(\sqrt{3}-(-1))^2 + (\sqrt{3}-1)^2} = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$

$= \sqrt{2(3+1)}$ [$\because (a+b)^2 + (a-b)^2 = 2(a^2+b^2)$] $= \sqrt{8}$ units.

$CA = \sqrt{(1-\sqrt{3})^2 + (-1-\sqrt{3})^2} = \sqrt{(1-\sqrt{3})^2 + (1+\sqrt{3})^2} = \sqrt{8}$ units

$\therefore AB = BC = CA = \sqrt{8}$

Hence, the points (1, –1), (–1, 1) and $(\sqrt{3}, \sqrt{3})$ when joined, form an equilateral triangle.

Example

Show that the points (2, 0), (–6, –2), (–4, –4) and (4, –2) form a parallelogram.

Solution

Let the given points be A(2, 0), B(–6, –2), C(–4, –4) and D(4, –2)

Then, $AB = \sqrt{(-6-2)^2 + (-2-0)^2} = \sqrt{68}$ units.

$BC = \sqrt{(-4+6)^2 + (-4+2)^2} = \sqrt{8}$ units.

$CD = \sqrt{(4-4)^2 + (-4+2)^2} = \sqrt{68}$ units.

$DA = \sqrt{(4-2)^2 + (-2-0)^2} = \sqrt{8}$ units.

$AC = \sqrt{(-4-2)^2 + (4-0)^2} = \sqrt{52}$ units.

$BD = \sqrt{(4+6)^2 + (-2+2)^2} = 10$ units.

Clearly,

AB = CD, BC = DA and AC ≠ BD.

i.e., the opposite sides of the quadrilateral are equal and diagonals are not equal.

Hence, the given points form a parallelogram.

Example

The vertices of a ΔABC are A(1, 2), B(3, –4) and C(5, –6). Find its circumcentre and circumradius.

Solution

Let S(x, y) be the circumcentre of ΔABC

$\therefore SA^2 = SB^2 = SC^2$.

Consider

$SA^2 = SB^2$.

$\Rightarrow (x-1)^2 + (y-2)^2 = (x-3)^2 + (y+4)^2$.

$x^2 - 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 + 8y + 16$

$-2x - 4y + 1 + 4 = -6x + 9 + 8y + 16$

$4x - 12y - 20 = 0$

$x - 3y = 5$ -------- (1)

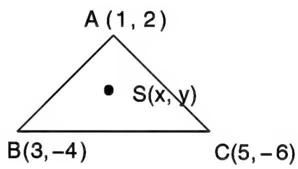


Figure 15.5

$SB^2 = SC^2$.

$\Rightarrow (x-3)^2 + (y+4)^2 = (x-5)^2 + (y+6)^2$

$x - y = 9$ -------- (2)

Solving (1) and (2), we have

x = 11 and y = 2

$\therefore$ The required circumcentre of ΔABC is (11, 2).

Circumradius = SA = $\sqrt{(11-1)^2 + (2-2)^2}$ = 10 units.

Example

Find the area of the circle whose centre is (–3, 2) and (2, 5) is a point on the circle.

Solution

Let the centre of the circle be A (–3, 2) and the point of circumference be B (2, 5)

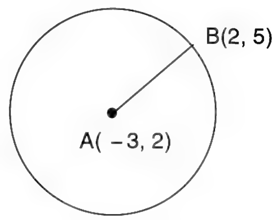


Figure 15.6

Radius of the circle = AB = $\sqrt{(2+3)^2 + (5-2)^2}$ = $\sqrt{25+9}$

r = $\sqrt{34}$ units.

$\therefore$ The area of circle = πr^2.

= $\pi(\sqrt{34})^2$ = 34π sq. units.

Example

Find the area of square whose one pair of the opposite vertices are (3, 4) and (5, 6).

Solution

Let the given vertices be A (3, 4) and C(5, 6).

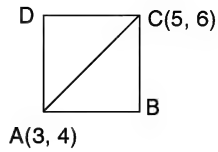


Figure 15.7

Length of AC $= \sqrt{(3-5)^2 + (4-6)^2} = \sqrt{8}$ units

Area of the square $= \dfrac{AC^2}{2} = \dfrac{(\sqrt{8})^2}{2} = 4$ sq. units.

Example

Find the orthocentre of the ΔABC formed by the vertices A(2, 2), B(6, 3) and C(4, 11).

Solution

The given vertices of ΔABC are A(2, 2), B(6, 3) and C(4, 11).

Length of AB $= \sqrt{(6-2)^2 + (3-2)^2} = \sqrt{17}$ units

length of BC $= \sqrt{(6-4)^2 + (3-11)^2} = \sqrt{68}$ units

Length of AC $= \sqrt{(4-2)^2 + (11-2)^2} = \sqrt{85}$ units

Clearly, $AC^2 = AB^2 + BC^2$.

ABC is a right triangle, right angle at B.

Hence orthocentre is the vertex containing right angle i.e., B(6, 3).

Straight lines

Inclination of a line

The angle made by a straight line with positive direction of X-axis in the anti-clockwise direction is called its inclination.

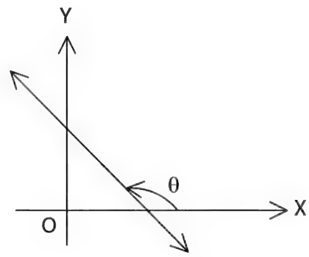


Figure 15.8

Slope or gradient of a line

if θ is the inclination of a line L, then its slope is denoted by m and is given by
$m = \tan\theta$

Example

The inclination of the line ℓ in the figure below is 45°

∴ The slope of the line is $m = \tan 45° = 1$

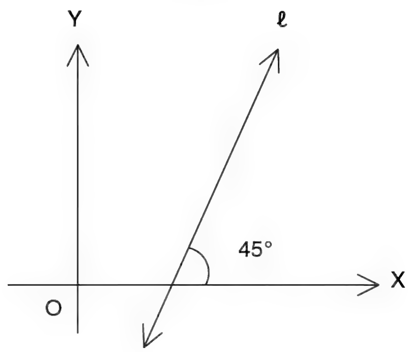


Figure 15.9

Example

The line L in the following figure makes an angle of 45° in clockwise direction with x-axis. So, the inclination of the line L is 180° – 45° = 135°.

$\therefore$ The slope of the line L is m = tan 135° = –1

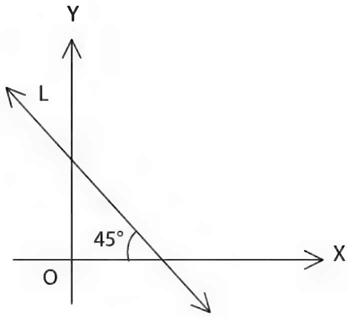


Figure 15.10

Some results on the slope of a line

1. The slope of a horizontal line is zero.

   Hence (i) Slope of X-axis is zero.

   (ii) Slope of any line parallel to X-axis is also zero.

2. The slope of a vertical line is not defined.

   Hence (i) Slope of Y-axis is undefined

   (ii) Slope of any line parallel to Y-axis is also undefined.

Theorem 1

Two non-vertical lines are parallel if and only if their slopes are equal.

Proof

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively.

If θ_1 and θ_2 are the inclinations of the lines, L_1 and L_2 respectively, then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Now, since $L_1 // L_2$. Then, $\theta_1 = \theta_2$

($\because$ They form a pair of corresponding angles)

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

Conversely, let $m_1 = m_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2$

$\Rightarrow L_1 // L_2$ ($\because$ θ_1 and θ_2 form a pair of corresponding angles)

Hence, two non-vertical lines are parallel if and only if their slopes are equal.

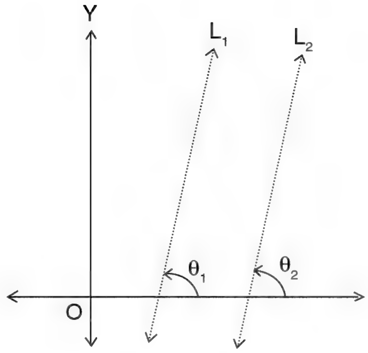


Figure 15.11

Theorem 2

Two non-vertical lines are perpendicular to each other if and only if the product of their slopes is –1.

Proof

Let L_1 and L_2 be two non-vertical lines with slopes, m_1 and m_2.

If θ_1 and θ_2 are the inclinations of the lines L_1 and L_2 respectively, then

$m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

If $L_1 \perp L_2$, then

$\theta_2 = 90° + \theta_1$ ($\because$ The exterior angle of a triangle is equal to the sum of two opposite interior angles)

$\Rightarrow \tan\theta_2 = \tan(90° + \theta_1)$

$\Rightarrow \tan\theta_2 = -\cot\theta_1$

$\Rightarrow \tan\theta_2 = \dfrac{-1}{\tan\theta_1} \; [\because \theta_1 \neq 0]$

$\Rightarrow \tan\theta_1 \cdot \tan\theta_2 = -1$

$\therefore m_1 m_2 = -1$

Conversely, let $m_1 m_2 = -1$

$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$

$\Rightarrow \tan\theta_2 = \dfrac{-1}{\tan\theta_1} \; [\because \theta_1 \neq 0)$

$\Rightarrow \tan\theta_2 = -\cot\theta_1$

$\Rightarrow \tan\theta_2 = \tan(90° + \theta_1)$

$\Rightarrow \theta_2 = 90° + \theta_1$

$\Rightarrow L_1 \perp L_2$

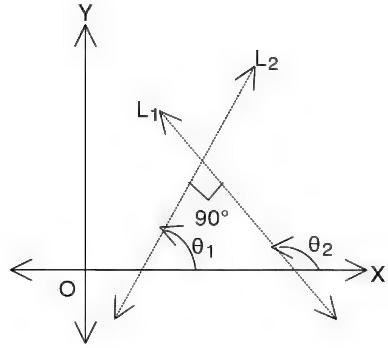


Figure 15.12

Hence, two non-vertical lines are perpendicular to each other if and only if the product of their slopes is –1.

The slope of a line passing through the points (x_1, y_1) and (x_2, y_2)

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points.

Let AB be the straight line passing through the points A and B.

Let θ be the inclination of the line $\overleftrightarrow{AB}$

Draw the perpendiculars AL and BM on to X-axis from A and B respectively. Also draw $AN \perp BM$.

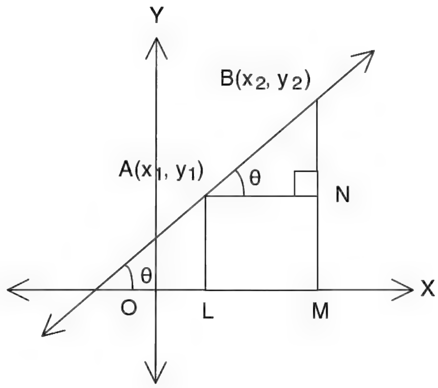


Figure 15.13

Then, $\angle NAB = \theta$

Also, $BN = BM - MN = BM - AL = y_2 - y_1$

$AN = LM = OM - OL = x_2 - x_1$

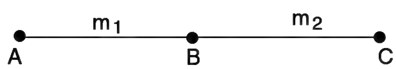
$\therefore$ The slope of the line L is, $m = \tan\theta = \dfrac{BN}{AN} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Hence, the slope of a line passing through the points (x_1, y_1) and (x_2, y_2) is, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

The following table gives the inclination (θ) of the line and its corresponding slope (m) for some particular values of θ

θ	0°	30°	45°	60°	90°	120°	135°	150°
$m = \tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-1/\sqrt{3}$

NOTE: If the points A, B and C are collinear, then the slope of AB = the slope of BC.



i.e., if $m_1 = m_2$, then A, B and C are collinear.

Example

Find the slope of line joining the points $(5,-3)$ and $(7,-4)$.

Solution

Let $A(5,-3)$ and $B(7,-4)$ be the given points.

Then, the slope of $\overleftrightarrow{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-4-(-3)}{7-5} = \dfrac{-1}{2}$

Example

Find the value of k if the slope of the line joining the points $(k, 4)$ and $(-3,-2)$ is $\dfrac{1}{2}$.

Solution

Let the given points be $A(k, 4)$ and $B(-3,-2)$.

Given, the slope of $\overleftrightarrow{AB} = \dfrac{1}{2}$

$$\frac{-2-4}{-3-k} = \frac{1}{2}$$

$$\Rightarrow -12 = -3 - k$$

$$\Rightarrow k = -3 + 12$$

$$\Rightarrow k = 9$$

Example

Find the value of m, if the line passing through the points $A(2,-3)$ and $B(3, m+5)$ is perpendicular to the line passing through the points $P(-2, 3)$ and $Q(-4,-5)$.

Solution

The slope of $\overleftrightarrow{AB}$ i.e., $(m_1) = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m_1 = m + 8$

The slope of PQ i.e., $(m_2) = \dfrac{y_2 - y_1}{x_2 - x_1} = 4$

Since, $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are perpendicular to each other $\Rightarrow m_1 m_2 = -1$

i.e., $(m+8) \times (4) = -1$

$\Rightarrow m + 8 = \dfrac{-1}{4}$

Hence, $m = \dfrac{-33}{4}$

Example

If the points (–3, 6), (–9, *a*) and (0, 15) are collinear, then find *a*.

Solution

Let the given points be A(–3, 6), B(–9, *a*) and C(0, 15).

The slope of AB $= \dfrac{a - 6}{-9 + 3} = \dfrac{6 - a}{6}$

The slope of BC $= \dfrac{a - 15}{-9 - 0} = \dfrac{15 - a}{9}$

Since the points A, B and C are collinear.

The slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{BC}$

$\Rightarrow \dfrac{6 - a}{6} = \dfrac{15 - a}{9}$

$\Rightarrow 18 - 3a = 30 - 2a$

$a = -12$

Hence, a = –12.

Intercepts of a straight line

Say a straight line L meets X-axis in A and Y-axis in B.
Then, OA is called the x-intercept and OB is called the y-intercept.

Note: OA and OB are taken as positive or negative based on whether the line meets positive or negative axes.

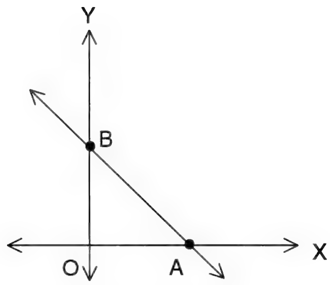


Figure 15.14

Example

The line L in the given figure meets X-axis at A(4, 0) and Y-axis at B(0, –5).

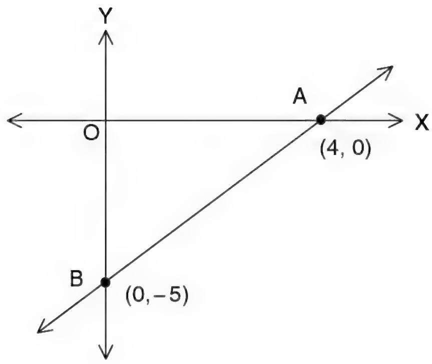


Figure 15.15

Hence, the x-intercept = 4 and y-intercept = –5.

Equation of a line in general form

An equation of the form, $ax + by + c = 0$. (where $|a| + |b| \neq 0$ i.e., a and b are not simultaneously equal to zero), which is satisfied by every point on a line and not by any point outside the line, is called the equation of a line.

Equations of some standard lines

1. **Equation of X-axis**

   We know that the y-coordinate of every point on X-axis is zero so, if P(x, y) is any point on X-axis, then $y = 0$.

   Hence, the equation of X-axis is $y = 0$.

2. **Equation of Y-axis**

   We know that the x-coordinate of every point on Y-axis is zero. So, if P(x, y) is any point on Y-axis, then $x = 0$, hence, the equation of Y-axis is $x = 0$.

3. **Equation of a line parallel to X-axis**

   Let L be a line parallel to x-axis and at a distance of k units away from x-axis.

   Then, the Y-coordinate of every point on the line L is k.

   So, if P(x, y) is any point on the line L, then $y = k$

   Hence, the equation of a line parallel to X-axis at a distance of k units from it is $y = k$

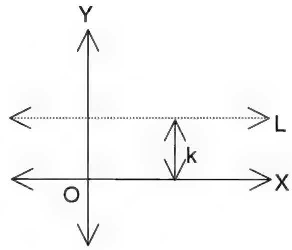


Figure 15.16

NOTE: For the lines lying below X-axis, k is taken as negative.

4. **Equation of a line parallel to Y-axis**

   Let L^1 be a line parallel to Y-axis and at a distance of k units away from it. Then the x-coordinate of every point on the line L^1 is k.

   So, if P(x, y) is any point on the line L^1, then $x = k$

   Hence, the equation of a line parallel to Y-axis and at a distance of k units from it is $x = k$

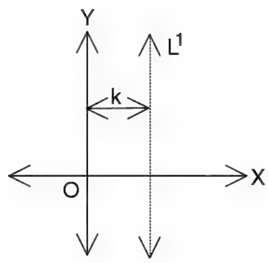


Figure 15.17

NOTE: For the lines lying towards the left side of y-axis, k is taken as negative.

Oblique line

A straight line which is neither parallel to X-axis nor parallel to Y-axis is called an oblique line or an inclined line.

Different forms of equations of oblique lines

1. **Gradiant form (or) slope form**

   The equation of a straight line with slope m and passing through origin is given by $y = mx$.

2. **Point-slope form**

   The equation of a straight line passing through the point (x_1, y_1) and with slope m is given by $y - y_1 = m(x - x_1)$.

3. **Slope-intercept form**

   The equation of a straight line with slope m and having y-intercept c is given by $y = mx + c$.

4. **Two-point form**

The equation of a straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

5. **Intercept form**

The equation of a straight line with x-intercept as *a* and y-intercept as *b* is given by $\frac{x}{a} + \frac{y}{b} = 1$.

Note: Area of triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is $\frac{1}{2}|ab|$ sq units.

Example

Find the equation of a line parallel to X-axis and passing through the point (3, −4).

Solution

We know that the equation of a line parallel to X-axis can be taken as y = k

Given, the line passes through the point (3, −4)

$\Rightarrow k = -4$

Hence, the equation of the required line is y = −4

i.e., y + 4 = 0

Example

Find the equation of a line having a slope of $-\frac{3}{4}$ and passing through the point (3, −4)

Solution

We know that, the equation of a line passing through the point (x_1, y_1) and having a slope m is given by $y - y_1 = m(x - x_1)$

Hence, the equation of the required line is

$$y - (-4) = -\frac{3}{4}(x - 3)$$

$$\Rightarrow 4(y + 4) = -3(x - 3)$$

$$\Rightarrow 3x + 4y + 7 = 0$$

Example

Find the equation of a line making intercepts 3 and −4 on the coordinate axes respectively.

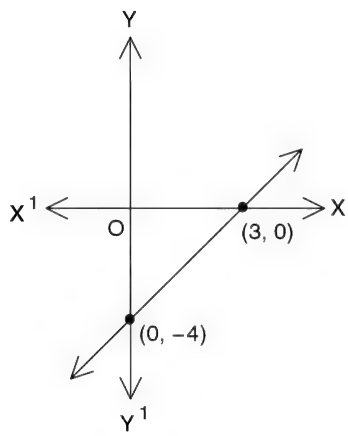


Figure 15.18

Solution

Given,

x-intercept (a) = 3

y-intercept (b) = −4

∴ The equation of the required line is

$$\frac{x}{a}+\frac{y}{b}=1$$

i.e., $\frac{x}{3}+\frac{y}{-4}=1$

$4x - 3y = 12$ (or) $4x - 3y - 12 = 0$

Area of triangle

Consider $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ as the vertices of ΔABC. Drop perpendiculars AP, BQ and CR on to x-axis.

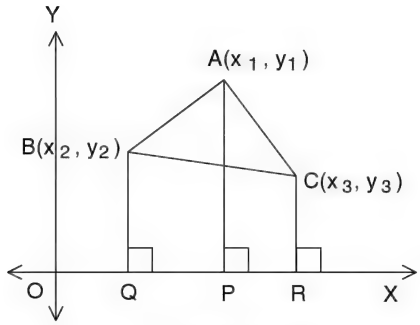


Figure 15.19

Area of ΔABC = Area of trapezium APQB + Area of trapezium APRC – Area of trapezium BCRQ.

$$= \frac{1}{2}QP\,(AP + BQ) + \frac{1}{2}PR\,(AP + CR) - \frac{1}{2}QR\,(BQ + CR)$$

Here $QP = x_1 - x_2$, $PR = x_3 - x_1$, $QR = x_3 - x_2$,

$AP = y_1$, $BQ = y_2$, $CR = y_3$

$$= \frac{1}{2}(x_1 - x_2)(y_1 + y_2) + \frac{1}{2}(x_3 - x_1)(y_1 + y_3) - \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$$

$$= \frac{1}{2}(x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2)$$

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

As the area is always positive

$\therefore$ Area of ΔABC "Δ" $= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$ sq. units.

"or"

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} \text{ sq. units.}$$

Note:

1. Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and $(0, 0)$ is $\Delta = \frac{1}{2}|x_1y_2 - x_2y_1|$
2. Area of ΔABC is zero, if the points A, B and C are collinear.
3. Area of triangle DEF formed by the mid-points of the sides of the ΔABC is $\frac{1}{4}$ th of the area of ΔABC

   i.e., Area of ΔABC = 4(Area of ΔDEF)
4. If G is the centroid of ΔABC, then Area of ΔABC = 3(Area of ΔAGB) = 3(Area of ΔBGC) = 3(Area of ΔACG)

Area of a quadrilateral

Area of a quadrilateral with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is given by $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

☛ *Example*

Find the area of the triangle whose vertices are A(1, –2), B(3, 4) and C(2, 3).

Solution

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 1-3 & -2-4 \\ 3-2 & 4-3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & -6 \\ 1 & 1 \end{vmatrix} = \frac{1}{2}\left| -2-(-6) \right| = 2 \text{ sq.units.}$$

☛ *Example*

Find the value of p, if the points A(2, 3), B(–1, 6) and C(p, 4) are collinear.

Solution

Given, the points A(2, 3), B(–1, 6) and C(p, 4) are collinear.

$\therefore$ Area of $\Delta ABC = 0$.

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2-(-1) & 3-6 \\ -1-p & 6-4 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3 & -3 \\ -1-p & 2 \end{vmatrix} = 0$$

$\Rightarrow 6 - 3(1 + p) = 0 \Rightarrow p = 1$

Hence, $p = 1$.

Section formulae

1. Section formula

Consider two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $C(x, y)$ be any point on AB that divide AB in the ratio m : n. Draw perpendiculars AL, CN and BM to x-axis.

AP and CQ are perpendiculars drawn to CN and BM.

Now it is clear that ΔAPC and ΔCQB are similar.

$$\therefore \frac{AC}{CB} = \frac{AP}{CQ} = \frac{CP}{BQ} \text{ -------- (1)}$$

here, $LN = x - x_1$ and $NM = x_2 - x$

from (1), we have

$$\frac{m}{n} = \frac{x - x_1}{x_2 - x}, \ (\quad AP = LN \text{ and } CQ = NM)$$

$\Rightarrow mx_2 - mx = nx - nx_1$.

$\Rightarrow mx_2 + nx_1 = mx + nx$.

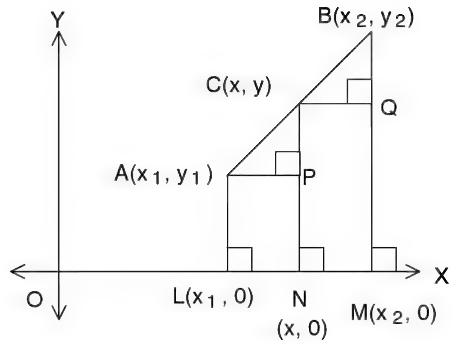


Figure 15.20

$\therefore x = \dfrac{mx_2 + nx_1}{m + n}$

Similarly we can obtain $\dfrac{m}{n} = \dfrac{y - y_1}{y_2 - y} \Rightarrow y = \dfrac{my_2 + ny_1}{m + n}$

Hence the coordinates of 'C' are $\left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}\right)$

In this case we notice that point C lies in between A and B. So we say that C divides AB in the ratio m : n internally.

Note:

1. When C does not lie between A and B i.e., as shown below, then we say that C divides AB in m : n ratio externally then the coordinates of C are $\left(\dfrac{mx_2 - nx_1}{m - n}, \dfrac{my_2 - ny_1}{m - n}\right)$

2. Let P(x, y) divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio m : n.

   Then, $\dfrac{m}{n} = \dfrac{x - x_1}{x_2 - x}$ (or) $\dfrac{y - y_1}{y_2 - y}$

4. X-axis divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $-y_1 : y_2$ (or) $y_1 : -y_2$.
5. Y-axis divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $-x_1 : x_2$.

Example

Find the coordinates of the point P which divides the line segment joining the points A(3, –2) and B(2, 6) internally in the ratio 2 : 3.

Solution

Given, P(x, y) divides AB internally in the ratio 2 : 3.

So, $P = \left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}\right)$

Here, $(x_1, y_1) = (3, -2)$, $(x_2, y_2) = (2, 6)$ and m : n = 2 : 3

$\therefore P = \left(\dfrac{2(2) + 3(3)}{2+3}, \dfrac{2(6)+3(-2)}{2+3}\right) = \left(\dfrac{13}{5}, \dfrac{6}{5}\right)$

Hence, $\left(\dfrac{13}{5}, \dfrac{6}{5}\right)$ is the required point.

Example

Find the coordinates of a point P which divides the line segment joining the points A(1, 3) and B(3, 4) externally in the ratio 3 : 4.

Solution

Given, P divides AB externally in the ratio 3 : 4.

So, $P = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

Here, $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (3, 4)$ and $m : n = 3 : 4$

$\therefore\ P = \left(\frac{3(3)-4(1)}{3-4}, \frac{3(4)-4(3)}{3-4}\right) = (-5, 0)$

Hence, P = (–5, 0).

Example

Find the ratio in which the point P(3, –2) divides the line segment joining the points A(1, 2) and B(–1, 6).

Solution

The ratio in which P divides AB is AP : PB = (3 – 1) : (–1 –3) = 2 : –4 = –1 : 2

Hence, P divides AB externally in the ratio 1 : 2.

Example

Find the ratio in which the line joining the points (2, –3) and (3, 1) is divided by X-axis and Y-axis.

Solution

The ratio in which X-axis divides is $-y_1 : y_2$

i.e., –(–3) : 1 = 3 : 1

The ratio in which Y-axis divides is $-x_1 : x_2$

= –2 : 3

i.e., 2 : 3 externally.

Mid-point

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points and M be the mid-point of AB.
Then, M divides AB in the ratio 1 : 1 internally.

So, $M = \left(\frac{1.x_2 + 1.x_1}{1+1}, \frac{1.y_2 + 1.y_1}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Hence, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Points of trisection

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points. Then, the two points which divide AB in the ratio 1 : 2 and 2 : 1 internally are called the points of trisection of AB.

Further, if P and Q are the points of trisection of AB respectively,

Then $P = \left(\frac{2x_1+x_2}{3}, \frac{2y_1+y_2}{3}\right)$ and $Q = \left(\frac{x_1+2x_2}{3}, \frac{y_1+2y_2}{3}\right)$

Note:

1. If the mid–points of ΔABC are $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, x_3)$, then its vertices are $A(-x_1 + x_2 + x_3, -y_1 + y_2 + y_3)$, $B(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$ and $C(x_1 + x_2 - x_3, y_1 + y_2 - y_3)$.
2. The fourth vertex of a parallelogram whose three vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) in order is $(x_1 - x_2 + x_3, y_1 - y_2 + y_3)$

Example

Find the mid-point of the line segment joining the points (1, –3) and (6, 5).

Solution

Let A(1, –3) and B(6, 5) be the given points and M be the mid-point of AB.

Then, $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{1+6}{2}, \frac{-3+5}{2}\right) = \left(\frac{7}{2}, 1\right)$

Hence, the mid point of AB is $\left(\frac{7}{2}, 1\right)$.

Example

Find the points of trisection of the line segment joining the points (3, –2) and (4, 1).

Solution

Let A(3, –2) and B(4, 1) be the given points.

Let P and Q be the points of trisection of AB respectively.

Then, $P = \left(\frac{2(3)+4}{3}, \frac{2(-2)+1}{3}\right)$ and $Q = \left(\frac{3+2(4)}{3}, \frac{-2+2(1)}{3}\right)$

$\Rightarrow P = \left(\frac{10}{3}, -1\right)$ and $Q = \left(\frac{11}{3}, 0\right)$

Hence, the points of trisection are $\left(\frac{10}{3}, -1\right)$ and $\left(\frac{11}{3}, 0\right)$.

Example

Find the fourth vertex of the rhombus formed by (–1, –1), (6, 1) and (8, 8).

Solution

Let the three vertices of Rhombus be A(–1, –1), B(6, 1) and C(8, 8) then fourth vertex D(x, y) is given by

$$\begin{aligned} D(x, y) &= (x_1 - x_2 + x_3, y_1 - y_2 + y_3) \\ &= (-1 - 6 + 8, -1 - 1 + 8) \\ &= (1, 6) \end{aligned}$$

Hence the required fourth vertex is D(1, 6)

Example

Find the area of the triangle formed by the mid-points of the sides of ΔABC. Where A(3, 2), B(–5, 6) and C = (8, 3).

Solution

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 3-(-5) & 2-6 \\ -5-8 & 6-3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 8 & -4 \\ -13 & 3 \end{vmatrix} = \frac{1}{2}\left|8(3) - 4(13)\right| = \frac{1}{2}\left|24 - 52\right|$$

$$= \frac{1}{2}\left|-28\right| = 14 \text{ sq. units.}$$

Hence the area of triangle formed by the mid-points of the sides of ΔABC $= \frac{1}{4}(\text{Area of } \Delta ABC)$

$$= \frac{1}{4}(14) = 3.5 \text{ sq.units.}$$

Centroid

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and G be its centroid. Then, the coordinates of G are given by, $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

Example

Find the centroid of ΔABC whose vertices, are A(1, –3), B(–3, 6) and C(–4, 3).

Solution

Given, A(1, –3), B(–3, 6) and C(–4, 3).

$$\text{So, Centroid of } \Delta ABC = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{1-3-4}{3}, \frac{-3+6+3}{3}\right) = (-2, 2)$$

Hence, (–2, 2) is the centroid of ΔABC.

Example

Find the centroid of the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 10$ as sides.

Solution

Let OAB be the triangle formed by the given lines

O is the point of intersections of $x = 0$ and $y = 0$ i.e., origin $\Rightarrow O(0, 0)$

A is the point of intersection of $x = 0$ and $x + y = 10$ i.e., A(0, 10)

and B is the point of intersection of $y = 0$ and $x + y = 10$, i.e., B(10, 0)

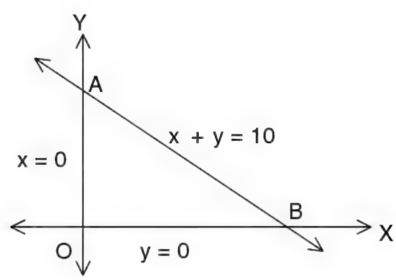


Figure 15.21

$$\therefore \text{ Centroid of } \Delta OAB \text{ is } G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$= \left(\frac{0 + 0 + 10}{3}, \frac{0 + 10 + 0}{3}\right)$$

$$= \left(\frac{10}{3}, \frac{10}{3}\right)$$

Example

Find the third vertex of ΔABC if two of its vertices are $A(-3, 2)$, $B(1, 5)$ and its centroid is $G(3, -4)$.

Solution

Let $C(x, y)$ be the third vertex.

Given, centroid of $\Delta ABC = (3, -4)$

$$\Rightarrow \left(\frac{-3+1+x}{3}, \frac{2+5+y}{3}\right) = (3, -4)$$

$$\Rightarrow \frac{-2+x}{3} = 3, \frac{7+y}{3} = -4$$

$\Rightarrow x = 11, y = -19.$

$\therefore$ The third vertex is $(11, -19)$.

Equation of a line parallel or perpendicular to the given line

Let $ax + by + c = 0$ be the equation of a straight line, then,

(i) The equation of a line passing through the point (x_1, y_1) and parallel to the given line is given by $a(x - x_1) + b(y - y_1) = 0$.

(ii) The equation of a line passing through the point (x_1, y_1) and perpendicular to the given line is given by $b(x - x_1) - a(y - y_1) = 0$.

Example

Find the equation of a line passing through the point $A(2, -3)$ and parallel to the line $2x - 3y + 6 = 0$.

Solution

Here, $(x_1, y_1) = (2, -3)$, $a = 2$ and $b = -3$.

$\therefore$ Equation of the line passing through $A(2, -3)$ and parallel to the line $2x - 3y + 6 = 0$ is

$a(x - x_1) + b(y - y_1) = 0.$

i.e., $2(x - 2) - 3(y + 3) = 0$

$\Rightarrow 2x - 3y - 13 = 0.$

Hence, the equation of the required line is $2x - 3y - 13 = 0$

Example

Find the equation of a line passing through the point (5, 2) and perpendicular to the line $3x - y + 6 = 0$.

Solution

Here, $(x_1, y_1) = (5, 2)$, $a = 3$ and $b = -1$.

$\therefore$ Equation of the line perpendicular to $3x - 6y + 6 = 0$ and passing through the point (5, 2) is $b(x - x_1) - a(y - y_1) = 0$

i.e., $-1(x - 5) - 3(y - 2) = 0$

$\Rightarrow (x - 5) + 3(y - 2) = 0$

$\Rightarrow x + 3y - 11 = 0.$

Hence, the equation of the required line is $x + 3y - 11 = 0$.

Example

The line $(5x - 8y + 11)\lambda + (8x - 3y + 4) = 0$ is parallel to X-axis. Find λ.

Solution

The given line is

$(5x - 8y + 11)\lambda + (8x - 3y + 4) = 0$

i.e., $x(5\lambda + 8) - y(8\lambda + 3) + (11\lambda + 4) = 0$

As the given line is parallel to X-axis, its slope = 0

i.e., $\dfrac{-(5\lambda + 8)}{-(8\lambda + 3)} = 0$

$\Rightarrow 5\lambda + 8 = 0$

Hence, $\lambda = \dfrac{-8}{5}$

Median of the triangle: A line drawn from the vertex, which bisects the opposite side is called a median of the triangle.

Example

Find the median to the side BC of the triangle whose vertices are A(–2, 1), B(2, 3) and C(4, 5).

Solution

Let E be the mid-point of side BC.

$\therefore E = \left(\dfrac{2 + 4}{2}, \dfrac{3 + 5}{2}\right) = (3, 4)$

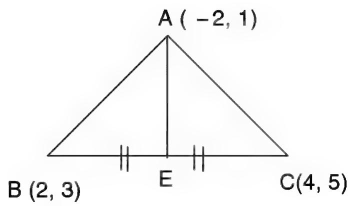


Figure 15.22

Equation of line AE is

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

i.e., $y - 1 = \dfrac{4 - 1}{3 - (-2)}(x - 1)$

$$y - 1 = \frac{3}{5}(x - 1)$$

$5y - 5 = 3x - 3$

$\therefore$ The required equation of the median is

$3x - 5y + 2 = 0$

Altitude of the triangle

A perpendicular dropped from the vertex to the opposite side in a triangle is called an altitude.

Example

Find the equation of the altitude drawn to side BC of ΔABC, whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

Solution

Let AD be the altitude drawn to side BC.

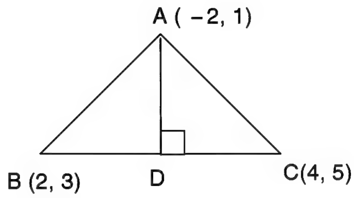


Figure 15.23

$$\text{Slope of BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = 1$$

$\therefore$ Slope of AD $= -1$ ($\because AD \perp BC$)

Equation of AD is $y - y_1 = m(x - x_1)$

$$y - 1 = -1(x + 2)$$

$$y - 1 = -x - 2$$

$$\Rightarrow x + y + 1 = 0$$

$\therefore$ The required equation of altitude is $x + y + 1 = 0$.

Example

Find the equation of the perpendicular bisector drawn to the side BC of ΔABC, whose vertices are $A(-2, 1)$, $B(2, 3)$ and $C(4, 5)$.

Solution

$$\therefore D = \left(\frac{2+4}{2}, \frac{3+5}{2}\right)$$

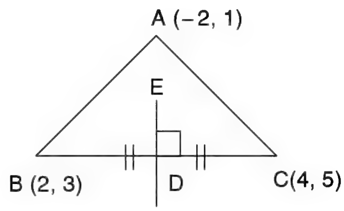


Figure 15.24

$D = (3, 4)$

$$\text{Slope of BC} = \frac{5-3}{4-2} = 1$$

$\Rightarrow$ Slope of ED $= -1$ ($\because ED \perp BC$)

Equation of ED is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 4 = -1(x - 3)$$

$$\Rightarrow y - 4 = -x + 3$$

$$\Rightarrow x + y - 7 = 0$$

$\therefore$ The required equation of the perpendicular bisector is $x + y - 7 = 0$.

test your concepts

Very short answer type questions

1. If the inclination of a line is 45°, then the slope of the line is _____.
2. If the point (x, y) lies in the third quadrant, then x is _____ and y is _____.
3. The point of intersection of the lines $x = 2$ and $y = 3$ is _____.
4. The point of intersection of medians of triangle is called its ____________.
5. The line $y + 7 = 0$ is parallel to _____ axis.
6. The distance of a point (2, 3) from y-axis is _____.
7. The slope of the line perpendicular to $5x + 3y + 1 = 0$ is ____________.
8. If (x, y) represents a point and $|x| > 0$ and $y > 0$ then in which quadrants can the point lie?
9. The line $4x + 7y + 9 = 0$ meet the x-axis at _____ and y-axis at _____.
10. If (x, y) represents a point and $xy < 0$, then the point may lie in _____ or _____ quadrant.
11. The area of the triangle, with vertices A(2, 0), B(0, –4)and origin is
12. The area of the triangle formed by the points (0, 0), (0, a), (b, 0) is ____________.
13. The area of the triangle formed by the line $ax + by + c = 0$ with coordinate axes is ____________.
14. If the points (5, 5), (7, 7) and (a, 8) are collinear, then the value of a is _____.
15. Area of a triangle whose vertices are (0, 0), (x_1, y_1), (x_2, y_2) is ____________.
16. The end vertices of one diagonal of a parallelogram are (1, 3) and (5, 7), then the mid-point of the other diagonal is ____________.
17. The orthocentre of the triangle formed by the points (0, 1), (1, 2) and (0, 2) is _____.
18. The two straight lines, $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular to each other, then m_1m_2 = ____________.
19. If A(4, 0), B(0, –6) are the two vertices of a triangle OAB where O is origin, then the circumcentre of the triangle OAB is
20. If the centre and radius of a circle is (3, 4) and 7, then the position of the point (5, 3) w.r.t. the circle is ____________.
21. If A(2, 3), B(x, y) and C(4, 3) are the vertices of a right angled triangle, right angled at A, then x =
22. The coordinates of the points P which divides (1, 0) and (0, 0) in 1 : 2 ratio are ____________.
23. The points A, B and C represents the vertices of a ΔABC. AD is the median drawn from A to BC, then the centroid of the triangle divides AD in ____________ ratio.
24. If (5, 7) and (9, 3) are the ends of the diameter of a circle, then the centre of the circle is _____.

25. What is the formula for calculating the area of a quadrilateral whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)?

26. The centroid of the triangle formed by the lines $x = 0, y = 0$ and $x + y = 6$ is

27. The line $ax + by + c = 0$ is such that $a = 0$ and $bc \neq 0$, then the line is perpendicular to ______ axis.

28. If A is one of the points of trisection of the line joining B and C, then A divides BC in the ratio ____________ (or) ____________.

29. The ratio that the line joining points $(3, -6)$ and $(4, 9)$ is divided by x-axis is ______.

30. If $(1, 2)$, $(3, 4)$ and $(0, 6)$ are the three vertices of a parallelogram taken in that order, then the fourth vertex is

Short answer type questions

31. Find λ, if the line $(3x - 2y + 5) + \lambda(3x - y + 4) = 0$ passes through the mid-point of the line joining the points $A(2, 3)$ and $B(4, 9)$.

32. Find the distance between the points $(3, -5)$ and $(-4, 7)$.

33. Find the slope of the line perpendicular to $\overleftrightarrow{AB}$ where $A(5, -6)$ and $B(2, -7)$.

34. If $A(-2, -1)$, $B(-4, 5)$ and $C(2, 3)$ are three vertices of the parallelogram ABCD, then find the vertex D.

35. Find the equation of line parallel to y-axis and passing through the point $(3, -5)$.

36. Find the equation of a line, whose inclination is $30°$ and making an intercept of $\frac{-3}{5}$ on y-axis.

37. Find the equation of a straight line whose slope is -5 and making an intercept 3 on the y-axis.

38. Find the centroid of a triangle whose vertices are $(3, -1)$, $(2, 4)$ and $(-8, 6)$.

39. Find the equation of a line passing through the point $(2, -3)$ and parallel to the line $2x - 3y + 8 = 0$.

40. If $A(2, -2)$, $B(3, 4)$ and $C(7, 2)$ are the mid points of the sides PQ, QR and RP respectively of ΔPQR, Find its vertices.

41. Let $(-3, 2)$ be one end of a diameter of a circle with centre $(4, 6)$. Find the other end of the diameter.

42. Let $A(-1, 2)$ and $D(3, 4)$ be the ends points of the median AD of ΔABC. Find the centroid of ΔABC.

43. Find the point of intersection of the lines $3x + 5y + 2 = 0$ and $4x + 7y + 3 = 0$.

44. Find the coordinates of the point which divides the line joining the points $A(-3, 2)$ and $B(2, 6)$ internally in the ratio $3 : 2$.

45. Find the point on Y-axis which is equidistant from $A(3, -6)$ and $B(-2, 5)$.

Essay type questions

46. Find the equation of a line parallel to the line $2x + 3y - 6 = 0$ and where sum of intercepts is 10.

47. Find the equation of a line passing through the point of intersection of the lines $5x - y - 7 = 0$ and $3x - 2y - 7 = 0$ and parallel to the $x + 3y - 5 = 0$.

48. Find the equation of a line with Y-intercept -4 and perpendicular to a line passing through the points $A(1, -2)$ and $B(-3, 2)$.

49. Find the circumcentre of the triangle formed by the points $(2, 3)$, $(1, -5)$ and $(-1, 4)$.

50. Let $A(3, 2)$, $B(-4, 1)$, $C(-3, 1)$ and $D(2, -4)$ be the vertices of a quadrilateral ABCD. Find the area of the quadrilateral formed by the mid-points of the sides of the quadrilateral ABCD.

CONCEPT APPLICATION

Concept Application Level—1

1. The lines, $x = 2$ and $y = 3$ are

(1) parallel to each other.
(2) perpendicular to each other.
(3) neither parallel nor perpendicular to each other.
(4) None of these

2. The lines, $x = -2$ and $y = 3$ intersect at the point ________.

(1) $(-2, 3)$ (2) $(2, -3)$ (3) $(3, -2)$ (4) $(-3, 2)$

3. The slope of the line joining the points $(2, k - 3)$ and $(4, -7)$ is 3. Find k.

(1) -10 (2) -6 (3) -2 (4) 10

4. The centre of a circle is $C(2, -3)$ and one end of the diameter AB is $A(3, 5)$. Find the coordinates of the other end B.

(1) $(1, -11)$ (2) $(5, 2)$ (3) $(1, 8)$ (4) None of these

5. The angle made by the line $\sqrt{3}x - y + 3 = 0$ with the positive direction of X- axis is ___.

(1) $30°$ (2) $45°$ (3) $60°$ (4) $90°$

6. The points on X-axis which are at a distance of $\sqrt{13}$ units from $(-2, 3)$ is _______.

(1) $(0, 0), (-2, -3)$ (2) $(0, 0), (-4, 0)$ (3) $(0, 0), (2, 3)$ (4) None of these

7. The point P lying in the fourth quadrant which is at a distance of 4 units from X-axis and 3 units from Y-axis is ________.

(1) (4, –3) (2) (4, 3) (3) (3, –4) (4) (–3, 4)

8. The radius of a circle with centre (– 2, 3) is 5 units. The point (2, 5) lies

(1) on the circle (2) inside the circle (3) outside the circle (4) None of these

9. The points (a, b + c), (b, c + a) and (c, a + b)

(1) are collinear (2) form a scalene triangle

(3) form an equilateral triangle (4) None of these

10. Find λ, if the line $3x - \lambda y + 6 = 0$ passes through the point (–3, 4).

(1) $\frac{3}{4}$ (2) $\frac{-3}{4}$ (3) $\frac{4}{3}$ (4) $\frac{-4}{3}$

11. If A(–2, 3) and B(2, 3) are two vertices of ΔABC and G(0, 0) is its centroid, then the coordinates of C are

(1) (0, –6) (2) (–4, 0) (3) (4, 0) (4) (0, 6)

12. Let ΔABC be a right angled triangle in which A(0, 2) and B(2, 0). Then the coordinates of C can be

(1) (0, 0) (2) (2, 2) (3) Either (1) or (2) (4) None of these

13. If A (4, 7), B(2, 5), C(1, 3) and D(–1, 1) are the four points, then the lines AC and BD are _______.

(1) perpendicular to each other

(2) parallel to each other

(3) neither parallel nor perpendicular to each other

(4) None of these

14. Find the area of the triangle formed by the line $5x - 3y + 15 = 0$ with coordinate axes.

(1) 15 cm^2 (2) 5 cm^2 (3) 8 cm^2 (4) $\frac{15}{2}$ cm^2

15. Equation of a line whose inclination is 45° and making an intercept of 3 units on X-axis is

(1) $x + y - 3 = 0$ (2) $x - y - 3 = 0$ (3) $x - y + 3 = 0$ (4) $x + y + 3 = 0$

16. The centre of a circle is C(2, k). If A(2, 1) and B(5, 2) are two points on its circumference, then the value of k is

(1) 6 (2) 2 (3) –6 (4) –2

17. The lines $x = -1$ and $y = 4$ are _______.

(1) perpendicular to each other

(2) parallel to each other

(3) neither parallel nor perpendicular to each other

(4) None of these

18. The distance between the points (2k + 4, 5k) and (2k, –3 + 5k) in units is

(1) 1 (2) 2 (3) 4 (4) 5

19. The equation of the line with inclination 45° and passing through the point (–1, 2) is

(1) $x + y + 3 = 0$ (2) $x - y + 3 = 0$

(3) $x - y - 3 = 0$ (4) $x + y - 3 = 0$

20. The distance between the points $(3k + 1, -3k)$ and $(3k - 2, -4 - 3k)$ (in units) is

(1) 3k (2) 5k (3) 5 (4) 3

21. The angle made by the line $x - \sqrt{3}\,y + 1 = 0$ with the positive Y-axis is

(1) 60° (2) 30° (3) 45° (4) 90°

22. If ΔABC is a right angled triangle in which A(3, 0) and B(0, 5), then the coordinates of C can be

(1) (5, 3) (2) (3, 5)

(3) (0, 0) (4) Both (2) and (3)

23. If the roots of the quadratic equation $2x^2 - 5x + 2 = 0$ are the intercepts made by a line on the coordinate axes, then the equation of the line can be

(1) $4x + y = 2$ (2) $2x + 5y + 2$

(3) $x + 4y = 2$ (4) Both (1) and (3)

24. The inclination of the line $\sqrt{3}\,x - y + 5 = 0$ with X-axis is

(1) 90° (2) 45° (3) 60° (4) 30°

25. The equation of the line parallel to $3x - 2y + 7 = 0$ and making an intercept –4 on X-axis is

(1) $3x - 2y + 12 = 0$ (2) $3x - 2y - 12 = 0$

(3) $3x + 2y - 12 = 0$ (4) $3x + 2y + 12 = 0$

26. A triangle is formed by the lines $x + y = 8$, X-axis and Y-axis. Find its centroid.

(1) $\left(\frac{8}{3}, \frac{8}{3}\right)$ (2) (8, 8) (3) (4, 4) (4) (0, 0)

27. The point which divides the line joining the points A(1, 2) and B(–1, 1) internally in the ratio 1 : 2 is _______.

(1) $\left(\frac{-1}{3}, \frac{5}{3}\right)$ (2) $\left(\frac{1}{3}, \frac{5}{3}\right)$ (3) (–1, 5) (4) (1, 5)

28. Find the area of the triangle formed by the line $3x - 4y + 12 = 0$ with the coordinate axes.

(1) 6 units^2 (2) 12 units^2 (3) 1 units^2 (4) 36 units^2

29. The equation of a line whose sum of intercepts is 5 and the area of the triangle formed by the line with positive coordinate axis is 2 sq.units can be

(1) $x + y = 4$ (2) $x + 4y = 4$ (3) $y + 4x + 4 = 0$ (4) $y = x + 4$

30. Find the equation of a line which divides the line segment joining the points (1, 1) and (2, 3) in the ratio 2 : 3 perpendicularly.

(1) $5x - 5y + 2 = 0$ (2) $5x + 5y + 2 = 0$

(3) $x + 2y - 5 = 0$ (4) $x + 2y + 7 = 0$

Concept Application Level—2

31. The equation of the line making an angle of 45° with X-axis in positive direction and having y-intercept as –3 is

(1) $3x - y + 1 = 0$ (2) $3x + y - 1 = 0$ (3) $x - y + 3 = 0$ (4) $x - y = 3$

32. The ratio in which the line joining points $(a + b, b + a)$ and $(a - b, b - a)$ is divided by the point (a, b) is __________.

(1) b : a internally (2) 1 : 1 internally (3) a : b externally (4) 2 : 1 externally

33. Which of the following lines is perpendicular to $x + 2y + 3 = 0$?

(1) $2\sqrt{2}x - y + 3 = 0$ (2) $\sqrt{2}x + \sqrt{2}y - 5 = 0$

(3) $2\sqrt{2}x - \sqrt{2}y + 3 = 0$ (4) $x + \sqrt{2}y + 4 = 0$

34. The orthocentre of the triangle formed by the points (0, 3), (0, 0) and (1, 0) is

(1) $\left(\frac{1}{3}, 1\right)$ (2) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (3) (0, 0) (4) $\left(0, \frac{3}{2}\right)$

35. The equation of the line passing through the point of intersection of the lines $x + 2y + 3 = 0$ and $2x - y + 5 = 0$ and parallel to X-axis is

(1) $5y + 1 = 0$ (2) $5x - 13 = 0$ (3) $5x + 13 = 0$ (4) $5y - 1 = 0$

36. Find the equation of a line which divides the line segment joining the points (1, –2) and (3, –1) in the ratio 3 : 1 perpendicularly.

(1) $x - 2y - 5 = 0$ (2) $6x + 4y - 5 = 0$ (3) $3x + 2y - 5 = 0$ (4) $8x + 4y - 15 = 0$

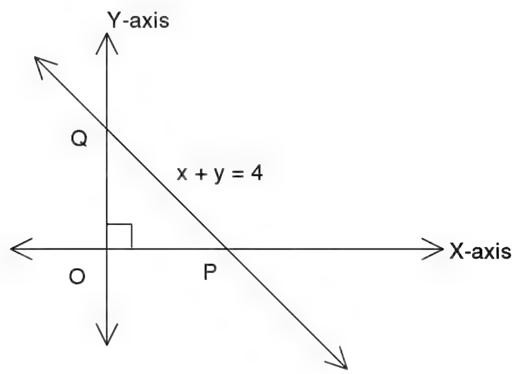
37. If $x + p = 0$, $y + 4 = 0$ and $x + 2y + 4 = 0$ are concurrent, then p =

(1) 4 (2) –2 (3) –4 (4) 2

38. The perpendicular bisector of the side PQ is

(1) $x - y = 0$ (2) $x + y - 2 = 0$

(3) $3x - 2y - 2 = 0$ (4) $x + 2y - 6 = 0$



39. The orthocentre of the triangle formed by the vertices A (4, 6), B (4, 3) and C(2, 3) is

(1) (2, 3) (2) (4, 3) (3) (4, 6) (4) (3, 4)

40. The circumcentre and the orthocentre of the triangle formed by the sides $y = 0, x = 0$ and $2x + 3y = 6$ are respectively

(1) (3, 2), (0, 0)
(2) $\left(\frac{3}{2}, 1\right)$, (0, 0)
(3) (–3, 1), (0, 0)
(4) $\left(-\frac{3}{2}, 1\right)$, (0, 0)

41. The equation of median drawn to the side BC of ΔABC whose verticles are A (1, –2), B(3, 6) and C(5, 0) is

(1) $5x - 3y - 11 = 0$
(2) $5x + 3y - 11 = 0$
(3) $3x - 5y + 11 = 0$
(4) $3x - 5y - 11 = 0$

42. Find the equation of the line passing through (1, 1) and forming an area of 2 sq.units with positive coordinate axis.

(1) $2x + 3y = 5$
(2) $x - y + 2 = 0$
(3) $x + y - 2 = 0$
(4) $x - y + 1 = 0$

43. If the vertices of a triangle are A(3, –3), B(–3, 3) and C($-3\sqrt{3}$, $-3\sqrt{3}$), then the distance between the orthocentre and the circumcentre is

(1) $6\sqrt{2}$ units
(2) $6\sqrt{3}$ units
(3) 0 units
(4) None of these

44. The circumcentre of the triangle formed by the lines $x + 4y = 7$, $5x + 3y = 1$ and $3x - 5y = 21$ is

(1) (–3, 2)
(2) (3, 1)
(3) (3, –1)
(4) (–3, –2)

45. If the line $(3x - 8y + 5) + a(5x - 3y + 10) = 0$ is parallel to X-axis, then a is

(1) $-\frac{8}{3}$
(2) $-\frac{3}{5}$
(3) – 2
(4) $-\frac{1}{2}$

Concept Application Level—3

46. In what ratio does the line $4x + 3y - 13 = 0$ divide the line segment joining the points (2, 1) and (1, 4)?

(1) 3 : 2 internally
(2) 2 : 3 externally
(3) 2 : 3 internally
(4) 3 : 2 externally

47. If A(3, 4), B(1, –2) are the two vertices of triangle ABC and G(3, 5) is the centroid of the triangle, then the equation of AC is

(1) $4x - 5y - 7 = 0$
(2) $4x - 5y + 8 = 0$
(3) $9x - 2y - 23 = 0$
(4) $9x - 2y - 19 = 0$

48. If $ax + 4y + 3 = 0$, $bx + 5y + 3 = 0$ and $cx + 6y + 3 = 0$ are concurrent lines, then $a + c =$

(1) 3b
(2) 2b
(3) b
(4) 4b

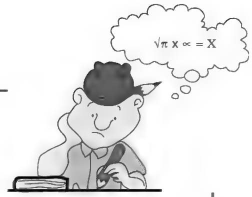
49. If (5, 3), (4, 2) and (1, –2) are the mid points of sides of triangle ABC, then the area of ΔABC is

(1) 2 sq. units
(2) 3 sq. units
(3) 1 sq. units
(4) 4 sq. units

50. Find the distance between the orthocentre and circum centre of the triangle formed by joining the points (5, 7), (4, 10) and (6, 9).

(1) $\sqrt{\frac{5}{4}}$ units
(2) $\sqrt{\frac{5}{2}}$ units
(3) $\sqrt{10}$ units
(4) $\sqrt{5}$ units

KEY



Very short answer type questions

1. 1

2. Negative, Negative.

3. (2, 3)

4. Centroid

5. X-axis

6. 2 units.

7. $\frac{3}{5}$

8. First quadrant or second quadrant.

9. $\left(-\frac{9}{4}, 0\right)$ and $\left(0, -\frac{9}{7}\right)$

10. Second quadrant or fourth quadrant

11. 4 sq. units

12. $\frac{1}{2}|ab|$ sq. units

13. $\frac{1}{2}\left|\frac{c^2}{ab}\right|$

14. 8

15. $\frac{1}{2}|x_1y_2 - x_2y_1|$

16. (3, 5)

17. (0, 2)

18. –1

19. (2, –3)

20. inside the circle

21. 2

22. $\left(\frac{2}{3}, 0\right)$

23. 2 : 1

24. (7, 5)

25. $\frac{1}{2}|(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)|$ sq.units

26. (2, 2)

27. Y-axis

28. 2 : 1; 1 : 2

29. 2 : 3

30. (–2, 4)

Short answer type questions

31. $\left(\frac{-9}{10}, -\frac{7}{10}\right)$

32. $\sqrt{193}$ units

33. –3

34. (4, –3)

35. $x = 3$

36. $5\sqrt{3}\,y - 5x + 3\sqrt{3} = 0$.

37. $5x + y - 3 = 0$

38. (–1, 3)

39. $2x - 3y - 13 = 0$

40. P(6, –4), Q(–2, 0) and R(8, 8).

41. (11, 10)

42. $\left(\frac{5}{3}, \frac{10}{3}\right)$

43. (1 – 1)

44. $\left(0, \frac{22}{5}\right)$

45. $\left(0, \frac{-8}{11}\right)$

Essay type questions

46. $2x + 3y - 12 = 0$

47. $x + 3y + 5 = 0$

48. $x - y - 4 = 0$

49. $\lambda = -\frac{2}{7}$

50. 9 sq.units.

Short answer type questions

31. (i) Let the given points be A, B and C.
(ii) Check whether ABC forms a right triangle. If yes, then
(iii) Circumcenter is the mid-point of hypotenuse.

32. The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

33. (i) Find the slope of AB using $\frac{y_2 - y_1}{x_2 - x_1}$.
(ii) Slope of line perpendicular to AB $= \frac{-1}{m}$.

34. Fourth vertex is given by $(x_1 \, x_2 + x_3, y_1 \, y_2 + y_3)$.

35. (i) Equation of line parallel to y-axis is $x = k$
(ii) Evaluate k by substituting the given point.

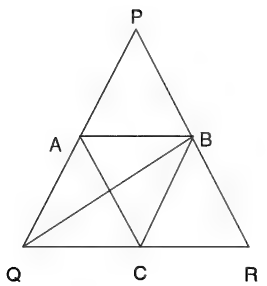
36. (i) Find slope by using $m = \tan\theta$.
(ii) Use, $y = mx + c$ and proceed.

37. The required equation is $y = mx + c$, where m is the slope and c is the y-intercept.

38. Use, centroid $= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

39. (i) Slope of line parallel to the given line is same as its slope.
(ii) Use, slope–point form and find the equation of the line.

40.



(i) Let P, Q and R be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.
(ii) ABCQ is the parallelogram. $\Rightarrow$ Equate mid-points of diagonals of ABCQ and evaluate Q.
(iii) C is the mid-point of Q R, use mid-point formula and find R.
(iv) Similarly find P.

41. (i) Let the other end of the diameter be (x, y).
(ii) Find the mid-point of diameter i.e., centre and equate it to the given centre.

42. (i) Centroid divides the median in the ratio 2 : 1 from the vertex.
(ii) Use section formula to find the centroid.

43. Use elimination method of solving linear equations.

44. Use, $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$ and find the point P.

45. (i) Let the point on y-axis be $P(0, y)$.
(ii) Equate PA, PB and solve for y.

Essay type questions

46. (i) Take the required line as $2x + 3y + k = 0$
(ii) Find the intercepts made by the above line.
(iii) Equate the sum of the intercepts above step to 10 to find k and proceed.

47. (i) Find the point of intersection of the given lines.
(ii) Slope of a line parallel to the given line is same as it.
(iii) Then, use the slope-point form and find the equation of the line.

48. (i) Find the slope of the line perpendicular to AB.
(ii) The required line passes through $(0, -4)$.
(iii) Now use slope point form and find the equation of the line.

49. Substitute the mid-point of AB in the given line and find the value of λ.

50. (i) Find the mid-points of the sides of the quadrilateral ABCD.
(ii) Find the area of the triangles formed by joining any one of the diagonals.
(iii) Find the sum of the above areas, to get the required area.

KEY

Concept Application Level—1,2,3

1. 2	2. 1
3. 1	4. 1
5. 3	6. 2
7. 3	8. 2
9. 1	10. 2
11. 1	12. 3
13. 2	14. 4
15. 2	16. 1
17. 1	18. 4
19. 2	20. 3
21. 1	22. 4
23. 4	24. 3
25. 1	26. 1
27. 2	28. 1
29. 2	30. 3
31. 4	32. 2
33. 3	34. 3
35. 1	36. 4
37. 3	38. 1
39. 2	40. 2
41. 1	42. 3
43. 3	44. 2
45. 2	46. 3
47. 4	48. 2
49. 1	50. 2

Concept Application Level—1,2,3

Key points for select questions

3. Use slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
4. Use the mid-point formula.
5. Slope(m) = $\tan\theta$.
6. Assume the point on X-axis as (a, 0) and use the distance formula.
7. x-coordinate of the point represents the distance of the point from Y-axis.
8. Use the distance formula.
9. Use the condition for collinearity.
10. Substitute the given point in the line.
11. Use the centroid formula.
12. Use right triangle properties.
13. Find the slopes of AC and BD.
14. Find the intercepts made on X–axis and Y–axis. Then Area = $\frac{1}{2}|ab|$
15. Slope = $\tan\theta$ and intercept on X-axis = 3.
16. Radius of any circle is constant.
17. $x = -1$ is a vertical line and $y = 4$ is a horizontal line.
18. Use the distance formula.
19. Find slope and use point slope form of the line.
20. Use distance formula.
21. Slope $m = \tan\theta$ and the angle between positive X–axis and Y–axis is 90°.
22. Use $AC^2 = AB^2 + BC^2$.
23. Find the roots and substitute in $\dfrac{x}{a} + \dfrac{y}{b} = 1$.
24. The inclination of a line with the positive X-axis is called the slope.
25. (i) If two lines are parallel, then their slopes are equal.
    (ii) The equation of any line parallel to $ax + by + c = 0$ can be taken as $ax + by + k = 0$.

(iii) Find the x-intercept of the line and equate it to the given x-intercept then get the value of k.

26. (i) Evaluate the vertices of the triangle and proceed.

(ii) Find the point, where the line cuts X-axis and Y-axis.

(iii) These two points and the origin are the ertices of the triangle.

(iv) Then use the formula centroid

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

27. Use the section formula.

28. Area of the triangle formed by the line $= \frac{1}{2}$ $|ab|$, a, b are intercepts of the line.

29. (i) $a + b = 5$ and $\frac{1}{2}|ab| = 2$

(ii) The equation of a line in the intercept form is $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow a + b = 5$

(iii) Then use the formula to find the area of the triangle formed by a line with coordinate axis, is $A = \frac{1}{2}|ab|. \Rightarrow |ab| = 4$.

30. (i) Evaluate the point using section formula then proceed.

(ii) Find the point on the required line that divides the line joining the points in the ratio 2 : 3.

(iii) If two lines are perpendicular then $m_1 \times m_2 = -1$.

(iv) Find the slope of the required line.

(v) Hence find the equation of the line by using slope-point formula.

31. (i) Use $y = mx + c$ where $m = \tan\theta$.

(ii) Slope of the required line is tan45°.

(iii) Use the slope and y-intercept form.

32. (i) Use the section formula.

(ii) The ratio in which (x, y) divides the line joining the points (x_1, y_1) and (x_2, y_2) is $-(x - x_1)$:

$(x - x_2)$ or $-(y - y_1) : (y - y_2)$. If the ratio is negative, it divides externally otherwise divides internally.

33. (i) Find the slope of the given line.

(ii) Use the concept, if two lines are perpendicular to each other, then $m_1 \times m_2 = -1$.

34. First prove the given triangle is a right angled triangle. In a right angled triangle, orthocentre is the vertex containing right angle.

35. First find the point of intersection (x_1, y_1) of first two lines. The equation of the line parallel to X-axis is $y = y_1$.

36. (i) Find the point diving in the ratio 3 : 1.

(ii) Find the equation of line by using point-slope form.

37. (i) Solve equation (2) and equation (3)

(ii) If three lines are concurrent then the point of intersection of any two lines, always lie on the third line.

38. (i) Find the coordinates of the points P and Q.

(ii) Then find the mid-point of PQ say M.

(iii) Now the required line is perpendicular to PQ and passes through M.

39. (i) Prove the given vertices form a right angled triangle.

(ii) In a right angled triangle, the vertex containing right angle is the orthocentre.

40. Clearly the given lines forms a right angled triangle. In a right angled triangle circumcentre is the mid-point of hypotenuse and orthocentre is the vertex containing 90°.

41. Find the equation of the line passing through A and mid-point of BC.

42. The area of triangle formed by $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is $\frac{1}{2}|ab|$.

43. (i) ABC forms an equilateral triangle.

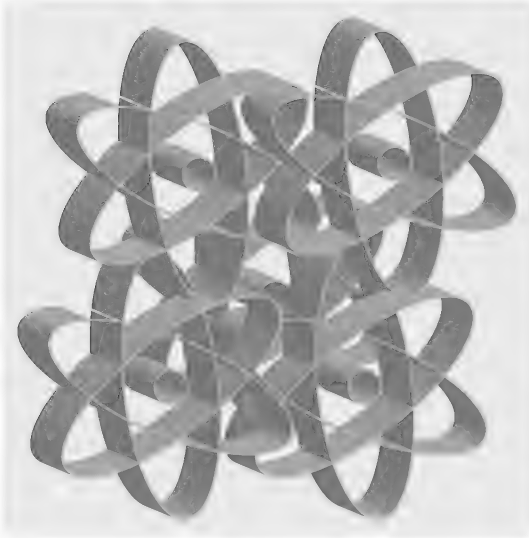
(ii) In an equilateral triangle all the geometric centres except excentre coincide.

44. Prove that the given lines form a right triangle, then find the ends of the hypotenuse there by find the mid-point of them.

45. If a line is parallel to X-axis, then its x-coefficient is zero.

CHAPTER 16

Mathematical Induction and Binomial Theorem



INTRODUCTION

The process of mathematical induction is a indirect method which helps us to prove complex mathematical formulae, that cannot be easily proved by direct methods.

For example, to prove that 'n(n + 1) is always divisible by 2' for n being a natural number, we can substitute n = 1, 2, 3…… in n(n + 1), and check in each case if the result is divisible by 2. After checking, for a few of values, we can say that the formula is likely to be correct. Since, we cannot substitute all possible values of n, to prove the formula we use the principle of mathematical induction to prove the given formula.

The principle of mathematical induction

If P(n) is a statement such that,

(i) P(n) is true for n =1

(ii) P(n) is true for n = k + 1, when it is true for n = k, where k is a natural number then the statement P(n) is true for all natural numbers.

1. Let us prove some results using this principle

☛ *Example*

Prove that $1 + 2 + 3 + \ldots. + n = \dfrac{n(n+1)}{2}$.

Solution

Let P(n): $1+2+\ldots.+n = \dfrac{n(n+1)}{2}$ be the given statement.

Step 1: Put n = 1

Then, L.H.S. = 1 and R.H.S. = $\frac{1(1+1)}{2} = 1$.

$\therefore$ L.H.S. = R.H.S. $\Rightarrow$ P(n) is true for n = 1.

Step 2: Assume that P(n) is true for n = k.

$\therefore 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

Adding (k+1) on both sides, we get $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) =$

$(k + 1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \frac{(k+1)(\overline{k+1}+1)}{2}$

$\Rightarrow$ P(n) is true for n = k + 1

$\therefore$ By the principle of mathematical induction P(n) is true for all natural numbers n.

Hence, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for all $n \in N$

☛ *Example*

Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Solution

Let P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ be the given statement

Step 1: Put n =1

Then, L.H.S. = 1

R.H.S. = $(1)^2 = 1$

$\therefore$ L.H.S. = R.H.S.

$\Rightarrow$ P(n) is true for n = 1.

Step 2: Assume that P(n) is true for n = k.

$\therefore 1 + 3 + 5 + \ldots + (2k - 1) = k^2$

Adding 2k+1 on both sides, we get

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$

$\therefore 1 + 3 + 5 + \ldots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$

$\Rightarrow$ P(n) is true for n = k + 1.

$\therefore$ By the principle of mathematical induction P(n) is true for all natural numbers 'n'.

Hence, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$, for all $n \in N$

☛ *Example*

Prove that $1.2 + 2.3 + 3.4 + \ldots + n.(n + 1) = \frac{n(n+1)(n+2)}{3}$

Solution

Let P(n): $1.2 + 2.3 + 3.4 + \ldots + n.(n + 1) = \frac{n(n+1)(n+2)}{3}$ be the given statement

Step 1: Put n = 1

Then, L.H.S.= 1.2 = 2

$$\text{R.H.S.} = \frac{1(1+1)(1+2)}{3} = \frac{2\times 3}{3} = 2$$

$\therefore$ L.H.S. = R.H.S.

$\Rightarrow$ P(n) is true for n = 1.

Step 2: Assume that P(n) is true for n = k.

$$\therefore\ 1.2 + 2.3 + 3.4 + \ldots. + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Adding (k + 1)(k + 2) on both sides, we get 1.2 + 2.3 + 3.4 + ……. + k(k + 1) + (k + 1)(k + 2)

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = (k+1)(k+2)\left(\frac{k}{3}+1\right) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$\therefore\ 1.2 + 2.3 + 3.4 + \ldots. + k.(k+1) + (k+1)(k+2) = \frac{(k+1)(\overline{k+1}+1)(\overline{k+1}+2)}{3}$$

$\Rightarrow$ P(n) is true for n = k + 1.

$\therefore$ By the principle of mathematical induction P(n) is true for all natural numbers

$$\text{Hence, } 1.2 + 2.3 + 3.4 + \ldots + n.(n+1) = \frac{n(n+1)(n+2)}{3}, n\in N$$

Example

Prove that $3^{n+1} > 3(n+1)$

Solution

Let P(n) : $3^{n+1} > 3(n+1)$

Step 1: Put n = 1

Then, $3^2 > 3(2)$

$\Rightarrow$ p(n) is true for n = 1

Step 2: Assume that P(n) is true for n = k

Then, $3^{k+1} > 3(k+1)$

Multiplying throughout with '3'.

$3^{k+1}\cdot 3 > 3(k+1)\cdot 3 = 9k + 9 = 3(k+2) + (6k+3) > 3(k+2)$

$\Rightarrow 3^{\overline{k+1}+1} > 3(\overline{k+1}+1)$

P(n) is true for n = k + 1

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in N$.

Hence, $3^{n+1} > 3(n+1), \forall\ n \in N$

Example

Prove that 7 is a factor of $2^{3n} - 1$ for all natural numbers n.

Solution

Let P(n) : 7 is a factor of $2^{3n} - 1$ be the given statement

Step 1: When n = 1,

$2^{3(1)} - 1 = 7$ and 7 is a factor of itself.

$\therefore$ P(n) is true for n = 1

Step 2: Let P(n) be true for n = k.

$\Rightarrow$ 7 is a factor of $2^{3k} - 1$.

$\Rightarrow 2^{3k} - 1 = 7M$, where $M \in N$.

$\Rightarrow 2^{3K} = 7M + 1$ ----------(1)

Now consider $2^{3(k+1)} - 1 = 2^{3k+3} - 1 = 2^{3k}.2^3 - 1$

$= 8(7M+1) - 1$ (using (1)) $= 56M + 7$ (As $2^{3k} = 7m + 1$)

$\therefore 2^{3(k+1)} - 1 = 7(8M + 1)$

$\Rightarrow$ 7 is a factor of $2^{3(k+1)} - 1$

$\Rightarrow$ P(n) is true for n = k + 1

$\therefore$ By the principle of mathematical induction, P(n) is true for all natural numbers n.

Hence, 7 is a factor $2^{3n} - 1$ for all $n \in N$.

Binomial expression: An algebraic expression containing only two terms is called a binomial expression.

For example, x + 2y, 3x + 5y, 8x – 7y etc,

We know that, $(a + b)^2 = a^2 + 2ab + b^2$.

$(a + b)^3 = (a + b)(a + b)^2$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Now using a similar approach we can arrive at the expressions for $(a + b)^4$, $(a + b)^5$ etc. However, when the index is large, this process becomes very cumbersome. Hence, we need a simpler method to arrive at the expression for $(a + b)^n$, for n = 1, 2, 3.....

The binomial theorem is the appropriate tool in this case. It helps us arrive at the expression for $(a + b)^n$, for any value of n, by using a few standard coefficients also know as binomial coefficients.

Now, consider the following cases in which we find the expansions when a binomial expression is raised to different powers.

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

In the above examples, the coefficients of the variables in the expansions of the powers of the binomial expression are called binomial coefficients.

When the binomial coefficients are listed, for different values of n, we see a definite pattern being followed. This pattern is given by the Pascal Triangle.

Pascal triangle

This definite pattern, show above, can be used to write the binomial expansions for higher powers such as n = 6, 7, 8.... so on. The binomial theorem gives us a general algebraic formula by means of which any power of a binomial expression can be expanded into a series of simpler terms.

The exponent in the binomial	The coefficients of the terms in the expansion
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Before we take up the binomial theorem, let us review the concepts of factorial notation and the ^nC_r representation.

Factorial notation and ^nC_r representation

The factorial of n is denoted by n! and is defined as $n! = 1 \times 2 \times 3 \times \ldots\ldots \times (n-1) \times n$

For example, $4! = 1 \times 2 \times 3 \times 4$ and $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$.

Also, $0! = 1$ and $n! = n\,(n-1)!$.

For $0 \le r \le n$, we define ^nC_r as $^nC_r = \dfrac{n!}{(n-r)!\,r!}$

for example, $^6C_2 = \dfrac{6!}{(6-2)!\,2!} = \dfrac{6 \times 5 \times 4!}{4! \times 2!} = 15$

also, $^nC_0 = {}^nC_n = 1$; $^nC_1 = {}^nC_{n-1} = n$ and $^nC_r = {}^nC_{n-r}$

for example, $^{10}C_2 = {}^{10}C_8$ and if $^nC_3 = {}^nC_5$, then $n = 3 + 5 = 8$

Binomial theorem

If n is a. positive integer,

$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots\ldots\ldots + {}^nC_r x^{n-r} y^r + \ldots\ldots\ldots + {}^nC_n y^n$

Important inferences from the above expansion

1. The number of terms in the expansion is n + 1.
2. The exponent of x goes on decreasing by '1' from left to right and the power of 'y' goes on increasing by 1 from left to right.
3. In each term of the expansion the sum of the exponents of x and y is equal to the exponent (n) of the binomial expression.

4. the coefficients of the terms that are equidistant from the beginning and the end have numerically equal, i.e., ${}^nC_0 = {}^nC_n$; ${}^nC_1 = {}^nC_{n-1}$; ${}^nC_2 = {}^nC_{n-2}$ and so on.
5. The general term in the expansion of $(x + y)^n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$.
6. On substituting '– y' in place of 'y' in the expansion, we get
   $(x-y)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 - {}^nC_3\, x^{n-3}\, y^3 + \dots + (-1)^n\, {}^nC_n y^n$
   The general term in the expansion $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$.

Example

Expand $(x + 2y)^5$.

Solution

$(x + 2y)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^{5-1}\,(2y) + {}^5C_2\, x^{5-2}\,(2y)^2 + {}^5C_3\, x^{5-3}\,(2y)^3 + {}^5C_4\, x^{5-4}\,(2y)^4 + {}^5C_5\,(2y)^5$.

$\Rightarrow (x + 2y)^5 = x^5 + 5x^4\,(2y) + 10x^3\, 4y^2 + 10x^2 8y^3 + 5x\, 16y^4 + 2^5\, y^5$

$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

Example

Find the 3rd term in the expansion of $(3x–5y)^7$.

Solution

The general term in $(x - y)^n$ is $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r} y^r$

$\therefore T_3 = T_{2+1} = (-1)^2\, {}^7C_2\, (3x)^{7-2}\, (5y)^2 = {}^7C_2\, (3x)^5\, (5y)^2$.

Middle terms in the expansion of $(x + y)^n$

Depending on the nature of n, i.e., whether n is even or odd, there may exist one or two middle terms.

Case 1

When n is an even number, then there is only one middle term in the expansion $(x + y)^n$, which is $\left(\frac{n}{2}+1\right)$th term.

Case 2

When n is odd number, there will be two middle terms in the expansion of $(x + y)^n$, which are $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Example

Find the middle term in the expansion of $(2x + 3y)^8$.

Solution

Since n is even number, $\left(\frac{8}{2}+1\right)$ th term i.e., 5th term is the middle term in $(2x + 3y)^8$.

$T_5 = T_{4+1} = {}^8C_4\, (2x)^{8-4}\, (3y)^4 = {}^8C_4\, (2x)^4\, (3y)^4$

Example

Find the middle terms in the expansion of $(5x-7y)^7$.

Solution

Since n is an odd number, the expansion contains two middle terms.

$\left(\frac{7+1}{2}\right)$th and $\left(\frac{7+3}{2}\right)$th terms are the two middle terms in the expansion of $(5x-7y)^7$.

$T_4 = T_{3+1} = (-1)^3\, {}^7C_3\, (5x)^{7-3}\, (7y)^3 = -{}^7C_3\, (5x)^4\, (7y)^3$

$T_5 = T_{4+1} = (-1)^4 \cdot {}^7C_4\, (5x)^{7-4} \cdot (7y)^4 = -{}^7C_4\, (5x)^3\, (7y)^4$

Term independent of x

In an expansion of form $\left(x^p + \frac{1}{x^q}\right)^n$, the term for which the exponent of x is 0 is said to be the term that is independent of x or a constant term.

For example, in the expansion $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}$, the 2nd term is independent of 'x'.

Example

Find the term independent of x in $\left(x + \frac{1}{x}\right)^4$.

Solution

Let T_{r+1} be the term independent term of x in the given expansion.

$\therefore\ T_{r+1} = {}^4C_r\, x^{4-r} \left(\frac{1}{x}\right)^r = {}^4C_r\, \frac{x^{4-r}}{x^r} = {}^4C_r\, x^{4-2r}$

For the term independent of x the power of x should be zero.

$\therefore\ 4 - 2r = 0$ or $r = 2$.

$\Rightarrow T_{2+1} = T_3$ term, is the independent term of the expansion.

Note: If r is not a positive integer, then the expansion does not contain constant term.

Example

Find the coefficient of x^2 in $\left(x^2 + \frac{1}{x^3}\right)^6$.

Solution

Let T_{r+1} be the term containing x^2.

$T_{r+1} = {}^6C_r\, (x^2)^{6-r} \left(\frac{1}{x^3}\right)^r = {}^6C_r\, x^{12-2r} \frac{1}{x^{3r}} = {}^6C_r\, x^{12-5r}$

As the coefficient of x is 2

$12 - 5r = 2 \Rightarrow r = 2$. $\therefore$ Coefficient of $x^2 = {}^6C_2 = 15$.

The greatest coefficient in the expansion of $(1 + x)^n$ (where n is a positive integer)

The coefficient of the (r + 1)th term in the expansion of $(1 + x)^n$ is ${}^nC_r$.
${}^nC_r$ is maximum when $r = n/2$ (if n is even) and

$r = \dfrac{n+1}{2}$ or $\dfrac{n-1}{2}$ (if n is odd)

Example

Find the total number of terms in the expansion of $(2 + 3x)^{15} + (2 - 3x)^{15}$.

Solution

$(2 + 3x)^{15} = {}^{15}C_0(2)^{15} + {}^{15}C_1(2)^{14}(3x)^1 + \ldots + {}^{15}C_{14}(2)^1(3x)^{14} + {}^{15}C_{15}(3x)^{15}$ and $(2 - 3x)^{15}$

$= {}^{15}C_0(2)^{15} - {}^{15}C_1(2)^{14}(3x^1) + \ldots\ldots + {}^{15}C_{14}(2)^1(3x)^{14} - {}^{15}C_{15}(3x)^{15}$

Adding the two equations, we see that the terms in even positions get cancelled, and we get

$(2 + 3x)^{15} + (2 - 3x)^{15} = 2[{}^{15}C_0(2)^{15} + {}^{15}C_2(2)^{13}(3x)^2 + \ldots + {}^{15}C_{14}(2)^1(3x)^{14}]$

$\therefore$ Total number of terms = 8.

Alternately, the number of terms in $(a + x)^n + (a - x)^n$, if n is odd is $\dfrac{n+1}{2}$. Hence in this case, the number of terms are $\dfrac{15+1}{2} = 8$.

Example

If the expansion $\left(x^2 + \dfrac{1}{x^3}\right)^n$ is to contain an independent term, then what should be the value of n?

Solution

General term, $T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r$, for $(x + y)^n$

$\Rightarrow$ general term of $\left(x^2 + \dfrac{1}{x^3}\right)^n$ is ${}^nC_r \cdot x^{2n-2r} \cdot \dfrac{1}{x^{3r}} = {}^nC_r \cdot x^{2n-5r}$

For a term to be independent of x, $2n - 5r$ should be equal to zero,
i.e., $2n - 5r = 0$.

$\Rightarrow r = \dfrac{2}{5}n$, since r can take only integral values, n has to be a multiple of 5.

Example

If the coefficient of x^7 in $\left(ax + \dfrac{1}{x}\right)^9$ and x^{-7} in $\left(bx - \dfrac{1}{x}\right)^9$ are equal, find the relation between a and b?

Solution

For $\left(ax + \dfrac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r(ax)^{9-r}\left(\dfrac{1}{x}\right)^r = {}^9C_r(a)^{9-r}(x)^{9-2r}$ as $9 - 2r = 7, r = 1$

$\therefore$ Coefficient of x^7 is ${}^9C_1(a)^{9-1} = 9(a)^8$

Now, for $\left(bx - \frac{1}{x}\right)^9$, $T_{r+1} = {}^9C_r(bx)^{9-r}\left(-\frac{1}{x}\right)^r = {}^9C_r(b)^{9-r}(-1)^r x^{9-2r}$ as $9 - 2r = -7, r = 8$.

$\therefore$ Coefficient of x^{-7} is ${}^9C_8\, b^{9-8}(-1)^8 = 9b$

$\therefore$ $9a^8 = 9b$ i.e., $a^8 - b = 0$

Example

Find the term independent of 'x' in the expansion of $(1 + x^2)^4\left(1+\frac{1}{x^2}\right)^4$.

Solution

$$(1+x^2)^4\left(1+\frac{1}{x^2}\right)^4 = ({}^4C_0 + {}^4C_1x^2 + \ldots.. + {}^4C_4x^8) \times ({}^4C_0 + {}^4C_1x^{-2} + \ldots.. + {}^4C_4x^{-8})$$

The term independent of 'x' is the term containing the coefficient $({}^4C_0 \cdot {}^4C_0 + {}^4C_1 \cdot {}^4C_1 + \ldots. {}^4C_4 \cdot {}^4C_4)$

$= ({}^4C_0)^2 + ({}^4C_1)^2 + ({}^4C_2)^2 + ({}^4C_3)^2 + ({}^4C_4)^2$

$= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$

Example

Find the sum of the co-efficients of the terms of the expansion $(1 + x + 2x^2)^6$.

Solution

Substituting $x = 1$, we have $(1 + 1 + 2)^6$, which gives us the sum of the co-efficients of the terms of the expansion.

$\therefore$ Sum $= 4^6$

Example

Find the value of x, if the fourth term in the expansion of $\left(\frac{1}{x^2} + x^2.2^x\right)^6$ is 160.

Solution

4th term $\Rightarrow T_{3+1} = {}^6C_3 \cdot \left(\frac{1}{x^2}\right)^3 \cdot (x^2)^3 \cdot (2^x)^3$

$\therefore$ ${}^6C_3 \cdot (2^x)^3 = 160$

i.e., $20 \cdot 2^{3x} = 160$

$\therefore 2^{3x} = 8 \Rightarrow 2^{3x} = 2^3$

$\therefore$ $x = 1$

test your concepts

Very short answer type questions

1. If p(n) is a statement which is true for n = 1 and true for (n + 1) then _________.
2. According to the principle of mathematical induction, when can we say that a statement X(n) is true for all natural numbers n?
3. If $p(n) = n(n + 1)(n + 2)$ then highest common factor of p(n), for different values of n where n is any natural number is _________.
4. Is $2^{3n} - 1$ a prime number for all natural numbers n?
5. The product of (q – 1) consecutive integers where q > 1 is divisible by ________.
6. An algebraic expression with two terms is called a _______.
7. In pascal triangle, each row of coefficients is bounded on both sides by ______.
8. The number of terms in the expansion of $(x + y)^n$ is _______ (where n is a positive integer).
9. In the expansion of $(x + y)^n$, if the exponent of x in second term is 10, what is the exponent of y in 11th term.
10. What is the coefficient of a term in a row of pascal triangle if in the preceding row, the coefficient on the immediate left is 5 and on the immediate right is 10.
11. In the expansion of various powers of $(x + y)^n$, if the expansion contains 49 terms, then it is the expansion of _______.
12. In the expression of $(x + y)^{123}$, the sum of the exponents of x and y in 63rd term is ______.
13. $(n - r)! =$ _______
14. The value of ${}^{n+1}C_r =$ _______
15. If ${}^nC_r = 1$ and n = 6, then what may be the value(s) of r be?
16. In the expansion of $(x + y)^n$, $T_{r+1} =$ _______
17. ${}^7C_2 =$ _______
18. ${}^{1230}C_0 =$ _______
19. The coefficient of x in the expansion of $(2x + 3)^5$ is _______ .
20. The coefficient of y^7 in the expansion of $(y + z)^7$ is ______.
21. $(x + y)^3 =$ _______
22. The term which does not contain 'a' in the expansion of $\left(\frac{x}{a} + 6x\right)^{12}$ is ______.
23. If ${}^{12}C_r (4)^{12-r}(x)^{12-3r}$ is a constant term in an expansion, then r = ________.
24. Write the first, the middle and the last terms in the expansion of $(x^2 + 1)^3$.

25. Constant term in the expansion of $(x + 3)^{16}$ is _______.

26. The sum of the first n even natural numbers is _______.

27. The sum of the first n odd natural numbers is _______.

28. The elements in the fifth row of Pascal triangle is _______.

29. If ${}^nC_3 = {}^nC_{15}$, then ${}^{20}C_n$ is _______.

30. The inequality $2^n > n$ is true for _______.

Short answer type questions

Directions for questions 31 to 39: By mathematical Induction prove the following.

31. $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1, n \in N$.

32. $a - b$ divides $a^n - b^n, n \in N$

33. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}, n \in N$

34. $2.5 + 3.8 + 4.11 + \ldots +$ upto n terms $= n(n^2 + 4n + 5), n \in N$

35. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}, n \in N$

36. $a + (a + d) + (a + 2d) + \ldots$ upto n terms $= \dfrac{n}{2}[2a + (n-1)d]$

37. $\dfrac{1}{2.5} + \dfrac{1}{5.8} + \dfrac{1}{8.11} + \ldots + \dfrac{1}{(3n-1)(3n+2)} = \dfrac{n}{6n+4}, n \in N$

38. 9 is a factor of $4^n + 15n - 1, n \in N$

39. $2 + 5 + 8 + \ldots + (3n - 1) = \dfrac{n(3n+1)}{2}, n \in N$

40. Expand $(3x^2 + \dfrac{5}{y^2})^6$

41. Expand $(5x + 3y)^8$

42. Find the middle term or terms of the expansion of $(x + 5y)^9$.

43. Find the middle term or terms of the expansion of $\left(x + \dfrac{1}{x}\right)^6$.

44. Find the 7th term in the expansion of $\left(5x - \dfrac{1}{7y}\right)^9$.

45. Prove that $2^{n+1} > 2n + 1; n \in N$

Essay type questions

46. Find the value of $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$

47. Find the coefficient of x^{-5} in the expansion of $\left(2x^2 - \frac{1}{5x}\right)^8$.

48. Find the term independent of x in $\left(6x^2 - \frac{1}{7x^3}\right)^{10}$.

49. Find the coefficient of x^3 in the expansion of $\left(x^2 + \frac{1}{3x^3}\right)^4$.

50. Find the term independent of x in $\left(2x^5 + \frac{1}{3x^2}\right)^{21}$.

CONCEPT APPLICATION

Concept Application Level—1

1. $n^2 + n + 1$ is a/an _____ number for all $n \in N$.
   (1) even (2) odd (3) prime (4) None of these

2. If the expansion $\left(x^3 + \frac{1}{x^2}\right)^n$ contains a term independent of x, then the value of n can be
   (1) 18 (2) 20 (3) 24 (4) 22

3. $1 + 5 + 9 + \ldots\ldots + (4n - 3)$ is equal to
   (1) $n(4n - 3)$ (2) $(2n - 1)$ (3) $n(2n - 1)$ (4) $(4n - 3)^2$

4. For all $n \in N$, which of the following is a factor of $2^{3n} - 1$?
   (1) 3 (2) 5 (3) 7 (4) None of these

5. For what values of n is $14^n + 11^n$ divisible by 5?
   (1) when n is an even positive integer
   (2) For all values of n
   (3) When n is a prime number
   (4) When n is a odd positive integer

6. The smallest positive integer n for which $n! < \frac{(n-1)^n}{2}$ holds is
   (1) 4 (2) 3 (3) 2 (4) 1

7. The third term from the end in the expansion of $\left(\frac{4x}{3y} - \frac{3y}{2x}\right)^9$ is

(1) ${}^9C_7 \frac{3^5}{2^3} \frac{y^5}{x^5}$ (2) $-{}^9C_7 \frac{3^5}{2^3} \frac{y^5}{x^5}$ (3) ${}^9C_7 \frac{3^3}{2^5} \frac{y^5}{x^3}$ (4) None of these

8. In the 8th term of $(x + y)^n$, the exponent of x is 3, then the exponent of x in 5th term is

(1) 5 (2) 4 (3) 2 (4) 6

9. The sum of the elements in the sixth row of pascal triangle is

(1) 32 (2) 63 (3) 128 (4) 64

10. In $(x + y)^n - (x - y)^n$ if the number of terms is 5, then find n.

(1) 6 (2) 5 (3) 10 (4) 9

11. If the third term in the expansion of $\left(x + x^{\log_2 x}\right)^6$ is 960, then the value of x is

(1) 2 (2) 3 (3) 4 (4) 8

12. Find the sum of coefficients of all the terms of the expansion $(ax + y)^n$.

(1) ${}^nC_0a^n + {}^nC_1a^{n-1}x^{n-1}y + {}^nC_2a^{n-2}x^{n-2}y^2 + \ldots\ldots. + {}^nC_ny^n$

(2) ${}^nC_0a^n + {}^nC_1a^{n-1} + {}^nC_2a^{n-2} + \ldots\ldots.. + {}^nC_n$

(3) 2^n

(4) None of these

13. If the sum of the coefficients in the expansion $(4ax - 1 - 3a^2x^2)^{10}$ is 0, then the value of a can be

(1) 2 (2) 4 (3) 1 (4) 7

14. Find the coefficient of x^4 in the expansion of $\left(2x^2 + \frac{3}{x^3}\right)^7$.

(1) ${}^7C_2\, 2^5\, 3^3$ (2) ${}^7C_2\, 2^5\, 3^2$ (3) ${}^7C_2\, 3^5\, 2^2$ (4) ${}^7C_3\, 2^5\, 3^2$

15. $n^2 - n + 1$ is an odd number for all

(1) $n > 1$ (2) $n > 2$ (3) $n \geq 1$ (3) $n \geq 5$

16. $7^{n+1} + 3^{n+1}$ is divisible by

(1) 10 for all natural numbers n (2) 10 for odd natural numbers n

(3) 10 for even natural numbers n (4) None of these

17. For $n \in N$, $2^{3^n} + 1$ is divisible by

(1) 3^{n+11} (2) 3^{n-11} (3) 3^{n+1} (4) 3^{n+111}

18. $2^n - 1$ gives the set of all odd natural numbers for all $n \in N$. Comment on the given statement.

(1) True for all values of n (2) False

(3) True for only odd values of n (4) True for only prime values of n

19. In the 5th term of $(x + y)^n$, the exponent of y is 4, then the exponent of y in the 8th term is

(1) 1 (2) 7 (3) 5 (4) 9

20. If the coefficients of 6th and 5th terms of expansion $(1 + x)^n$ are in the ratio 7 : 5, then find the value of n.

(1) 11 (2) 12 (3) 10 (4) 9

21. The third term from the end in the expansion of $(3x - 2y)^{15}$ is

(1) $-{}^{15}C_{13} 3^{13} 2^2 x^{13} y^2$ (2) ${}^{15}C_{13} 3^{13} 2^2 x^{13} y^2$ (3) ${}^{15}C_2 3^2 2^{13} x^2 y^{13}$ (4) $-{}^{15}C_2 3^2 2^{13} x^2 y^{13}$

22. Find the sixth term in the expansion of $\left(2x^2 - \frac{3}{7x^3}\right)^{11}$.

(1) $-{}^{11}C_5 \frac{2^6 3^5}{7^5} x^3$ (2) ${}^{11}C_5 \frac{2^6 3^5}{7^5} x^{-3}$ (3) $-{}^{11}C_5 \frac{2^6 3^5}{7^5} x^{-3}$ (4) None of these

23. The term independent of x in the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{10}$ is

(1) ${}^{10}C_6$ (2) ${}^{10}C_7$ (3) ${}^{10}C_9$ (4) $-{}^{10}C_6$

24. Which term is the constant term in the expansion of $\left(2x - \frac{1}{3x}\right)^6$?

(1) 2nd term (2) 3rd term (3) 4th term (4) 5th term

25. The sum of the coefficients in the expansion of $(x + y)^7$ is

(1) 119 (2) 64 (3) 256 (4) 128

26. The number of terms which are not radicals in the expansion $(\sqrt{7} + 4)^6 + (\sqrt{7} - 4)^6$, after simplification is

(1) 6 (2) 5 (3) 4 (4) 3

27. The coefficient of x^4 in the expansion of $\left(4x^2 + \frac{3}{x}\right)^8$ is

(1) ${}^8C_5 12^5$ (2) ${}^8C_4 12^4$ (3) ${}^8C_3 12^3$ (4) ${}^8C_6 12^6$

28. In the expansion of $(a + b)^n$, the coefficients of 15th and 11th terms are equal. Find the number of terms in the expansion.

(1) 26 (2) 25 (3) 20 (4) 24

29. The number of terms in the expansion of $[(2x + 3y)^4 (4x - 6y)^4]^9$ is

(1) 36 (2) 37 (3) 10 (4) 40

30. If sum of the coefficients of the first two odd terms of the expansion $(x + y)^n$ is 16, then find n.

(1) 10 (2) 8 (3) 7 (4) 6

Concept Application Level—2

31. The number of rational terms in the expansion of $\left(x^{\frac{1}{5}}+y^{\frac{1}{10}}\right)^{45}$ is

(1) 5 (2) 6 (3) 4 (4) 7

32. The remainder when $9^{49} + 7^{49}$ is divided by 64 is

(1) 24 (2) 8 (3) 16 (4) 38

33. If $p(n) = (n - 2)(n - 1)n(n + 1)(n + 2)$, then greatest number which divides $p(n)$ for all $n \in N$ is

(1) 12 (2) 24 (3) 120 (4) None of these

34. For $n \in N$, $a^{2n-1} + b^{2n-1}$ is divisible by

(1) $a + b$ (2) $(a + b)^2$ (3) $a^3 + b^3$ (4) $a^2 + b^2$

35. Find the coefficient of the independent term in the expansion of $\left(x^{\frac{1}{2}}+7x^{-\frac{1}{3}}\right)^{10}$.

(1) ${}^{10}C_4\,7^4$ (2) ${}^{10}C_6\,7^6$ (3) ${}^{10}C_6\,7^5$ (4) ${}^{10}C_4\,7^7$

36. Find the term which has the exponent of x as 8 in the expansion of $\left(x^{\frac{5}{2}}-\frac{3}{x^3\sqrt{x}}\right)^{10}$.

(1) T_2 (2) T_3 (3) T_4 (4) Does not exist

37. The greatest number which divides $25^n - 24n - 1$ for all $n \in N$ is

(1) 24 (2) 578 (3) 27 (4) 576

38. If three consecutive coefficients in the expansion of $(1 + x)^n$, where n is a natural number are 36, 84 and 126 respectively, then n is

(1) 8 (2) 9
(3) 10 (4) Cannot be determined

39. Find the value of k for which the term independent of x in $\left(x^2+\frac{k}{x}\right)^{12}$ is 7920.

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\sqrt{2}$ (4) 2

40. Find the coefficient of x^7 in the expansion of $\left(7x+\frac{2}{x^2}\right)^{13}$.

(1) $78 \times 8^8 \times 4$ (2) $78 \times 7^6 \times 4^2$ (3) $78 \times 7^{11} \times 4$ (4) $78 \times 7^{11} \times 4^2$

41. The value of $(\sqrt{5}+2)^6 + (\sqrt{5}-2)^6$ is

(1) a positive integer (2) a negative integer
(3) an irrational number (4) a rational number but not an integer

42. The ratio of the coefficients of x^4 to that of the term independent of x in the expansion of $\left(x^2 + \frac{9}{x^2}\right)^{18}$ is

(1) 1 : 6 (2) 3 : 8 (3) 1 : 10 (4) 1 : 8

43. $\sum_{r=2}^{16} {}^{16}C_r =$

(1) $2^{15} - 15$ (2) $2^{16} - 16$ (3) $2^{16} - 17$ (4) $2^{17} - 17$

44. Number of non-zero terms in the expansion of $(5\sqrt{5}x + \sqrt{7})^6 + (5\sqrt{5}x - \sqrt{7})^6$ is

(1) 4 (2) 10 (3) 12 (4) 14

45. Find the value of $(98)^4$ by using the binomial theorem.

(1) 92236846 (2) 92236816
(3) 92236886 (4) 92236806

Concept Application Level—3

46. The number of irrational terms in the expansion of $\left(x^{2/3} + y^{1/4}\right)^{81}$ is __________.

(1) 70 (2) 12(3) 75 (4) 13

47. Find the independent term in the expansion of $\left(x^4 + \frac{3}{8x^3\sqrt{x}}\right)^{15}$

(1) ${}^{15}C_4\left(\frac{3}{8}\right)^{16}$ (2) ${}^{15}C_{12}\left(\frac{3}{8}\right)^{4}$
(3) ${}^{15}C_8\left(\frac{3}{8}\right)^{8}$ (4) ${}^{15}C_7\left(\frac{3}{8}\right)^{12}$

48. $\sum_{r=1}^{30} r\frac{{}^{30}C_r}{{}^{30}C_{r-1}} =$

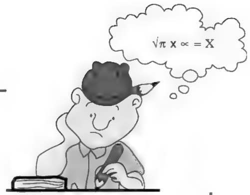
(1) 930 (2) 465 (3) 310 (4) 630

49. For all $n \in N$, $41^n - 40n - 1$ is divisible by

(1) 41 (2) 40 (3) 300 (4) 500

50. If m and n are the coefficients of x^{a^2} and x^{b^2} respectively in $(1+x)^{a^2+b^2}$, then

(1) n = 2m (2) m + n = 0
(3) 2n = m (4) m = n



KEY

Very short answer type questions

1. p(n) is true for all natural numbers.
2. When x(n) is true for n = 1 and also true for n + 1.
3. $3! = 6$
4. not a prime number
5. $(q - 1)!$
6. Binomial
7. 1
8. $n + 1$
9. 10
10. 15
11. $(x + y)^{48}$
12. 123
13. $1 \cdot 2 \cdot 3 \ldots\ldots (n - r - 1) . (n - r)$
14. $\dfrac{(n+1)!}{(n-r+1)!r!}$
15. 0 or 6
16. ${}^{n}C_{r}\, x^{n-r} y^{r}$
17. 21
18. 1
19. 810
20. 1
21. $x^3 + 3x^2y + 3xy^2 + y^3$
22. t_{13}
23. 4
24. $3x^4; 3x^2$
25. 3^{16}
26. $n(n + 1)$
27. n^2
28. 1,5,10,10,5,1
29. 190
30. all integers

Short answer type questions

40. $(3x^2)^6 + 6(3x^2)^5 \dfrac{5}{y^2} + 15(3x^2)^4 \left(\dfrac{5}{y^2}\right)^2 + 20(3x^2)^3 \left(\dfrac{5}{y^2}\right)^3 + 15(3x^2)^2 \left(\dfrac{5}{y^2}\right)^4 + 6(3x^2) \left(\dfrac{5}{y^2}\right)^5 + \left(\dfrac{5}{y^2}\right)^6.$

41. $(5x)^8 + 8(5x)^7(3y) + 28(5x)^6(3y)^2 + 56(5x)^5(3y)^3 + 70(5x)^4(3y)^4 + 56(5x)^3(3y)^5 + 28(5x)^2(3y)^6 + 8(5x)(3y)^7 + (3y)^7$

42. ${}^{9}C_{4}\, 5^4x^5y^4$ and ${}^{9}C_{5}\, 5^5x^4y^5$

43. ${}^{6}C_{3}$.

44. ${}^{9}C_{6}\, (5x)^3 \left(\dfrac{1}{7y}\right)^6$

Essay type questions

46. 152
47. $\dfrac{-16}{5^7}$
48. ${}^{10}C_{4}\, \dfrac{6^6}{7^4}$
49. $\dfrac{4}{3}$
50. ${}^{21}C_{15}\, \dfrac{2^6}{3^{15}}$

key points for selected questions

Very short answer type questions

26. Use the sum of the n terms of

$$AP = \frac{n}{2}[2a + (n - 1)d]$$

27. Use sum of the n terms of

$$A.P = \frac{n}{2}[2a + (n - 1)d]$$

28. The elements in the 'nth' row of Pascal triangle are $n_{C_0}, n_{C_1}, n_{C_2}, \ldots\ldots\ldots, n_{C_n}$.

29. If $n_{C_r} = n_{C_p}$, then r = p or n = r + p.

30. Substitute n = ±1, ±2, ±3, ±4.......... on both sides of the inequality.

Short answer type questions

31. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

32. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

33. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

34. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

35. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

36. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

37. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

38. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

39. Prove for n = 1, assume it is true for n = k and prove for n = k + 1.

40. Use the formula which is used to expand $(x + y)^n$

41. Use the formula which is used to expand $(x + y)^n$.

42. If 'n' is odd then the middle terms are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term.

43. If 'n' is even then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

44. $T_{r+1} = {}^nC_r\, x^{n-r} y^r$. (Here r = 6)

45. Prove for n = 1, assume it is true for n = k and prove for n = k + 1

Essay type questions

46. $(a + 1)^n - (a - 1)^n =$
$2\left[n_{C_1} a^{n-1} + n_{C_3} a^{n-3} + \ldots\ldots\ldots + n_{C_n} a^0\right]$

47. Use the formula to find T_{r+1} and equate the exponent of x to –5.

48. Exponent of x should be zero in T_{r+1}.

49. Use the formula to find T_{r+1} and equate the exponent of x to 3.

50. Exponent of x should be zero in T_{r+1}.

KEY

Concept Application Level—1,2,3

1. 2	2. 2	15. 3	16. 3
3. 3	4. 3	17. 3	18. 2
5. 4	6. 1	19. 2	20. 1
7. 2	8. 4	21. 4	22. 3
9. 4	10. 4	23. 1	24. 3
11. 1	12. 2	25. 4	26. 3
13. 3	14. 2	27. 2	28. 2
		29. 2	30. 4

31. 1	**32.** 3
33. 3	**34.** 1
35. 2	**36.** 4
37. 4	**38.** 2
39. 3	**40.** 3
41. 1	**42.** 3
43. 3	**44.** 1
45. 2	**46.** 3
47. 3	**48.** 2
49. 2	**50.** 4

Concept Application Level—1,2,3

Key points for select questions

1. Substitute different natural numbers for 'n' in the given expression.

2. Use the formula, $r = \dfrac{np}{p+q}$ to find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$.

3. For n = 2, find the sum of first two terms and check for which option this sum is obtained.

4. For n = 1, 2, 3 and 4, find the value of $2^{3n} - 1$ and obtain a general conclusion.

5. Evaluate the given expression for n = 1, 2, 3, 4 and 5.

6. Among the options, identify the value that satisfies the given inequality.

7. The rth term from the end in the expansion $(x + y)^n$ is $(n - r + 2)$th term from the beginning.

8. The exponent of x of the terms in the expansion of $(x + y)^n$ decreases as we go from left to right.

9. The sum of the elements in the nth row of pascal triangle is 2^n.

10. $(x + y)^n - (x - y)^n$ has $\dfrac{n}{2}$ terms when n is even and $\dfrac{n+1}{2}$ terms when n is odd.

11. Use $T_{r+1} = {}^nC_r\ x^{n-r}\, y^r$

13. $(x + y + z)^n$ has ${}^{n+2}C_2$ terms.

14. To find the coefficient of x^k in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np-k}{p+q}$

15. For each choice, check if the given expression is divisible by it for the given values of n.

16. Substitute n = 1, 2, 3, 4.......... in the given expression.

17. In the given expression substitute different values of n and then identify the factor.

18. Substitute n = 1, 2, 3, 4, ... in the given expression.

19. As we go from left to right, the exponent of y in the expansion of $(x + y)^n$ increases.

20. Use $T_{r+1} = {}^nC_r\ x^{n-r}\, y^r$

21. Use $T_{r+1} = {}^nC_r\ x^{n-r}\, y^r$.

22. Use $T_{r+1} = {}^nC_r\ x^{n-r}\, y^r$

23. To find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np}{p+q}$.

24. To find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np}{p+q}$.

25. Put x =1 and y = 1 in the given expression.

26. Use the binomial expansion of $(x + y)^n + (x - y)^n$.

27. Use the formula $r = \dfrac{np - k}{p + q}$ in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ to find the coefficient of x^k.

28. Use $T_{r+1} = {}^nC_r\ x^{n-r}\ y^r$

29. The number of terms in the expansion of $(x + y)^n$ is $n + 1$.

30. Put $y = 1$ and proceed.

31. Use $T_{r+1} = {}^nC_r\ x^{n-r}\ y^r$. Find the number of values of r for which both r and as $(n - r)$ are integers.

32. (i) $(8 + 1)^{49} + (8 - 1)^{49}$.

(ii) Use the Binomial expansion $(x + y)^n + (x - y)^n$ and simplify.

33. The product of n consecutive integers is always divisible by n!.

34. Substitute $n = 1, 2, 3, 4, \ldots\ldots$ and verify from the options.

35. To find the independent term, in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$. Use the formula $r = \dfrac{np}{p + q}$.

36. To find the coefficient of x^k in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np - k}{p + q}$.

37. Substitute $n = 1, 2, 3, 4 \ldots\ldots\ldots\ldots$ in the given expression.

38. Let the three consecutive coefficients in the expansion $(1 + x)^n$ be ${}^nC_r$, ${}^nC_{r+1}$ and ${}^nC_{r+2}$.

39. To find the independent term in the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ use the formula $r = \dfrac{np}{p + q}$.

40. To find the coefficient of x^k in the expansion $\left(ax^p + \dfrac{b}{x^q}\right)^n$, use $r = \dfrac{np - k}{p + q}$.

41. $(x + a)^n + (x - a)^n = 2\ ({}^nC_0\ x^n + {}^nC_2\ x^{n-2}\ a^2 + {}^nC_4\ x^{x-4}\ y^4 + \ldots\ldots)$

42. In the expansion of $\left(ax^p + \dfrac{b}{x^q}\right)^n$ to find

(1) the independent term, use the formula $r = \dfrac{np}{p + q}$ and

(2) the coefficient of x^k, use the formula $r = \dfrac{np - k}{p + q}$.

43. Use ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$

44. The number of terms in the expansion of $(x + y)^n + (x - y)^n$ is $\dfrac{n}{2} + 1$ when n is even and $\dfrac{n + 1}{2}$ whe n n is odd.

45. $(98)^4 = (100 - 2)^4$. Expand using binomial theorem.

49. Substitute $n = 1, 2, 3, 4, \ldots\ldots$ in the given expression and verify from the options.

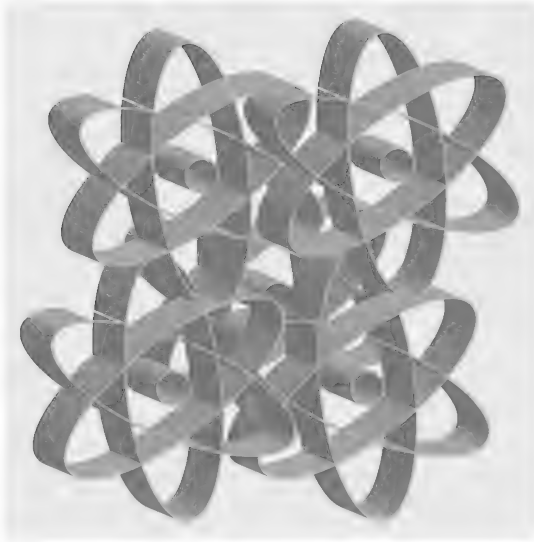
50. (i) Use ${}^nC_r = {}^nC_{n-r}$

(ii) $m = (a^2 + b^2)_{C_{a^2}}\ x^{a^2}$ and

$n = (a^2 + b^2)_{C_{b^2}}\ x^{b^2}$

(iii) Find.

CHAPTER 17



Modular Arithmetic

INTRODUCTION

Imagine that you have set out for a drive from your place. Let the watch in your hand show 5'O clock. After having driven for 6 hours, the time would be 11' O clock. We can deduce that the time is 11' O clock by adding 6 hours to 5. Now if the journey continues, after another 7 hours, the time would be 6' O clock.

It is not going to be 11' O clock + 7 = 18' O clock.

Thus 18 has to be replaced with 6.

How did we get this 6 from 18?

How is 6 related to 18?

The number 6 is the remainder obtained, when 18 is divided by 12. We say 18 is congruent to 6 modulo 12.

Similarly 21 is congruent to 9 modulo 12.

Congruence

Let a and b be two integers and m be a positive integer. We say that a is congruent to b modulo m if m is a factor of (a – b).

It is denoted by $a \equiv b \pmod{m}$ or $(a - b) \equiv 0 \pmod{m}$.

NOTE:

1. If m is not a factor of a – b, then $a \not\equiv b \pmod{m}$.
2. If m is a factor of a, then $a \equiv 0 \pmod{m}$.
   Eg: 5 is a factor of 15 So, $15 \equiv 0 \pmod{5}$.
3. If r is the remainder on dividing a by m, then $a \equiv r \pmod{m}$.
   Eg: $125 \equiv 5 \pmod{6}$ as 125 leaves a remainder of 5 on being divided by 6.

4. If $a \equiv b \pmod{m}$ and c is an integer, then $a + c \equiv b + c \pmod{m}$.
5. If $a \equiv b \pmod{m}$ and c is an integer, then $a.c \equiv b.c \pmod{m}$.
6. If $a \equiv b \pmod{m}$ and c is positive integer, then $a^c \equiv b^c \pmod{m}$.
7. If p is a prime number, then $x^p \equiv x \pmod{p}$.

Set of residues

When a positive integer is divided by 2, then the remainder will be either 0 or 1. Hence 0 and 1 are called residues of 'modulo 2'.

$\therefore$ $\{0, 1\}$ is the set of residues 'modulo 2'. It is denoted by Z_2

So, 1. $Z_4 = \{0, 1, 2, 3\}$

2. $Z_m = \{0, 1, 2, \ldots\ldots\ldots, (m - 1)\}$

Modular addition

Let a and b be two integers, and 'm' be a fixed positive integer. Then, an "addition modulo m" b is denoted by $a \oplus_m b$ is defined as the remainder when $a + b$ is divided by m.

Example

(i) $5 \oplus_6 4 = 3$

(ii) $3 \oplus_4 5 = 0$

NOTE:

1. If $a \oplus_m b = r$, then $a + b \equiv r \pmod{m}$.
2. $a \oplus_m b = b \oplus_m a$.

Modular multiplication

Let a and b be two integers, and 'm' be a fixed positive integer.

Then, a "multiplication modulo m" b is denoted by $a \otimes_m b$ is defined as the remainder when a.b is divided by m.

Examples

(i) $4 \otimes_5 4 = 1$

(ii) $8 \otimes_3 4 = 2$

NOTE:

1. If $a \otimes_m b = r$, then $ab \equiv r \pmod{m}$.
2. $a \otimes_m b = b \otimes_m a$.

Construction of caley's table

Consider $Z_2 = \{0, 1\}$

(i) All the possible results under addition modulo 2 are:

$0 \oplus_2 0 = 0, 1 \oplus_2 0 = 1,$

$1 \oplus_2 0 = 0$ and $1 \oplus_2 1 = 0$

These results can be tabulated as shown below:

$\oplus_2$	0	1
0	0	1
1	1	0

(ii) All the possible results under multiplication modulo 2 are:

$0 \otimes_2 0 = 0$, $1 \otimes_2 0 = 0$,

$0 \otimes_2 1 = 0$ and $1 \otimes_2 1 = 1$

These results can be tabulated as shown below:

$\otimes_2$	0	1
0	0	0
1	0	1

NOTE: Caley's table is the representation of modular arithmetic system.

Examples

1. Construct Caley's table for A = {1, 2, 3} under addition modulo 5.

Solution

$\oplus_5$	1	2	3
1	2	3	4
2	3	4	0
3	4	0	1

2. Construct Caley's table for the set {0,1,2,3,4} under multiplication modulo 6.

$\otimes_6$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	0	2
3	0	3	0	3	0
4	0	4	2	0	4

Linear congruence

A polynomial congruence of degree 1 is called a linear congruence.

A linear congruence is of the form $ax \equiv b \pmod{m}$. where $a \not\equiv 0 \pmod{m}$.

Examples

(i) $5x \equiv 8 \pmod 4$.

(ii) $6x \equiv 3 \pmod 5$.

Solution of linear congruence

An integer x_1 is said to be a solution of linear congruence $ax \equiv b \pmod m$ if $ax_1 \equiv b \pmod m$. i.e., $ax_1 - b$ is divisible by m.

1. Consider the linear congruence $5x \equiv 8 \pmod 4$

   Clearly, $x = 0$ is a solution as $5(0) - 8 = -8$ is divisible by 4.

2. Consider the linear congruence $6x \equiv 3 \pmod 5$

   Clearly, $x = 3$ is a solution as $6(3) - 3 = 15$ is divisible by 5.

test your concepts

Very short answer type questions

1. $15 \equiv -3 \pmod 9$ (True/False).
2. $5 \equiv 2 \pmod 4$ (True/False).
3. $4 \otimes_3 9 =$ ______.
4. 19th hour of the day is equivalent to ______ hour.
5. $6 \oplus_4 7 =$ ______.
6. In a certain month, the first Sunday falls on the fifth day of the month. Then the fourth Sunday falls on ______ day.
7. In a certain non-leap year, 1st February is Wednesday. Then the last day of the month is also Wednesday. (True/False).
8. If $63 \equiv 2 \pmod a$ and $a > 1$, then a is ______.
9. If x belongs to the set of residues modulo 4 and $2 + x \equiv 5 \pmod 4$, then $x =$ ______.
10. If $x \equiv y \pmod m$, then $6x - 5 \equiv 6y - 5 \pmod m$. (True/False).
11. In the set of integers modulo 5, $16 \oplus_5 7 =$ ______.
12. In the set of integers modulo 6, $35 \otimes_6 5 =$ ______.
13. If $a + 2 \equiv 3 \pmod 6$, then a is ______.
14. If $6x \equiv 5 \pmod 7$, then find x.
15. If $x - 4 \equiv 8 \pmod 5$, then x is ______.

Short answer type questions

16. If x belongs to the set of residues modulo 6 and $5 + x \equiv 3 \pmod 6$, then find x.

17. If x belongs to the set of residues modulo 4 and $6x - 3 \equiv -1 \pmod 4$, then find x.

18. If $46 \equiv 11 \pmod a$, and a is a prime number, then find the greatest possible value of a.

19. If 1st July 2006 was a Saturday, then what day of the week will be 18th July, 2007?

20. If you were born on 8th March, 1990 and the day of the week was a Thursday, then on what day of the week did your birthday fall in 1991?

21. Find the remainder when $(26)^{31}$ is divided by 31.

22. Find the remainder when 8^{15} is divided by 5.

23. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then list out all the pairs of distinct numbers from set A which are congruent to each other under modulo 5.

24. Construct Caley's table for Z_7 under multiplication modulo 7.

25. Construct Caley's table for the set $A = \{2, 4, 6, 8, 10\}$ under addition modulo 12.

Essay type questions

26. Construct Caley's table for the set $B = \{1, 3, 5, 7, 9\}$ under multiplication modulo 10.

27. Construct Caley's table for Z_8 under addition modulo 8.

28. Find the remainder when 3^{31} is divided by 31.

29. If $a \otimes_m b = 1$, then b is called there ciprocal of a under modulo m. Find the reciprocal of 8 under modulo 17.

30. How many two digit numbers satisfy the equation $3x \equiv 5 \pmod 7$?

CONCEPT APPLICATION

Concept Application Level—1

1. In the set of integers modulo 8, $28 \otimes_8 2 =$ _____.

(1) 0 (2) 1 (3) 2 (4) 3

2. If $25 \equiv 4 \pmod p$, where p is a prime number, then p is

(1) 3 (2) 5 (3) 7 (4) either (1) or (3)

3. Solve for x, if $5x \equiv 0 \pmod 4$.

(1) 0 (2) 3 (3) 2 (4) Both (1) and (2).

4. In the set of integers modulo 12, $38 \oplus_{12} 28 =$ ______.

(1) 6 (2) 5 (3) 4 (4) 3

5. The largest single digit number that satisfies $14x \equiv 4 \pmod 3$ is

(1) 5 (2) 7 (3) 8 (4) 9

6. If 8th August of 2009 is a Saturday, then 15th August of 2010 falls on _________.

(1) Saturday (2) Sunday (3) Wednesday (4) Thursday

7. If $23 \equiv 7 \pmod x$, then which of the following cannot be the value of x?

(1) 4 (2) 6 (3) 8 (4) 16

8. Find the remainder when 13^{15} is divided by 5.

(1) 4 (2) 3 (3) 2 (4) 1

9. Now the time is 1 : 30 p.m. If I woke up 8 hours ago, then I woke up at

(1) 4 : 30 a.m. (2) 5 : 30 a.m. (3) 3 : 30 a.m. (4) 6 : 30 a.m.

10. If $37 \equiv 18 \pmod p$, where p is a prime number, then find p.

(1) 3 (2) 7 (3) 19 (4) Either (1) or (2).

11. If $15 \equiv 3 \pmod x$, then which of the following cannot be the value of x?

(1) 3 (2) 4 (3) 6 (4) 8

12. Find the remainder when 5^{18} is divided by 19.

(1) 1 (2) 4 (3) 11 (4) 17

13. The largest two-digit number that satisfies $5x \equiv 6 \pmod 4$ is

(1) 96 (2) 97 (3) 98 (4) 99

14. If you were born on 15th April 1993 which was a Tuesday, then on which day of the week did your birthday fall in 1994?

(1) Tuesday (2) Wednesday (3) Thursday (4) Monday

15. Find the remainder when 11^{12} is divided by 7.

(1) 0 (2) 1 (3) 3 (4) 5

Concept Application Level—2

16. If $a \equiv b \pmod m$ and the remainder obtained when 'a' is divided by m is 2, then find the remainder when 'b' is divided by m.

(1) 2 (2) 1
(3) 0 (4) Cannot be determined

17. If $x \equiv y \pmod 2$, then which of the following is/are correct?

(a) x is even and y is odd
(b) Both x and y are odd.
(c) Both x and y are even.

(1) Only (c) (2) Only (a) (3) Both (b) and (c) (4) Both (a) and (b)

18. If 1st January, 2010 is a Friday, then the fifth Sunday of January, 2011 will fall on

(1) 26th day (2) 27th day (3) 29th day (4) 30th day

19. Anand started a work on Sunday at 9:30 a.m. and he finished it after 87 hours there of. Then he finished the work on

(1) wednesday 11: 30 p.m. (2) thursday 0: 30 a.m.
(3) wednesday 0: 30 a.m. (4) thursday 11: 30 p.m.

20. Which of the following are the common solutions of $3x \equiv 0 \pmod 6$ and $2x \equiv 0 \pmod 4$?

(a) 0 (b) 2 (c) 4

(1) Both (a) and (b) (2) Both (a) and (c) (3) Both (b) and (c) (4) All of (a), (b) and (c)

21. If $15x \equiv 2 \pmod 3$, then which of the following is a possible value of x?

(1) 3 (2) 315 (3) 0 (4) None of these

22. Which of the following is a common solution for $6x \equiv 0 \pmod 8$ and $8x \equiv 0 \pmod{10}$?

(1) 0 (2) 4 (3) 6 (4) Both (1) and (2)

23. Find the remainder when 2^{24} is divided by 35.

(1) 2 (2) 31 (3) 1 (4) 29

24. Which of the following is correct?

(1) $5 \oplus_3 2 \equiv 3 \otimes_3 6 \pmod 4$ (2) $4 \oplus_3 2 \equiv 3 \otimes_4 5 \pmod 6$
(3) $5 \oplus_5 3 \equiv 6 \otimes_8 9 \pmod 3$ (4) None of these

25. Which of the following is/are correct?

(1) $6 \oplus_4 3 \equiv 7 \otimes_9 8 \pmod 5$ (2) $10 \oplus_5 4 \equiv 9 \otimes_{11} 9 \pmod{11}$
(3) $14 \oplus_8 8 \equiv 15 \otimes_{16} 12 \pmod 4$ (4) Both (1) and (3)

Concept Application Level—3

26. Find the remainder when 3^{215} is divided by 43.

(1) 35 (2) 28 (3) 33 (4) 30

27. Kishore reached his school on Monday at 8 : 30 a.m. and then immediately started on a tour to GOA. After 106½ hours there on, he reached his house. Then Kishore reached his house on

(1) Saturday at 7 p.m. (2) Friday at 6 p.m.
(3) Saturday at 6 p.m. (4) Friday at 7 p.m.

28. If 1st August, 2012 is Wednesday, then find the day on which we shall celebrate our Independence Day in the year 2015.

(1) Saturday (2) Sunday (3) Friday (4) Thursday

29. Find the remainder when 5^{97} is divided by 97.

(1) 5 (2) 97 (3) 92 (4) None of these

30. If $a \equiv b \pmod m$, then which of the following is not always true?

(1) $a^2 \equiv b^2 \pmod m$ (2) $a + m \equiv b + m \pmod{2m}$
(3) $am \equiv bm \pmod{m^2}$ (4) None of these

KEY

Very short answer type questions

1. True	**2.** False
3. 0	**4.** 7th
5. 1	**6.** 26th
7. False	**8.** 61
9. 3	**10.** True
11. 3	**12.** 1
13. 1	**14.** 2
15. 2	

Short answer type questions

16. $x = 4$	**17.** 1 or 3
18. 7.	**19.** Wednesday.
20. Friday.	**21.** 31.
22. 2.	

23. [0, 5], [0, 10], [1, 6], [2, 7] [3, 8], [4, 9], [5, 10]

Essay type questions

28. 2	**29.** 3
30. 4	

key points for selected questions

Very short answer type questions

11. Recall the concept of modular addition.

12. Recall the concept of modular multiplication.

13. Check from the options.

14. Check from the options.

15. Check from the options.

Short answer type questions

16. Recall the concept of congruence modulo.

17. Recall the concept of congruence modulo.

18. If $p \equiv q \pmod{r}$, then $p - q$ is exactly divisible by r.

19. Apply congruence modulo 7.

20. Apply congruence modulo 7.

21. Apply the rule $a^p \equiv a \pmod{p}$ if p is a prime number.

22. Apply the rule if $a \equiv b \pmod{p}$ then $a^m \equiv b^m \pmod{p}$.

23. Recall the concept of congruence modulo.

24. Recall the method of multiplication under given modulo and construct the table.

25. Recall the method of addition under given modulo and construct the table.

Essay type questions

26. Recall the method of multiplication under given modulo and construct the table.

27. Recall the method of addition under given modulo and construct the table.

28. Apply congruence modulo 7.

Concept Application Level—1,2,3

1. 1	**16.** 1
2. 4	**17.** 3
3. 1	**18.** 4
4. 1	**19.** 2
5. 3	**20.** 4
6. 2	**21.** 4
7. 2	**22.** 1
8. 3	**23.** 3
9. 2	**24.** 3
10. 3	**25.** 2
11. 4	**26.** 2
12. 1	**27.** 4
13. 3	**28.** 1
14. 2	**29.** 1
15. 2	**30.** 2

Concept Application Level—1,2,3

Key points for select questions

1. Recall the concept of modular multiplication.

2. Check from the options.

3. Check from the options.

4. Recall the concept of modular addition.

5. Check from the options.

6. Use the concept, '365 under modulo 7'.

7. Check from the options.

8. Use the concept of congruence modulo.

9. Check from the options.

10. Check from the options.

11. Check from the options.

12. Start with the step $5^2 \equiv 6(\text{mod } 19)$ and proceed.

13. Check from the options.

14. Use the concept '365 under modulo 7'.

15. Use the concept of congruence modulo.

16. If $a \equiv b$ (mod m), then the remainder obtained when a is divided by m is equal to the remainder obtained when b is divided by m.

17. If $a - b$ is divisible by 2 then both a and b are either even numbers or odd numbers.

18. 1st Jan, 2010 is a Friday. 1st Jan, 2011 is a Saturday. First Sunday in 2011 is 2nd.

19. 87 hours = 3 days + 15 hours.

20. Verify whether the given options are common solutions are not.

21. 15x is always divisible by 3.

22. Verify whether each option is a solution of both the equations or not.

23. Use, if p is prime then $a^p \equiv a(\text{mod } p)$.

24. $a \oplus_m b$ means "The remainder when $a + b$ is divided by m" and $a \otimes_m b$ means. "The remainder when ab is divided by m".

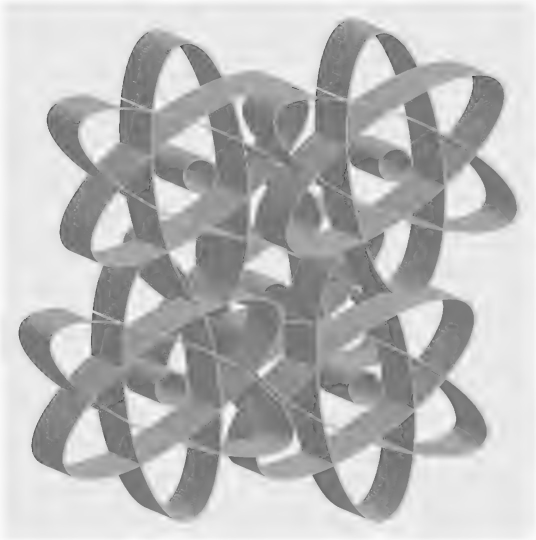
25. (i) Use the definitions of addition modulo m and multiplication modulo m.
(ii) Substitute the values in the options in the given inequations.
(iii) The point which satisfies the given inequations is the required point.

27. $106\frac{1}{2}$ hours = 4 days + $10\frac{1}{2}$ hours.

28. The number of days in between August 15th 2015 and August 1st 2012 is 1109.

29. We know that, when p is a prime number $a^p \equiv a$ (mod p).

30. (i) Use the condition $a \equiv b \ (\text{mod } m) \Rightarrow a - b$ is divisible by m.
(ii) Verify whether each option satisfies the above condition or not.
(iii) Refer the properties of congruence modulo.



CHAPTER 18

Linear Programming

INTRODUCTION

In business and industry certain problems arise, the solutions of which depend on the way in which a change in one variable may affect the other. Hence, we need to study the interdependence between variables such as cost of labour, cost of transportation, cost of material, availability of labour and profit. In order to study the interdependence we need to represent these variables algebraically. The conditions which these variables have to satisfy are represented as a set of linear inequations. We try to find the best or the optimum condition by solving these inequations. This process is called linear programming. In a situation wherein there exist only two variables, we can use the graphical method to arrive at the optimal solution. To start with, let us define some important terms related to linear equations and inequations.

Convex set

A subset X of a plane is said to be convex, if the line segment joining any two points P and Q in X, is contained in X.

Example

The triangular region ABC shown below is convex, as for any two points P and Q in the region, the line segment joining P and Q i.e., $\overline{PQ}$ is contained in the triangular region ABC.

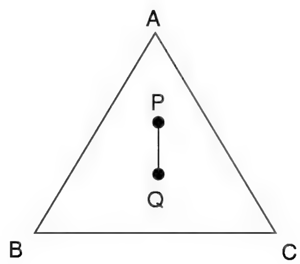


Figure 18.1

Example

The polygonal region ABCDE, shown below, is not convex, as there exist two points P and Q such that these points belong to the region ABCDE, but the line segment $\overline{PQ}$ is not wholly contained in the region.

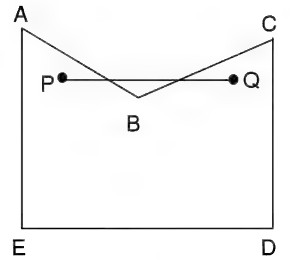


Figure 18.2

Objective function

Every linear programming problem involving two variables consists of a function of the form f = ax + by, which is to be either maximized or minimized subject to certain constraints. Such a function is called the objective function or the profit function.

Closed convex polygon

Closed convex polygon is the set of all points within, or on, a polygon with a finite number of vertices.

Example

If l_1 and l_2 are two lines, which meet the coordinate axes at A, E and D, C respectively, and B is the point of intersection of the lines, then OABC is a closed convex polygon.

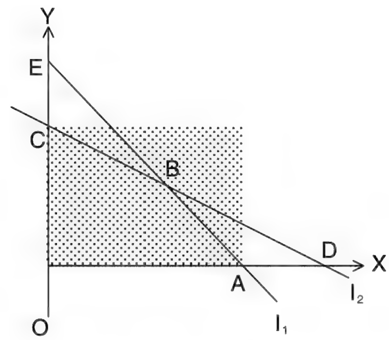


Figure 18.3

Open convex polygon

Consider an infinite region bounded by two non-intersecting rays and a number of line segments such that the end point of each segment is also the end point of another segment or one of the rays. If the angle between any two intersecting segments or between a segment and the intersecting ray, measured through the region, is less than 180°, the region is an open convex polygon.

For example, in the figure above, the region bounded by the rays $\overrightarrow{AP}$, $\overrightarrow{DQ}$ and the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ is an open convex polygon, because each of the angle ∠A, ∠B, ∠C and ∠D is less than 180°.

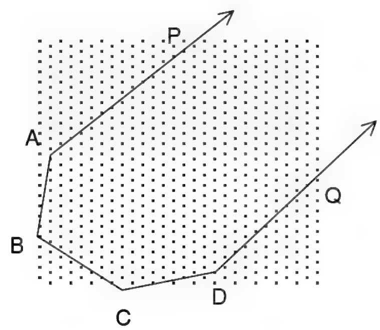


Figure 18.4

The fundamental theorem

When the values of the expression f = ax + by are considered over the set of points forming a non-empty closed convex polygon the maximum or minimum value of f occurs on at least one of the vertices of the polygon. To solve any linear programming problem, we use the fundamental theorem.

Feasible region

The common region determined by all the constraints of a linear programming problem is called the feasible region.

Feasible solution

Every point in the feasible region is called a feasible solution.

Optimum solution

A feasible solution which maximizes or minimizes the objective function is called an optimum solution.

Graphical method of solving a linear programming problem

When the solution set of the constraints is a closed convex polygon the following method can be adopted:

(i) Draw the graphs of the given system of constraints (system of inequations).

(ii) Identify the common region. If it is a closed convex polygon, find its vertices.

(iii) Find the value of the objective function at each of these vertices.

(iv) The vertex at which the objective function has the maximum or minimum value gives the required solution.

The following examples explain the method in detail.

Example

Maximize the function $f = 4x + 5y$ subject to the constraints $3x + 2y \leq 18$, $x + y \leq 7$ and $x \geq 0$, $y \geq 0$.

Solution

Given $f = 4x + 5y$

$3x + 2y \leq 18 \rightarrow (A)$

$x + y \leq 7 \rightarrow (B)$

and $x \geq 0, y \geq 0$

As $x \geq 0$ and $y \geq 0$ the feasible region lies in 1st quadrant. The graph of $3x + 2y \leq 18$ is shown with horizontal lines and the graph of $x + y \leq 7$ is shown with vertical lines.

$\therefore$ The feasible region is the part of the 1st quadrant in which there are both horizontal and vertical lines.

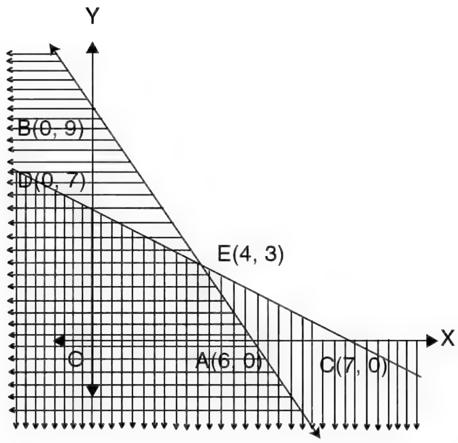


Figure 18.5

The feasible region is a closed convex polygon OAED in the above graph. The vertices of the closed polygon OAED are O(0, 0), A(6, 0), E (4, 3) and D (0, 7).

Now, $f = 4x + 5y$

At O(0, 0), $f = 0$

At A(6, 0), $f = 4(6) + 5(0) = 24$

At E(4, 3), $f = 4(4) + 5(3) = 31$

At D(0, 7), $f = 4(0) + 5(7) = 35$

$\therefore$ f is maximum at D (0, 7) and the maximum value is 35.

Example

A manufacturer makes two models A and B of a product. Each model is processed by two machines. To complete one unit of model A, machines I and II must work 1 hour and 3 hours respectively, and to complete one unit of model B, machine I and II must work 2 hours and 1 hour respectively. Machine I may not operate for more than 8 hours per day, and machine II for not more than 9 hours

per day. If profits on model A and B per unit are Rs 300 and Rs 350 respectively, then how many units of each model should be produced per day to maximize the profit?

Solution

Let the manufacturer produce x units of model A and y units of model B $\Rightarrow x \geq 0, y \geq 0$

$\therefore$ profit function $f = 300x + 350y$

To make x and y units of models A and B respectively, machine I should be used only 8 hours per day

$\therefore x + 2y \leq 8$ and machine II should be used for at the most 9 hours per day

$3x + y \leq 9$ and $x \geq 0, y \geq 0$

Hence we maximize $f = 300x + 350y$, subject to the constraints

$x + 2y \leq 8$

$3x + y \leq 9$

$x \geq 0, \ y \geq 0$

As $x \geq 0$ and $2y \geq 0$ the feasible region lies in 1st quadrant.

The graph of $x + 2y \leq 8$ is shown with horizontal lines and the graph of $3x + y \leq 9$ is shown with vertical lines.

$\therefore$ The feasible region is the part of the 1st quadrant in which there are both horizontal and vertical lines.

The shaded region is the closed polygon having vertices O(0, 0), A(3, 0), B(2,3) and C(0, 4)

Profit function $f = 300x + 350y$

At vertex O(0, 0), $f = 300(0) + 350(0) = 0$

At vertex A(3, 0), $f = 300(3) + 350(0) = 900$

At vertex B(2, 3), $f = 300(2) + 350(3) = 600 + 1050 = 1650$

At vertex C(0, 4), $f = 300(0) + 350(4) = 1400$

$\therefore$ f is maximum at the vertex B (2, 3)

Hence, the manufacturer has to produce 2 units of model A and 3 units of model B per day, in order to get the maximum profit.

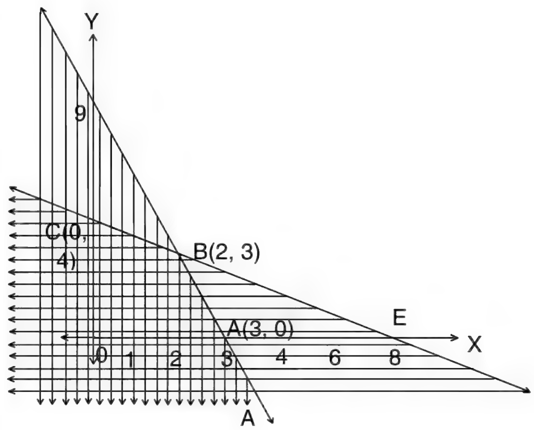


Figure 18.6

Example

A transport company has two main depots P and Q, from where buses are sent to three sub depots A, B and C in different parts of a region. The number of buses available at P and Q are 12 and 18 respectively. The requirements of A, B and C are 9, 13 and 8 buses respectively. The distance between the two main depots and the three sub-depots are given in the following table.

From \ To	A	B	C
P	15	40	50
Q	20	30	70

How should the buses be sent from P and Q to A, B and C so that the total distance covered by the buses is minimum?

Solution

Let x buses be sent from P to A, and y buses be sent from P to B.
Since P has only 12 buses so 12 – (x + y) buses are sent from P to C

Since 9 buses are required for A and x buses are sent from P, so (9 – x) buses should be sent from Q to A. Similarly, the number of buses to be sent from Q to B is 13 – y and the number of buses to be sent from Q to C is 18 – (9 – x + 13 – y) or x + y – 4

The above can be represented in the following table:

The number of buses to be sent			
to→ / From↓	A (9)	B (13)	C (8)
P (12)	x	y	12 – (x + y)
Q (18)	9 – x	13 – y	x + y – 4

All the variables are non-negative

The given conditions are

$x \geq 0$ and $x \leq 9$

$y \geq 0$ and $y \leq 13$

The region which satisfies the above set of inequalities is the rectangle OGBC.
The graph of $x + y \geq 4$ is shown with horizontal lines and the graph of $x + y \leq 12$ is shown with vertical lines.

∴ The feasible region is the area with both horizontal and vertical lines within the rectangular region OGBC i.e., the polygonal region FGHDE.

The distance covered by the buses

$$d = 15x + 40y + 50\,(12 - (x + y)) + 20\,(9 - x) + 30(13 - y) + 70(x + y - 4)$$

$$= 15x + 40y + 600 - 50x - 50y + 180 - 20x + 390 - 30y + 70x + 70y - 280$$

$$= 15x + 30y + 890$$

∴ we have to minimize the distance d.

In the graph, DEFGH is a closed convex polygon with vertices F(4, 0), G(9, 0), H(9, 3), D(0, 12) and E(0, 4)

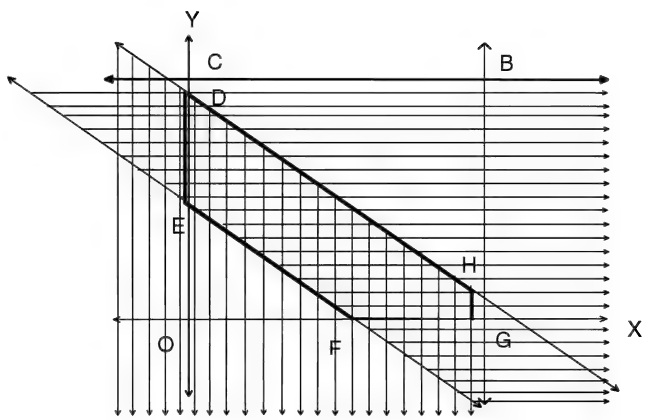


Figure 18.7

objective function $d = 15x + 30y + 890$

At vertex F (4, 0), $d = 15(4) + 30(0) + 890 = 60 + 890 = 950$

At vertex G(9, 0), $d = 15(9) + 30(0) + 890 = 1025$

At vertex H(9, 3), $d = 15(9) + 30(3) + 890 = 1215$

At vertex D(0, 12), d = 15(0) + 30(12) + 890 = 1250

At vertex E(0, 4), d = 15(0) + 30(4) + 890 = 1010

$\therefore$ d is minimum at F (4, 0)

$\therefore$ x = 4; y = 0

$\therefore$ 4 buses have to be sent from P to A and 8 buses have to be sent from P to C. Also 5 buses have to be sent from Q to A and 13 buses have to be sent from Q to B to minimize the distance travelled.

The number of buses to be sent			
to → / From ↓	A	B	C
P	4	0	8
Q	5	13	0

General graphical method for solving linear programming problems

If the polygon is not a closed convex polygon, the above method is not applicable. So we apply the following method:

(i) In this method, first we draw graphs of all systems of inequations representing the constraints

(ii) Let the objective function be ax + by. Now, for different values of the function, we draw the corresponding lines ax + by = c. These lines are called the isoprofit lines.

(iii) Take different values of c so that the line ax + by = c moves away from the origin till we reach a position where the line has at least one point in common with the feasible region. At this point the objective function has the optimum value.

This is explained with the following example.

Example

Minimize 3x + 2y subject to the constraints $x + y \geq 5$ and $x + 2y \geq 6$, $x \geq 0$, $y \geq 0$

Solution

As $x \geq 0$ and $y \geq 0$, the feasible region lies in the I quadrant.

The graph of $x + y \geq 5$ is shown with horizontal lines and the graph of $x + 2y \geq 6$ is shown with vertical lines.

$\therefore$ The feasible region is that part of the I quadrant in which there are both horizontal and vertical lines.

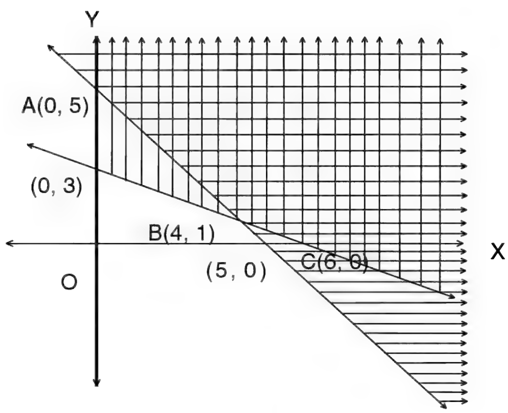


Figure 18.8

At O(0, 0), f(x, y) = 3x + 2y = 0

At B(4, 1), f(x, y) = 3(4) + 2(1) = 14

At A(0, 5), f(x, y) = 3(0) + 2(5) = 10 and b

at C(6, 0)

f(x, y) = 3(6) + 2(0) = 18.

Clearly f(x, y) is minimum at A.

test your concepts

Very short answer type questions

1. Does the point (1, 3) lie in the region specified by $x - y + 2 > 0$?

2. The region specified by the inequality $4x + 6y \leq 12$ contains the origin. (True/False)

3. Does the point (0, 0) lie in the region specified by $x + y > 6$?

4. In a rectangular coordinate system, the region specified by the inequality $y \geq 1$ lies below the x-axis. [True/False]

5. The feasible solution that maximizes an objective function is called _____.

6. If the line segment joining any two points A and B, belonging to a subset Y of a plane, is contained in the subset Y, then Y is called _____.

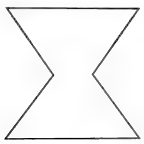
7. Does the line $y - x + 3 = 0$ pass through the point (3, 0)?

8. If $x > 0$ and $y < 0$, then the point (x, y) lies in the _____ quadrant of a rectangular co-ordinate system.

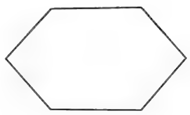
9. State whether the point (–3, 4) lies on the line, $3x + 2y + 1 = 0$ or not?

10. State which of the following figures are convex?

(i)



(ii)



(iii)

11. Define the Feasible region.

12. Define the Feasible solution.

13. The distances of a point P from the positive X axis and positive Y axis are 3 units and 5 units respectively. Find the coordinates of P.

14. State which of the following points belong to the region specified by the corresponding inequations.
(1, 2), $3x + 4y < 4$

15. State which of the following points belong to the region specified by the corresponding inequations.
(–4, 8), $5x + 6y + 30 > 0$

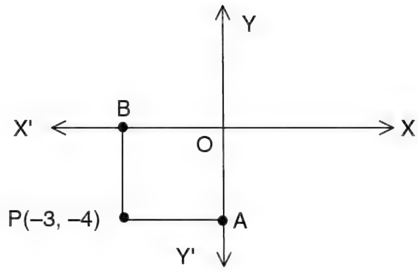
Short answer type questions

16. If (0, 0), (0, 4), (2, 4) and (3, 2) are the vertices of a polygonal region subject to certain constraints, then the maximum value of the objective function $f = 3x + 2y$ is _____.

17. If (3, 2), (2, 3), (4, 2) and (2, 4) are the vertices of a polygonal region subject to certain constraints, then the minimum value of the objective function $f = 9x + 5y$ is _____.

18. A profit of Rs 300 is made on class I ticket and Rs 800 is made on class II ticket. If x and y are respectively the number of tickets of class I and class II sold, then the profit function is _____.

19. In the following figure, find AP and BP.



20. Draw the graphs of the following inequations.

   $x - 4y + 8 \geq 0$

21. Draw the graphs of the following inequations.

   $4x - 5y - 20 \leq 0$

22. Draw the polygonal region represented by the given systems of inequations.

   $x \geq 1, y \geq 1, x \leq 4, y \leq 4$

23. Minimize $x + y$, subject to the constraints

   $2x + y \geq 6$
   $x + 2y \geq 8$
   $x \geq 0$ and $y \geq 0$

24. Define the following:

   Convex set

25. Define the following:

   Feasible solution

Essay type questions

26. A dietician wishes to mix two types of items in such a way that the mixture contains at least 9 units of vitamin A and at least 15 units of vitamin C. Item (1) contains 1 unit/kg of vitamin A and 3 units/kg of vitamin C while item (2) contains 3 units/kg of vitamin A and 5 units/kg of vitamin C. Item (1) costs Rs 6.00/kg and item (2) costs Rs 9.00/kg. Formulate the above information as a linear programming problem.

27. A manufacturer produces pens and pencils. It takes 1 hour of work on machine A and 2 hours on machine B to produce a package of pens while it takes 2 hours on machine A and 1 hour on machine B to produce a package of pencils. He earns a profit of Rs 4.00 per package on pens and Rs 3.00 per package on pencils. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at most 12 hours a day. Formulate the above information mathematically and then solve.

28. Santosh wants to invest a maximum of Rs 1,50,000 in saving certificates and national saving bonds, which are in denominations of Rs 4000 and Rs 5000 respectively. The rate of interest on saving certificate is 10% per annum and the rate of interest on national saving bond is 12% per annum. Formulate the above information as a linear programming problem.

29. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 8 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 10 minutes each for cutting and 15 minutes each for assembling. There are 4 hours available for cutting and 5 hours available for assembling. The profit is 60 paise on each item of type A and 75 paise on each item of type B. Formulate the above information as a linear programming problem.

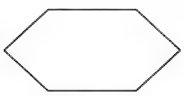
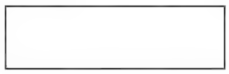
30. Find the ratio of the maximum and minimum values of the objective function $f = 3x + 5y$ subject to the constraints: $x \geq 0, y \geq 0, 2x + 3y \geq 6$ and $9x + 10y \leq 90$.

CONCEPT APPLICATION

Concept Application Level—1

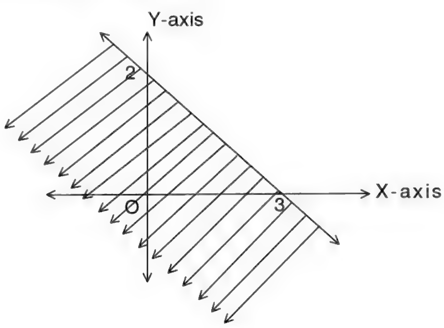
1. If an isoprofit line coincides with the edge of the polygon, then the problem has
   (1) no solution (2) one solution (3) infinite solutions (4) None of these

2. Which of the following is a convex set?
   (1) A triangle (2) A square (3) A circle (4) All the above

3. Which of the following is not a convex set?
   (1)  (2)  (3)  (4) None of these

4. Which of the following points belongs to the region indicated by the inequation $2x + 3y < -6$?
   (1) (0, 2) (2) (–3, 8) (3) (3, –2) (4) (–2, –2)

5. The inequation represented by the graph below is



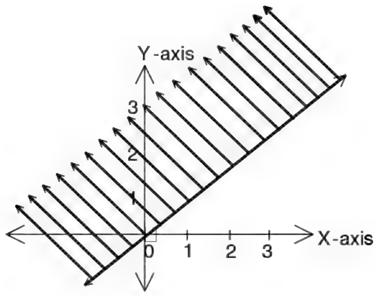
   (1) $2x + 3y + 6 \leq 0$ (2) $2x + 3y - 6 \geq 0$ (3) $2x + 3y \leq 6$ (4) $2x + 3y + 6 \geq 0$

6. The minimum value of $2x + 3y$ subjected to the conditions $x + 4y \geq 8, 4x + y \geq 12, x \geq 0$ and $y \geq 0$ is
   (1) $\frac{28}{3}$ (2) 16 (3) $\frac{25}{3}$ (4) 10

7. Find the maximum value of $x + y$ subject to the conditions $4x + 3y \le 12$, $2x + 5y \le 10$, $x \ge 0$, $y \ge 0$.

(1) 3 (2) $\frac{20}{7}$ (3) 4 (4) $\frac{23}{7}$

8. The inequation represented by the graph given below is



(1) $x \ge y$ (2) $x \le y$ (3) $x + y \ge 0$ (4) $x + y \le 0$

9. The solution of the system of inequalities $x \ge 0$, $x–5 \le 0$ and $x \ge y$ is a polygonal region with the vertices as

(1) (0, 0), (5, 0), (5, 5). (2) (0, 0), (0, 5), (5, 5).
(3) (5, 5), (0, 5), (5, 0). (4) (0, 0), (0, 5), (5, 0).

10. If the isoprofit line moves away from the origin, then the value of the objective function ______.

(1) increases (2) decreases (3) does not change (4) becomes zero

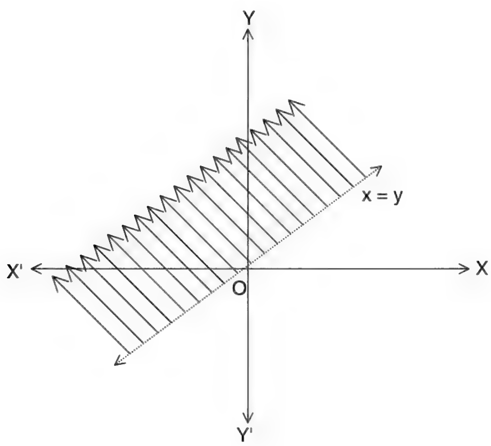
11. The solutions of the inequations $x \ge 0$, $y \ge 0$, $y = 2$ and $x = 2$ form the polygonal region with the vertices (0, 0) (0, 2) (2, 0) and (2, 2) and the polygon so formed by joining the vertices is a ______.

(1) parallelogram (2) rectangle (3) square (4) rhombus

12. Maximize $5x + 7y$, subject to the constraints $2x + 3y \le 12$, $x + y \le 5$, $x \ge 0$ and $y \ge 0$.

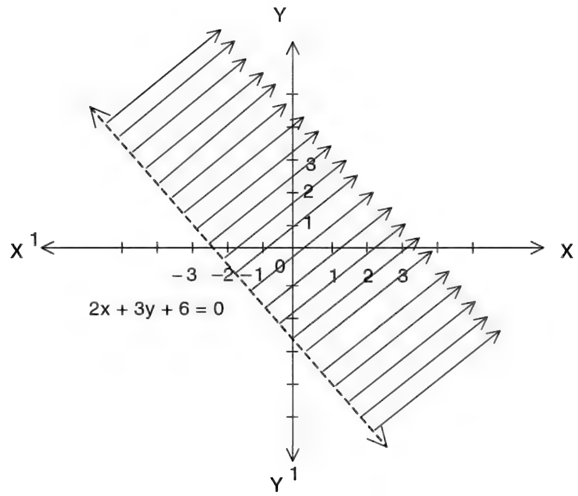
(1) 29 (2) 30 (3) 28 (4) 31

13. The inequation that best describes the graph given below is _______.



(1) $x > y$ (2) $x < y$ (3) $x \le y$ (4) $x \ge y$

14. The inequation that best describes the graph given below is ________.



(1) $2x + 3y + 6 \leq 0$ (2) $2x + 3y + 6 \geq 0$

(3) $2x + 3y + 6 > 0$ (4) $2x + 3y + 6 < 0$

15. The vertices of a closed convex polygon determined by the in equations $7x + 9y \leq 63$ and $5x + 7y \geq 35, x \geq 0, y \geq 0$ are

(1) (7, 0) (5, 0) (9, 0) (0, 9) (2) (7, 0) (9, 0) (0, 7) (0, 5)

(3) (9, 0) (6, 0) (5, 0) (0, 8) (4) (0, 9) (0, 5) (7, 0) (3, 0)

Concept Application Level—2

16. The vertices of a closed convex polygon representing the feasible region of the objective function are (6, 2), (4, 6), (5, 4) and (3, 6). Find the maximum value of the function $f = 7x + 11y$.

(1) 64 (2) 79 (3) 94 (4) 87

17. If the vertices of a closed convex polygon are A(8, 0), O(0, 0), B(20, 10), C(24, 5) and D(16, 20), then find the maximum value of the objective function $f = \frac{1}{4}x + \frac{1}{5}y$.

(1) $7\frac{1}{2}$ (2) 8 (3) 6 (4) 7

18. Find the profit function p if it yields the values 11 and 7 at (3, 7) and (1, 3) respectively.

(1) $p = -8x + 5y$ (2) $p = 8x - 5y$ (3) $p = 8x + 5y$ (4) $p = -(8x + 5y)$

19. A shopkeeper can sell upto 20 units of both the books and stationery. If he makes a profit of Rs 2 on each book and Rs 3 on each unit of stationery, then the profit function is ______, if x and y denote the number of units of books and stationery sold.

(1) $p = 2x - 3y$ (2) $p = 2x + 3y$

(3) $p = 3x - 2y$ (4) $p = 3x + 2y$

20. The vertices of a closed convex polygon representing the feasible region of the objective function f are (4, 0), (2, 4), (3, 2) and (1, 4). Find the maximum value of the objective function $f = 7x + 8y$.

(1) 39 (2) 46 (3) 49 (4) 38

21. The cost of each table and each chair cannot exceed Rs 7. If the cost of 3 tables and 4 chairs cannot exceed Rs 30, form the inequations for the above data.

(1) $x > 0, y > 0, x \le 7, y \ge 7, 3x + 4y \le 30$ (2) $x < 0, y < 0, x \le 7, y \le 7, 3x + 4y \le 30$

(3) $0 < x < 7, 0 < y \le 7, 3x + 4y \le 30$ (4) $x > 0, \le y > 0, x \ge 7, y \ge 7$ and $3x + 4y \le 30$

22. The vertices of the closed convex polygon determined by the inequations $3x + 2y \ge 6$, $4x + 3y \le 12$, $x \ge 0$ and $y \ge 0$ are

(1) (1, 0), (2, 0), (0, 2), (0, 1). (2) (2, 0), (3, 0), (0, 4) and (0, 3).

(3) (1, 0), (2, 0), (0, 2) and (2, 2). (4) (1, 0), (0, 2), (2, 2) and (1, 1).

23. Which of the following is a point in the feasible region determined by the linear inequations $2x + 3y \le 6$ and $3x - 2y \le 16$?

(1) (4, –3) (2) (–2, 4) (3) (3, –2) (4) (3, –4)

24. The maximum value of the function $f = 5x + 3y$ subjected to the constraints $x \ge 3$ and $y \ge 3$ is ______.

(1) 15 (2) 9 (3) 24 (4) does not exist

25. A telecom company manufactures mobile phones and landline phones. They require 9 hours to make a mobile phone and 1 hour to make a landline phone. The company can work not more than 1000 hours per day. The packing department can pack at most 600 telephones per day. If x and y are the sets of mobile phones and landline phones respectively then the inequalities are

(1) $x + y \ge 600, 9x + y \le 1000, x \ge 0, y \ge 0$ (2) $x + y \le 600, 9x + y \ge 1000, x \ge 0, y \ge 0$

(3) $x + y \le 600, 9x + y \le 1000, x \le 0, y \le 0$ (4) $9x + y \le 1000, x + y \le 600, x \ge 0, y \ge 0$

Concept Application Level—3

26. The cost of each table or each chair cannot exceed Rs 9. If the cost of 4 tables and 5 chairs cannot exceed Rs 120, then the inequations which best represents the above information are

(1) $x < 9, y < 9, 5x + 4x \ge 120$ (2) $x > 9, y > 9, 4x + 5y \ge 120$

(3) $0 < x \le 9, 0 < y \le 9, 4x + 5y \le 120$ (4) $0 < x \le 9, 0 < y \le 9, 5x + 4y \ge 120$

27. The vertices of a closed convex polygon determined by the inequations $5x + 4y \le 20$, $3x + 7y \le 21$, $x \ge 0$ and $y \ge 0$ are

(1) (0, 0) (7, 0) (0, 3) $\left(\frac{148}{69}, \frac{45}{23}\right)$ (2) (4, 0) (0, 3) (0, 5) $\left(\frac{148}{69}, \frac{45}{23}\right)$

(3) (0, 0), (4, 0) (0, 3) $\left(\frac{56}{23}, \frac{45}{23}\right)$ (4) (0, 0) (7, 0) (4, 0) (0, 3)

28. The profit function p which yields the values 61 and 57 at (4, 7) and (5, 6) respectively is _____

(1) $2x + 5y$ (2) $7x + 3y$ (3) $5x + 2y$ (4) $3x + 7y$

29. The vertices of a closed convex polygon representing the feasible region of the objective function f are (5, 1) (3, 5) (4, 3) and (2, 5). Find the maximum value of the function $f = 8x + 9y$.

(1) 61 (2) 69 (3) 59 (4) 49

30. The cost of each table or each chair cannot exceed Rs 13. If the cost of 5 tables and 7 chairs cannot exceed Rs 250, then the inequations which best represents the above information are

(1) $x > 13, y > 13, 5x + 7y > 250$ (2) $x > 0, y > 0, 5x + 7y < 250$

(3) $x < 13, y < 13, 5x + 7y \le 25$ (4) $0 < x \le 13, 0 < y \le 13, 5x + 7y \le 250$

KEY

Very short answer type questions

1. No
2. True
3. No
4. False
5. optimum solution
6. a convex set
7. Yes
8. fourth (Q_4)
9. The point lies on the line
10. (ii) Convex set
13. P = (5, 3)
15. –4, 8

Short answer type questions

16. 14
17. 33
18. $f = 300x + 800y$
19. AP = 3 units, BP = 4 units
23. $\frac{14}{3}$

Essay type questions

26. $x + 3y \ge 9$,
$3x + 5y \ge 15$ and
$x \ge 0, y \ge 0$.

27. The manufacturer can manufacture 4 packages of each pens and pencils daily to obtain the maximum profit of Rs 28.

28. $4x + 5y \le 150$ and $x \ge 0, y \ge 0$.

29. $8x + 10y \le 240$
$10x + 15y \le 300$
$x \ge 0, y \ge 0$

30. 5 : 1.

key points for selected questions

Very short answer type questions

9. Substitute the point in the given equation, if satisfies it belongs, otherwise no.

10. Recall the definition of convex set and proceed.

13. P(x, y) is a point in the coordinate plane which is at distance of x units from Y – axis and y units from x – axis.

Short answer type questions

19. AP and BP are the X and Y intercepts.

20. Substitute (0, 0) in the given inequation, if satisfied, then draw then region towards the origin, if not away from the origin.

21. Represent the corresponding boundary line on the graph.

22. (i) Represent the corresponding boundary line on the graph.

(ii) Substitute (0, 0) in the given inequation, if satisfied, then draw the region towards the origin, if not away from the origin.

(iii) Check the common region formed by the regions and name the polygon formed by it.

23. Substitute the vertices in the objective function and find the maximum value.

Essay type questions

26. According to the data frame the inequations.

27. According to the given data frame the inequations and draw the graphs for inequations. Identify the vertices of the convex polygon.

28. According to the data frame the inequations.

29. According to the given data frame the inequations.

30. Find the maximum and minimum value of the objective function.

KEY

Concept Application Level—1,2,3

1. 3	**2.** 4
3. 2	**4.** 4
5. 3	**6.** 1
7. 4	**8.** 2
9. 1	**10.** 1
11. 3	**12.** 1
13. 2	**14.** 3
15. 2	**16.** 3
17. 2	**18.** 1
19. 2	**20.** 2
21. 3	**22.** 2
23. 3	**24.** 4
25. 4	**26.** 3
27. 3	**28.** 4
29. 2	**30.** 4

Concept Application Level—1,2,3

Key points for select questions

2. Recall the definition of convex set.

3. Recall the definition of convex set.

4. Check the point which satisfies the given inequation.

5. The corresponding equation of the line is $\frac{x}{3}+\frac{y}{1} = 1$

6. Find the open convex polygon formed by the given inequations are proceed.

7. Find the closed convex polygon formed by the given inequations and proceed.

8. The corresponding equation of the line is $x = y$.

9. Check the points which belongs the given inequations.

10. Increases (Standard result).

11. Check which quadrilateral is formed by the points (0, 0), (0, 2), (2, 0) and (2, 2).

12. Represent the regions of given inequations and identify the vertices of the convex polygons.

17. (i) Substitute the given points in f.
 (ii) The maximum/minimum value of f occurs at one of the vertices of the closed convex polygon.
 (iii) Substitute the given points in f.
 (iv) Identify the maximum value of f.

18. (i) Profit function $p = ax + by$.
 (ii) Take the profit function as $p = ax + by$.
 (iii) Substitute the given points and obtain the equations in a and b.
 (iv) Solve the above equations to get a and b.

19. (i) $p = ax + by$ – find a and b
 (ii) Profit on books is Rs 2x and profit on stationary is Rs 3y.
 (iii) Profit function = Profit on books + Profit on stationary.

20. (i) Substitute the given points.
 (ii) The maximum/minimum value of f occurs at one of the vertices of the closed convex polygon.
 (iii) Substitute the given points in f.
 (iv) Identify the maximum value of f.

21. (i) Let the cost of each table and chair be x and y.
 (ii) Consider the cost of each table and each chair as Rs x and Rs y respectively.
 (iii) Frame the inequations according to the given conditions.

22. (i) Check the points which belong to the given inequation.
 (ii) Represent the given inequations on the graph.
 (iii) Detect the closed convex polygon and its vertices.

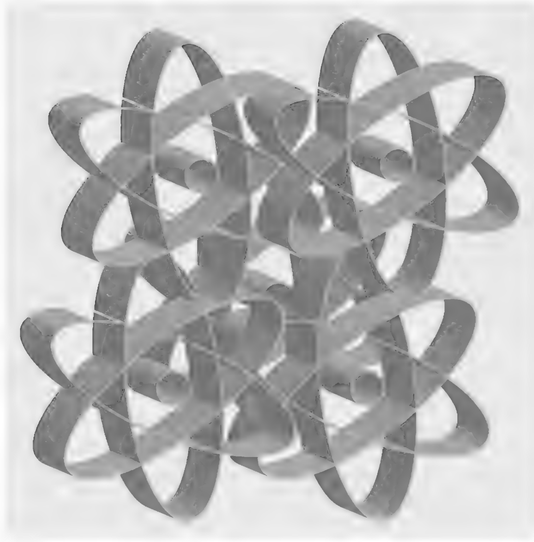
23. (i) Check the point which satisfies the given inequation.
 (ii) Substitute the values in the options in the given inequations.
 (iii) The point which satisfies the given inequations is the required point.

24. (i) Find the convex polygon.
 (ii) Represent the given inequations on the graph.
 (iii) Find the vertices of the closed convex polygon.
 (iv) Substitute the vertices of the polygon in f and check for the maximum value of f.

25. (i) Time taken to manufacture x mobiles and y landlines is 9x and y hours respectively.
 (ii) $x \geq 0$ and $y \geq 0$. As the number of mobiles cannot be negative.
 (iii) Use the above information and frame the inequations.

CHAPTER 19

Computing



INTRODUCTION

We are living in the age of information technology. Computers play a key role in our daily life. Computers are extensively used in various fields such as banking, insurance, transportation, science and technology, entertainment etc. Complex tasks can be solved easily with the help of computers.

The computer is a multipurpose electronic device which is used to store information, to process a large amount of information and to accomplish a task with high speed and accuracy.

The computer can be defined as an electronic device which accepts input data, processes it according to the set of instructions called programs, and gives the output information.

The idea of a computer was first developed by **Charles Babbage** in the 18th century. Since the time the computer was invented, the architecture of the computer has undergone many changes. Initially vacuum tubes were used in the computers. At that time, computers were very large and some computers were as large as that of a room. There after the vacuum tubes were replaced with transistors and the second generation of computers were made. Later on, small-scale integrated circuits were used in the third generation of computers. With rapid advancements made in science and technology, Very Large Scale Integrated (VLSI) circuits were fabricated. Present computers use VLSI circuits to achieve high speed, small size, better accuracy and vast memory. Today many mini-computers such as laptops, notebooks, and PDA (Personal Digital Assistant) are available in the market.

Historical development of computing

The method of computing started from early 4000-1200 B.C., where people used to keep a record of their transaction on clay tablets. Around 3000 B.C., Babylonians invented the abacus from where computing started. No significant development took place until the 17th century. **Blaire Pascal** invented a machine which was named as **Pascalens**. It was the first mechanical adding machine in the history of computing. Then punch cards were used in the early 18th century. In 1822, Charles Babbage developed the first mechanical computer and for this reason he is known as the father of the computer. In 1854, George Boole, developed

the Boolean logic. It is the basis of computer design. Before 20th century all machines were mechanical. The first electronic computer built in 1966, was named ENIAC (Electronic Numerical Integrator and Computer).

Integrated circuits were used in 1962 instead of individual transistors in the computers. The most famous machine at that time was the IBM 360 and DEC DPBQ. Later on, microprocessor was invented and this enabled the reduction of size and increase in the performance of computers. In 1981, IBM launched PC and from then onwards there has been a significant development in the field of micro electronics and micro processors.

Today, we use computers with 2 to 3 GHz processors, with ½ to 1 GB of memory and 80 to 120 GB storage (while 500 KB was considered significant even till a few years ago.)

In the evolution of computers, major changes have occurred in both the structure and the functioning of computers, which have had a tremendous impact on the way these machines appear and the extent to which people use these machines. We normally refer to these changes as introducing a new generation of computers. We can readily identify 5 such generations:

First generation (1945–1956): The computers of this generation were mechanical or electro mechanical in which vacuum tubes were used. These computers were huge, inflexible and slow when compared to the later generations of computers.

Second generation (1957–1963): In 1948, the transistor was invented and this had a major impact on the development of computers. Transistors replaced the large vacuum tubes and this led to a major reduction in the size of the computer. In the early 1960's several commercially successful computers were used in business establishments and universities. Several high level languages such as COBOL, FORTRAN etc were introduced.

Third generation (1964–1971): Integrated circuits (IC)s were manufactured in which hundreds of transistors were assembled in a tiny chip. This led to the third generation computers.

Fourth generation (1971 to the present): In this generation, LSI (large scale integrated), VLSI (Very large scale integrated) and ULSI (ultra large scale integrated) chips were introduced, which further increased the efficiency and reliability of the computers significantly.

In 1981, IBM introduced personal computers for use at home or office. These PC's could be linked together or networked to share both software and the hardware.

Fifth generation of Computers: (Present and beyond): With developments in artificial intelligence(AI), computers are enabled to hold conversation with human operators. They can use visual inputs and learn from their own experience. A robot is a good example of the application of artificial intelligence.

Characteristics of a computers

Some of the characteristics of computers are:

1. Speed
2. Reliability
3. Storage Capacity
4. Productivity

Architecture of a computer

A computer consists of three essential components. They are:

1. Input device (e.g., key board)
2. Central Processing Unit (C.P.U.)
3. Output device (e.g., Monitor)

Central Processing Unit is a very important component in a computer. It consists of:

(a) Memory unit

(b) Control unit

(c) Arithmetic and logic unit (A.L.U)

Block diagram of a computer is shown below:

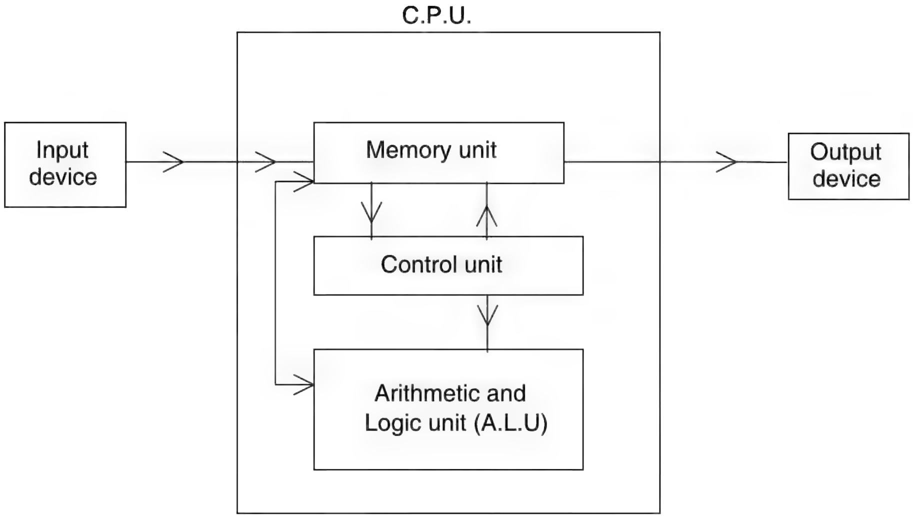


Figure 19.1

The instructions or processed data will be received by computer through input devices, and this information will be stored in the memory unit. If any arithmetical operations are to be performed, then with the help of the control unit, the arithmetic and logic unit (A.L.U) performs the operations and stores the result in the memory. Finally the results can be seen through output devices.

The three physical components-input devices, C.P.U, and output devices are together referred as **Hardware** of a computer.

Software

In order to accomplish a particular task by using a computer, we need to write a set of instructions in a language that can be understood by the computer. Any of our usual languages can not be read or understood by a computer. So, we need to feed information into a computer in a language called programming language. BASIC, PASCAL, C, C^{++}, Java are some of the popular programming languages.

A set of instructions that are written in a language that can be understood by a computer is called a program. A set of programs is called a **software**.

A program is to be written to accomplish a particular task. This program is fed into the memory of a computer by using an input device (like key board). The control unit reads the instructions (given in the program) from the memory and processes the data according to these instructions. The result can be displayed with a device such as a monitor or a printer. These devices are called output devices.

Note that the arithmetic and logic unit (A.L.U.) performs all the arithmetic and logical operations such as addition, subtraction, multiplication, division, comparison etc, under the supervision of the Control Unit (C.U.). The C.U. decodes these instructions to execute and the output unit receives the results from the memory unit and converts these results into a suitable form which the user can understand.

Algorithm

A comprehensive and detailed step-by-step plan or a design that is followed to solve a problem is called an algorithm. Thus, an algorithm is a set of systematic and sequential steps in arriving at a solution to a problem.

For example, if you want to buy some articles from a grocery store, then the following steps are to be followed:

1. Make a list of articles which you intend to purchase.
2. Go to the grocery store.
3. Give the storekeeper the list of articles.
4. List the prices of the articles on paper and add them to get the total amount of money you need to pay.
5. Pay the money and take the articles.
6. Verify whether you have received all the articles.
7. Return home with the grocery items.

Steps (1) to (7) form the algorithm for the task of buying grocery items. Even though it is a simple task, we follow several steps in a systematic way to achieve the task. Similarly to solve a task using a computer, we first need to make a blue print i.e., algorithm of the steps that are to be followed. Once an algorithm is ready, we can represent it on a flow chart.

Flow chart

Flow chart is the pictorial representation of an algorithm. Flow chart clearly depicts the points of input, decision-making, loops and output. Thus with the help of a flow chart we can plan more clearly and logically to solve a given task.

To draw a flow chart we use certain symbols or boxes to represent the information appropriately. Following are the notations used in a flow chart.

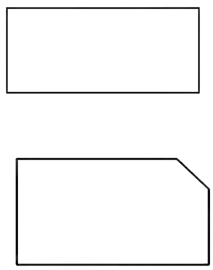


Figure 19.2

1. **The Operation box**

   This box is used to represent the operations such as addition, subtraction etc.

2. **The data box**

   This box is used to represent the data that is needed to solve a problem, also to represent information regarding the output of solution.

   Therefore this box is used for input and output.

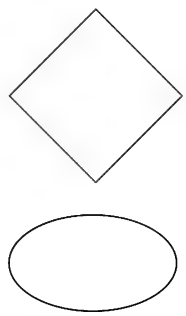


Figure 19.3

3. **Decision box**
   A diamond (or rhombus) shaped box is used whenever a decision is to be taken. The points of decision can be represented by using this box. Usually the answer to the decision is **yes** or **no**.

4. **Terminal box**
   This box indicates the start or termination of the program.

5. **Flow lines** "———→" The arrows which are used in flow chart are known as flow lines. These arrows are very important in flow chart.

6. **Connectors**

The circle is a connector in a flow chart. The connector is used in the flow chart only if the flow chart need to be continued on the another page. Connectors are always used in pairs. The flow chart will have an outwards connector on a page which can be continued with an inwards connector in the next page.

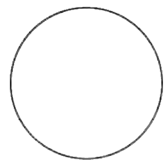


Figure 19.4

Both in and out connectors should contain the same alphabet.

Once a flow chart is ready, we can translate it into a programming language and feed it into the computer memory.

In order to accomplish a task on a computer, the following steps are to be followed.

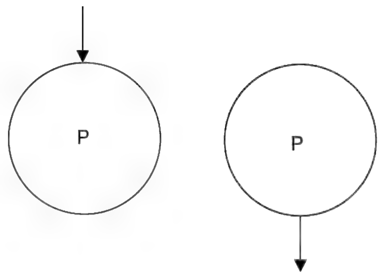


Figure 19.5

1. Identify and analyse the problem.
2. Design a systematic solution to the problem and write an algorithm.
3. Represent the algorithm in a flow chart.
4. Translate flow chart into a program.
5. Execute the program and get the output.

Examples

1. You are given the principle and the rate of simple interest per month. Write an algorithm to calculate the cummulative simple interest at the end of each year upto 10 years, and also draw a flow chart.

Algorithm

Step 1: Read the values of principle (P), rate of interest (R).

Step 2: Take T = 1

Step 3: $S.I = \dfrac{12 \times P \times T \times R}{100}$

Step 4: Print the S.I.

Step 5: Calculate T = T + 1

Step 6: If $T \leq 10$, then repeat the steps 3, 4, and 5.

Step 7: Otherwise stop the program.

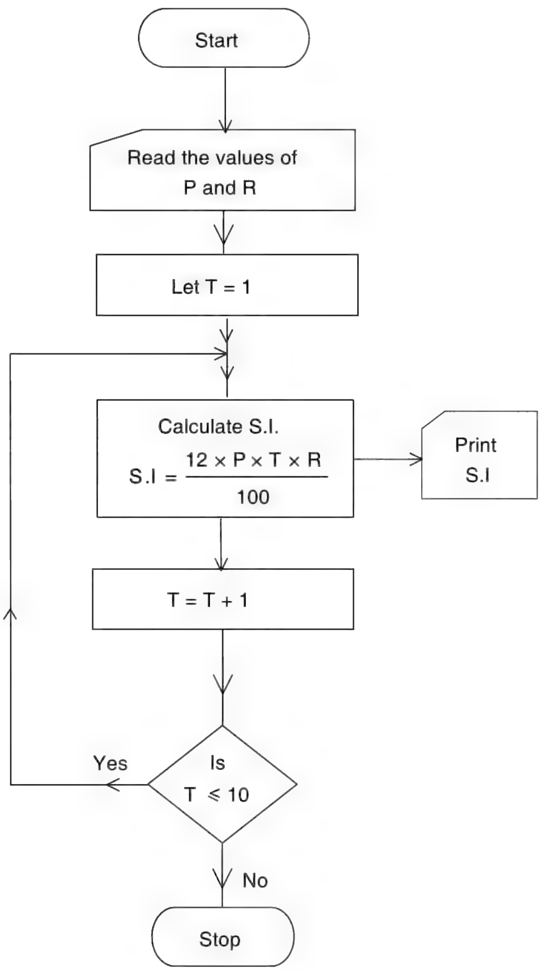


Figure 19.6

2. Write an algorithm and draw a flow chart to find the sum of first 50 natural numbers.

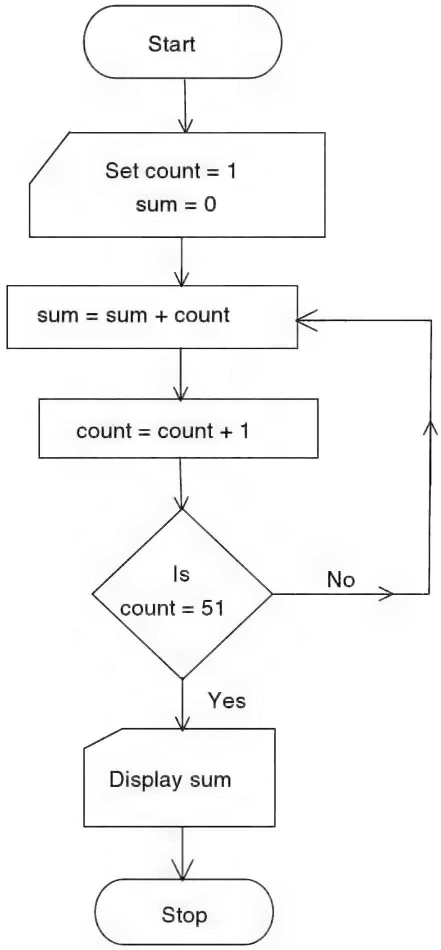


Figure 19.7

Algorithm

Step 1: Set count = 1, Sum = 0.

Step 2: Add count to sum.

Step 3: Increase count by one, i.e., count = count + 1

Step 5: Check whether count is 51.

Step 6: In step (5) if count is 51
Display sum and stop the program else
Go to step (2)

From this flow chart, we can observe that there is a loop among the boxes 3, 4 and 5.

Operators

Operators are used to perform various types of operations.

Example Addition can be done by '+' operator.

There are different types of operators:

1. Shift operators
2. Logical operators
3. Relational operators
4. Arithmetic Operators

Computer performance is measured in three ways:

1. Storage Capacity
2. Processing Speed
3. Data transfer Speed

Storage Capacity is measured in Bits, Bytes, Kilobytes, Megabytes or Giga Bytes.

1 nibble = 4 bits

1 Byte = 8 bits

1 Kilo Byte (KB) = 1024 Bytes = 2^{10} Bytes

1 Mega Byte (MB) = 1024 KB = 2^{20} Bytes

1 Giga Byte (GB) = 1024 MB. = 2^{30} Bytes

1 Tera Byte (TB) = 1024 GB. = 2^{40} Bytes

Processing Speed is measured in Hertze i.e., cycles/second. It explains about the processor speed.

Example 800 MHz, 1.5 GHz.

Data transfer speed is measured in Bytes per second.

Example 256 KB/Sec, 128 KB/Sec.

Examples

1. Write an algorithm to calculate the sum of the squares of the first five natural numbers and also draw the flow chart.

Algorithm

Step 1: Take N = 0, Sum = 0

Step 2: Calculate N = N + 1

Step 3: Calculate Temp = N * N

Step 4: Calculate Sum = Sum + temp.

Step 5: If N < 5 repeat step 2, 3 and 4, otherwise print the sum and stop the program.

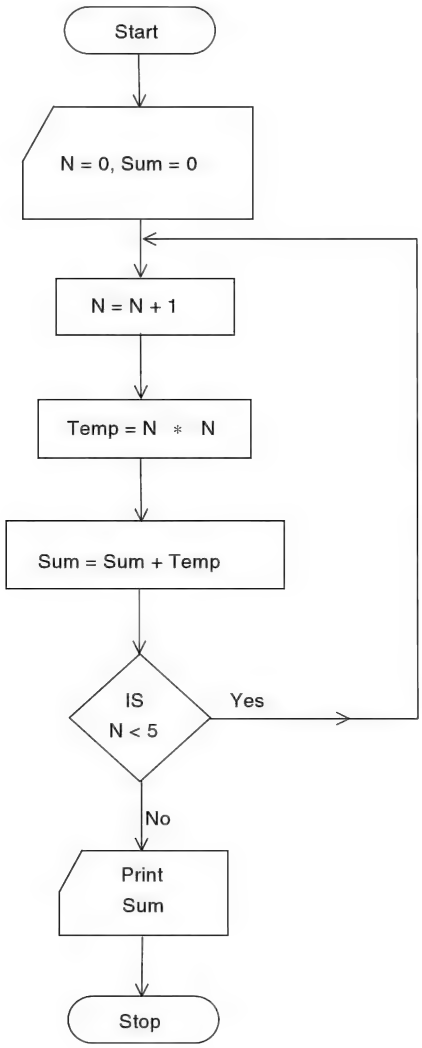


Figure 19.8

2. Write an algorithm to generate the fibonocci series up to n terms and also draw a flow chart.

Algorithm

Step 1: Let $F_1 = 0, F_2 = 1$ and $K = 2$

Step 2: Read N for number of terms

Step 3: Print F_1 and F_2

Step 4: Calculate $F_3 = F_1 + F_2$

Step 5: Print F_3

Step 6: Calculate $K = K + 1$

Step 7: $F_1 = F_2$ and $F_2 = F_3$

Step 8: If $K < N$, then go to step 4 else stop the program.

Flow Chart

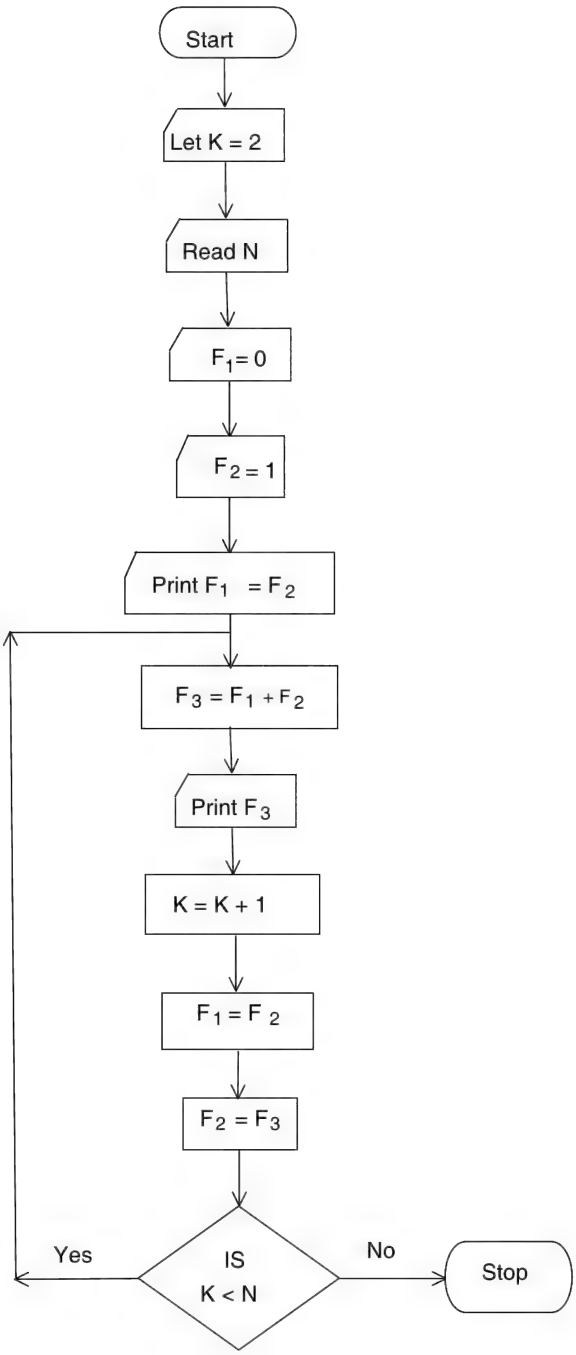


Figure 19.9

Basic

Basic (Beginners All purpose symbolic Instruction code) is a high level language and user friendly. The instructions can be given in simple English language along with some keywords and specific syntax. This language is useful in the field of Business, Engineering, Mathematics and other applications.

Constants

Constants are those which do not change their values in the program. They can be classified as

1. Numeric Constants
2. Alphanumeric Constants

1. **Numeric**
   All whole numbers ranging from – 32767 to +32767 are numeric constants.

   NOTE: Commas are not allowed in a constant. E.g., 23, 456 is not valid.

2. **Alphanumeric**
   Set of alphabets or numerics or alphanumerics which are enclosed within double quotation marks are treated as alphanumeric or string constants.
   E.g., "A", "576", "AP007"

Variables

It is a name which represents a number, a character or a string. Variables are of two types:

1. **Numeric variable**
   It must start with a letter.
   E.g., A, BASIS.

2. **Alphanumeric variables or string variables**
   It must start with a letter. It can be followed by other letters or numbers but must end with a dollar sign ($).
   E.g., A$, NAME 2$.

Rules for declaration of variables

1. Variable must start with an alphabet.
2. Keywords should not be used as a variable name.
3. In a variable name declaration, commas and blank spaces are not allowed.
4. The variable length should not exceed 40 characters.
5. The special characters like %, #, and $ are not allowed in variables.

E.g.,	(i)	9ALSAB	Invalid, it should start with an alphabet.
	(ii)	201	Valid
	(iii)	SAP GIM	No space is allowed in variable name, hence it is invalid.
	(iv)	K21A $	Invalid, $ can not be used here.
	(v)	R, PT20	Invalid, commas are not allowed.

Basic operators

Operators are used to relate variables and constants to form expressions.
Operators are of three types:

1. Arithmetic operators
2. Logical operators
3. Relational operators

Arithmetic operators

These are used for mathematical operations.
For addition '+'
Subtraction '–'
Multiplication '*'
Division '/'
Exponential '^'

Logical operators

There are three logical operators:

1. AND -: AND expression will be executed when both of the conditions are true.
2. OR -: OR expression will be executed when any one of the conditions is true.
3. NOT -: NOT expression will be executed when a condition is to be negated.

Relational operators

These operators are used to form a relational expression. These are:-

‘=’ equality, ‘<’ less than, ‘>’ greater than, ‘< =’ less than or equal to, ‘>=’ greater than or equal to, ‘< >’ Not equal to.

Example x < = y, A > B

Order in which arithmetic operators are evaluated

1. Parenthesis ( )
2. Exponentiation ^, ↑
3. Multiplication or Division *, /
4. Addition or subtraction (+, –)

   Operations of equal priority are performed from left to right.

Basic statements

BASIC statements are primarily of two types–executable and non executable. Executable statements are those which are executed by the computer, while non-executable statements are those which are ignored by the computer and used for the user to understand the nature of the program. The following statements are generally used in BASIC Programming.

1. REM
2. LET
3. INPUT
4. READ … DATA
5. END
6. GOTO
7. PRINT
8. BRANCHING
9. STOP

REM

To declare non executable statements, REM statement is used.

Syntax: Ln REM comment

Example 10 REM **BASIC LANGUAGE**

LET

To assign numeric or string values to a variable LET statement is used.

Syntax: Ln LET variable = constant/expression

Example 20 Let X = 10

30 Let SI = P * T * R/100

INPUT

To enter the data into the computer during the process of execution. The entered value will be stored in memory variable.

Syntax: Ln INPUT variables

☛ *Example* 10 INPUT X

NOTE: A single INPUT statement can have many variables either same data type or different data type.

The user has to enter the values in the same order in which the variable appears.

Input statement also allows the user to enter relevant data at the time of its execution.

10 INPUT "enter marks of student";A.

READ ... DATA

It is used to assign the values to the variables. In READ statement variables are declared and in DATA statement, the respective values are provided for the declared variables.

Syntax: Ln READ list of variables

Ln DATA list of values

NOTE:

1. The number values given through DATA should be more or equal to the number of variables of same data type or different data type declared in READ statement. Otherwise an error message will be displayed in the program and it will be terminated.
2. The constants in DATA statement must match the variable type.
3. The string constant in the DATA statement need not be enclosed within quotes.
4. There can be many READ and DATA statements in a program.

☛ *Example* 10 READ P, Q, R

20 DATA 10, 20, 30.

PRINT

It is used to display the output of the program.

Syntax: Ln PRINT variables

☛ *Example*

1. 100 PRINT x, 4$
2. 100 PRINT A$, B$ when A$ = NEW and B$ = YORK

   Result NEWYORK
3. 100 PRINT A; B; C when A = 15, B = –4 and C = 25

   Result: b15b – 4b – 25b

   Where b denotes blank space.

   Print statement can also be written with message enclosed within double quotes.

   150 PRINT "THIS IS AN ANIMAL"; B$.

END

It is used to terminate the execution of the program.

Syntax: Ln END

☛ *Example* 100 END

STOP

It terminates the execution of the program temporarily, it can be re-executed by typing CONT or by pressing the F5 key.

Syntax: Ln STOP

☛ *Example* 100 STOP

Conditional statements

The statements which are dependent on certain conditions are known as conditional statements. Only if the test of expression is true, the statements which are dependent on condition be executed. Otherwise, they will be skipped.

Conditional statements are of two types:

1. Branching statements
2. Looping statements

Branching Statements

1. IF – THEN statement
2. IF – THEN – ELSE statement

IF–THEN statement

This is a conditional branching statement. A condition will be specified here and if it is true, the action is carried out.

Syntax: Ln IF conditional THEN actions.

☛ *Example*

```
 5 REM OPERATION ON TWO NUMBERS.
10 Let A = 8
20 Let B = 20
30 IF B > A THEN C = B – A
40 IF A = B THEN C = B + A
50 IF A > B THEN C = A – B
60 Print C
70 END.
```

Solution

Here, $A = 8, B = 20$

i.e., $A < B$.

$C = B - A = 20 - 8 = 12$.

$\therefore$ The output of program is 12.

IF – then - else

It is a conditional statement. If a condition is satisfied then a particular action is executed otherwise another action is executed.

Syntax: Ln IF conditional

THEN action1

ELSE action 2

☛ *Example*

```
 5 REM Arranging the two numbers in ascending order.
10 Let A = 10
20 Let B = 20
30 If A > B THEN
40 PRINT B;A;
50 ELSE
60 PRINT A; B;
```

Solution

Here,

A = 10, B = 20

i.e., A < B

Control is transferred to else block.

∴ The output of the program 10, 20.

Looping statements

Here a condition is specified with a set of statements. The statements will be executed until the condition gets violated. Generally an incrementing or a decrementing statement will keep track of the loop.
Syntax: In variable [u = e_1] TO [e_2] STEP [e_3]

☛ *Example*

```
5 REM Summing of squares of odd numbers
10   LET S = O
20   FOR N = 1 to 10 STEP 2
30   LET S = S + N * N
40   NEXT N
50   PRINT S
60   END
```

Solution

Here, in the first time, the value of S is 1 × 1 = 1. STEP 2 implies that the value of N increases by 2 and becomes 3. This process keeps on going and finally prints the value of S as 165.

Unconditional statements

GO TO

It transforms the control to another part of the program which will be executed.

It is unconditional branching statement.

Syntax: Ln GOTO line number

Example 50 GOTO 200

test your concepts

Very short answer type questions

1. Computer works according to instructions. This set of instructions is called a _______.
2. What are the basic units of C. P. U.?
3. The language understood by the computer is called a _______.
4. In first generation of computers, _______ are used.
5. Software acts as a mediator between the user and Computer Hardware. [True/False]
6. Numeric variables must begin with _______ in BASIC language.
7. In fourth generation of computers, _______ are used.
8. RAM is a secondary memory. [True/False]
9. To display the output of the program, _______ keyword is used.
10. 1 MB = _______ KB.
11. The box indicating the decision in a flow chart is called a ________.
12. Every statement in BASIC language input starts with a _______.
13. In BASIC language, REM keyword is used to write _______.
14. _______ is pictorial representation of algorithm.
15. To assign any value to the variable, _______ key word is used.
16. BASIC is a _______ language.
17. To enter numerical or string data during the time of execution, _______ keyword is used.
18. Go to is used to skip the _______.
19. The maximum length of numeric constant in BASIC language is _______.
20. The usage of numbers that are allowed in BASIC language is _______ to _______.

Short answer type questions

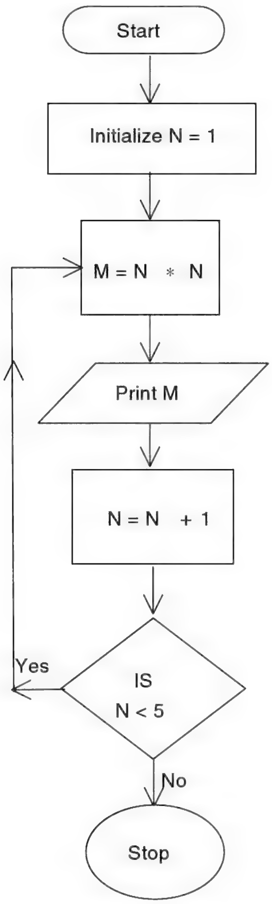
21. Write an algorithm to find the sum of squares of the first ten natural numbers.

22. Write an algorithm to find $S = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5$.

23. Write a program in BASIC to find the centroid of a triangle.

24. Write a program in BASIC to find the sum of first N natural numbers without using the formula $\frac{N(N+1)}{2}$.

25. What will be the output of the following flow-chart?



Essay type questions

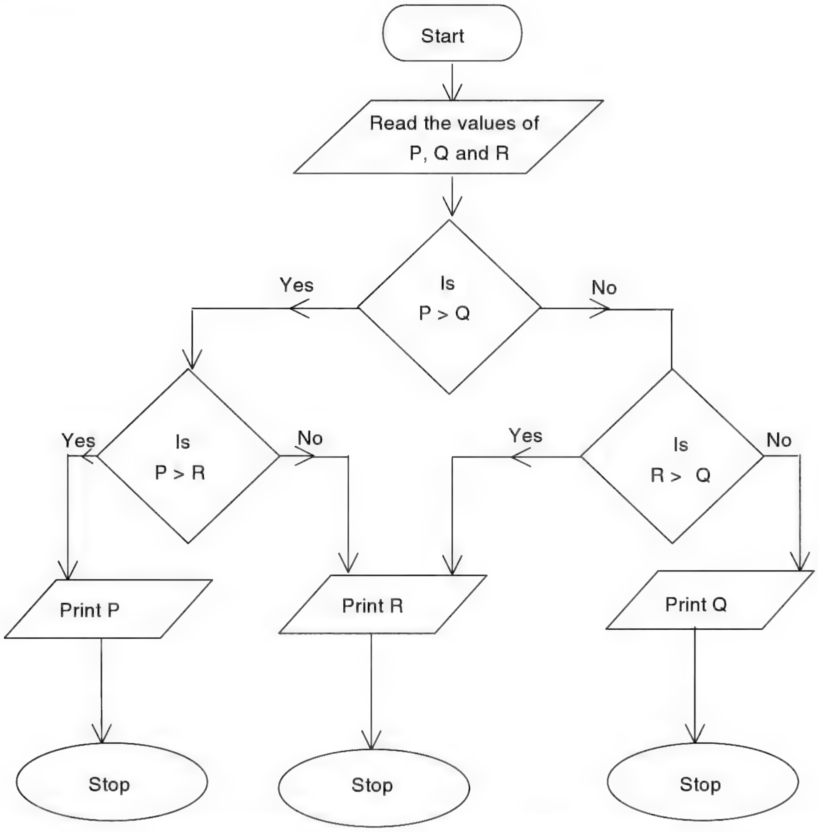
26. Write an algorithm to calculate the value of the expression $S = a * c - d * e + f \div m$, if the values of a, b, c, d, e, f and m are given.

27. Write an algorithm to calculate the value of the expression $R = \sqrt{x^2 + y^2 + z^2}$.

28. Write a program in BASIC to find the area of a triangle when the lengths of three sides are given.

29. Write an algorithm to find the largest number among the given ten natural numbers.

30. If 12, 120 and 105 are the values of P, Q and R respectively, then what is the output of the following flow chart?



CONCEPT APPLICATION

Concept Application Level—1

1. BASIC is

   (1) Business Arithmetic System Instruction Code.
   (2) Beginners All Purpose Symbolic Instruction Code.
   (3) Basic All Purpose System Instruction code.
   (4) Beginners All Purpose System Instruction Code.

2. Which of the following is correct?

(1) LET A = 20, 420 (2) LET B = 21, 445
(3) LET A = 40 (4) None of these

3. Low-level languages or machine languages use strings of

(1) Zero's and two's (2) One's and two's
(3) Zero's and One's (4) Both (2) and (3)

4. PRINT keyword is useful to assign the values during

(1) compilation (2) program
(3) the process of execution (4) None of these

5. Evaluate the expression, as done by a computer: $13 - 7 \times 4 \div 2 + 3 - 2 \times 5 - 8$

(1) 0 (2) 57 (3) –16 (4) –3

6. Which of the following statements is true?

(1) Every variable in the INPUT statement must have a corresponding constant or value in the READ statement.
(2) Every variable in the READ statement must have a corresponding constant or value in the DATA statement.
(3) There can be one READ statement and many Data statements.
(4) Every variable in the DATA statement must have many corresponding constants.

7. What is the output of the following program?

10 Read P, R and N
20 Data 1000, 10, 2;
30 Let $A = P * (1 + \frac{N * R}{100})$
40 Print A
50 End

(1) 1050 (2) 1120 (3) 1230 (4) 1200

8. Which of the following is not an alphanumeric?

(1) "S" (2) "SIX" (3) "S92" (4) "123"

9. Which of the following is correct?

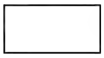
(1) 10 READ x$, y
20 DATA 20, "TIME"
(2) 10 READ x$, y
20 DATA "TIME", 20
(3) 10 READ x$, y$.
20 DATA 20, 30
(4) None of these

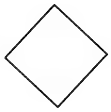
10. Find the result of the following program.

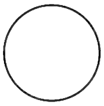
```
10 LET A = 20, B = 30
20 LET C = (A + B)/2
30 PRINT C
40 END
```

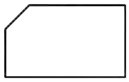
(1) 50
(2) 25
(3) 10
(4) None of these

11. In a flow chart representation, to connect a flow diagram from one page to another page of a program, which of the following diagram is used?

(1) 

(2) 

(3) 

(4) 

12. What will be the output of the following program?

```
10 REM AREA OF TRIANGLE IF
THREE SIDES ARE GIVEN.
20 READ a, b, c, s
30 DATA 12,18, 10
40 LET D = [s * (s – a) *(s – b)*(s – c)] ^ 1/2
50 PRINT "THE VALUE OF D = "; D
60 PRINT D
70 END
```

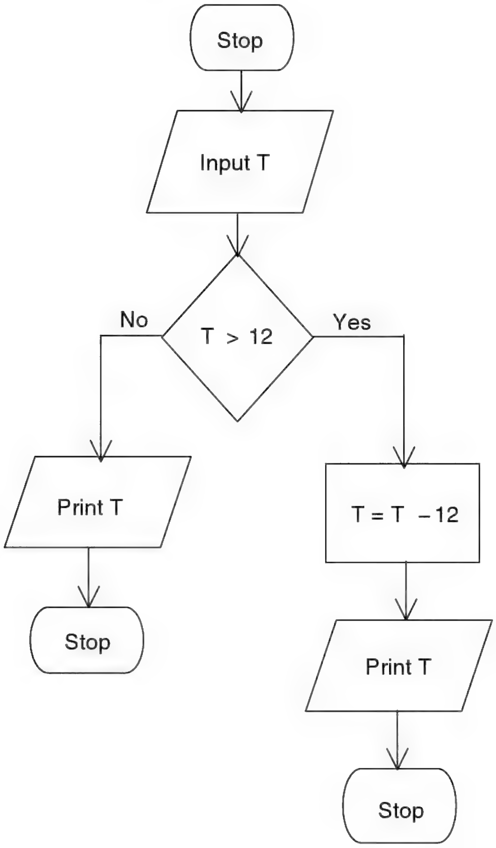
(1) 1850
(2) 3200
(3) 6400
(4) Error

13. What will be the output of the following program?

```
10 REM DISCRIMINANT OF THE
QUADRATIC EQUATION
20 READ a, b, c
30 DATA 10, 20, 5
40 LET D = b ^ 2 – 4 * a * c
50 PRINT "THE VALUE OF D IS = "; D
60 END
```

(1) 400
(2) 200
(3) 300
(4) 150

14. What will be the output of the following flow-chart, if T = 15?



(1) 3 (2) 12 (3) 15 (4) 5

15. Evaluate the following expressions as a computer would do.

I. $(\ell + m)/n + q \times t \div 5$

II. $80 + 30 - 27 * 4 - 25/\ (5 * 5)$

(1) $(m + \ell)/n + \frac{qt}{5}$, 1 (2) $(m + \ell)/n + qt/5$, 2

(3) $(m + \ell)/n$, 3 (4) $(m + \ell)/n + qt/5$, 2

16. Which of the following is an algorithm to find the area of a square?

(1) (i) Read an arm of the square.
(ii) Find the area by using A = 4 * a.
(iii) Display the area.

(2) (i) Find the area by using A = 4 * a.
(ii) Display the area.
(iii) Read an one arm of square (a).

(3) (i) Read an arm of square (a).
(ii) Find the area using A = (a * a).
(iii) Display the area.

(4) (i) Read an arm of the square (a).
(ii) Display the area.
(iii) Find the area by using A = (a * a).

17. REM is non-executable statement and is short for ________.

(1) REMAT (2) REMSTATE
(3) REMARKDATA (4) REMARK

18. What will be the output of the following program?

```
10 Read A, B, C, D
20 Data 8, 10, 6, 2
30 Let S = D – A/(B – C) + A
40 Print S
50 End
```

(1) 0.5 (2) 10
(3) 8 (4) 9

19. Find the output of the following program.

```
10 LET A = 36, B = 4
20 LET C = (A/B)^ (1/2)
30 PRINT C
40 END
```

(1) 9 (2) 36
(3) 3 (4) None of these

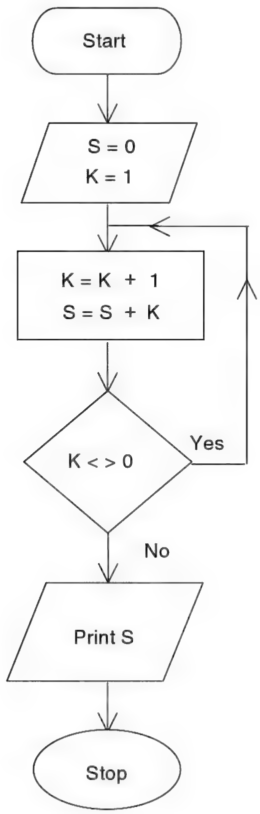
20. What will be the output of the following program?

10 REM AVERAGE OF FOUR NUMBERS
20 READ N, N_1, N_2, N_3, N_4
30 DATA 4, 16, 40, 42, 90
40 PRINT "AVERAGE = (N_1 + N_2 + N_3 + N_4)/ N
50 END

(1) AVERAGE = 43
(2) AVERAGE = 47
(3) AVERAGE = 39
(4) Error

Concept Application Level—2

21. What will be the output of the following flow-chart?



(1) 1 (2) Infinite loop (3) 27 (4) 54

22. Read the following algorithm:

10 Let P = 1, Sum = 0
20 If P ≤ N, then (N is the given Number)
30 Sum = Sum + P
40 P = P + 1
50 Repeat this loop
60 End

Above algorithm is used to ________.

(1) find first N natural numbers
(2) find sum of first N natural numbers
(3) find the sum of first N even numbers
(4) find the sum of first N odd numbers

23.
10 REM "Scholarship Test"
20 INPUT "Enter marks secured",
S 30
40 If S > 50 AND S < 60 THEN

```
Let  50 S = S + 1000
 60  If S > 70 AND < 80 THEN
Let  70 S = S + 2000
 80  If S > 75 THEN S = S + 2500
 90  If S > 80 THEN S = S + 3000
100  If S < 50 THEN
110  PRINT "You are not eligible for the Scholarship Test"
120  PRINT "Your Scholarship amount is:"
  S  100 END
```

If S = 72, then what is the scholarship that a student gets?

(1) Rs 2572 (2) Rs 3072

(3) Rs 2072 (4) Rs 4572

24. Study the following program:-

```
 10 LET P = 0
 20 LET a = 0
 30 LET a = a+1
 40 Read M
 50 If P > M then Go to 70
 60 LET P = M
 70 If a < 5 then Go to 30
 80 Print P;
 90 DATA 3, 5, 4, 2, 6
100 END
```

What is the output of the above program?

(1) 2 (2) 4

(3) 6 (4) 5

25. What is the output of the following program?

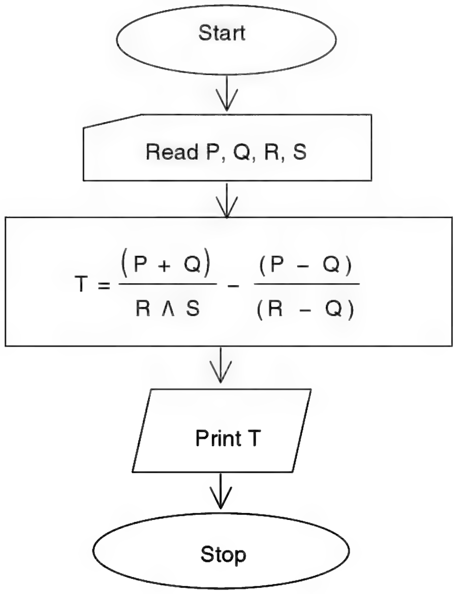
```
 20 Let I = 1
 30 Read x, y;
 40 Data 5, 4;
 50 Let S = X
 60 Print S
 70 If I = Y, then go to 100
 80 Let S = S * X
 90 Let I = I + 1
100 Go to 60
110 End
```

(1) 5 25 625 (2) 625 25 5

(3) 5 25 125 625 (4) 125 25 5

Concept Application Level—3

26.



If the values of P, Q, R and S are 8, 6, 7 and 2 respectively, then what is the output of above flow chart?

(1) $\frac{12}{7}$ (2) $\frac{14}{5}$ (3) $\frac{-12}{7}$ (4) $\frac{-27}{8}$

27.

```
10  Read A, B, C
20  Read S
30  Read L, M, N, X
40  Read P, Y
50  Data 5, 6, 7
60  Data 9
70  Data 2, √6, 2, 4
80  Data √6, 2
90  Let R = [S*(S – A)*(S – B)*(S – C)] ^ 0.5
100 Let T = L * M / (N * P) + X ∧Y
110 Let Z = R/T
120 Print Z
130 End
```

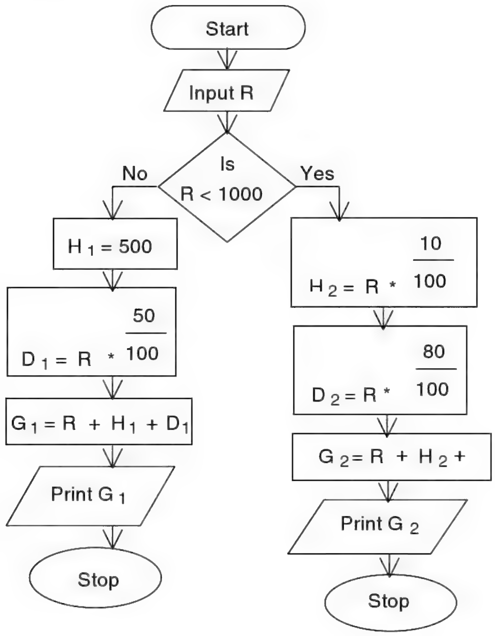
(1) $\frac{6\sqrt{3}}{17}$ (2) $\frac{6\sqrt{6}}{17}$ (3) $\frac{6\sqrt{5}}{17}$ (4) $\frac{6\sqrt{2}}{17}$

28. 10 LET x = 0
20 LET S = 1
30 x = x + 1
40 y = x * x
50 S = S + y
60 INPUT "Enter the value for K"; K
70 If x < K THEN GOTO 30 ELSE
PRINT "Sum of the squares of the numbers"; S
80 END

What is the output of the above program if K = 8?

(1) 120 (2) 204 (3) 140 (4) 205

29. Study the following flow-chart.



In the above flow chart, what will be the output, if R = 800?

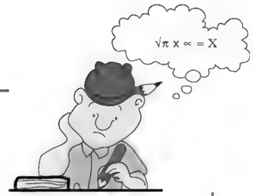
(1) 1044 (2) 844 (3) 735 (4) 1520

30. What is the output of the following program?

10 Let Sum = 0
20 Let N = 8
30 if N < 1 then Go to 70
40 Let Sum = Sum + N * N
50 Let N = N – 2
60 Go to 30
70 Print Sum
80 end

(1) 120 (2) 125 (3) 145 (4) 170

KEY



Very short answer type questions

1. Program
2. Control unit, A. L. U. and memory unit.
3. Machine level language or binary language or Low Level Language.
4. Vacuum tubes
5. True
6. a letter or alphabet
7. VLSI (very large scale integrated circuits)
8. False
9. PRINT
10. 1024 KB
11. (Decision box)
12. Statement number
13. Non-executable statements
14. Flow chart
15. LET
16. High level
17. INPUT
18. Statements
19. 8 digits
20. –32768 to 32767

Short answer type questions

25. 1 4 9 16.

Essay type questions

30. 120.

key points for selected questions

Short answer type questions

21. Initialize Sum = 0, i= 0.
Sum = Sum + i* i
If (i < = 10), i = i + 1

22. Take the variable for Sum, and initialize such that P = 0, Q = 1 and Y = 3. Make Sum = Sum + Q and let P = P + 1 and Q = Q *Y.
Repeat this loop of P till $P \leq 5$, print Sum.

23. Let the centriod be (X,Y)
Then, $X = (x_1 + x_2 + x_3)/3$
$Y = (y_1 + y_2 + y_3)/3$.
Where (x_1, y_1) (x_2, y_2) and (x_3, y_3) are the given vertices of the triangle.

24. Take the sum value of N. Then initialize the variable for Sum of first N numbers.
Let A = 1, If $A \leq N$, then Sum = Sum + A
A = A + 1, if until this condition is qualified else print Sum.

25. Perform the operations in the flow chart using given values.

Essay type questions

26. Take P = a * c
Q = d * e
$R = f \div m$

27. Use P = x * x + y * y + z * z, then $R = \sqrt{P}$.

28. Use the formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$,
where $s = \dfrac{(a+b+c)}{2}$.

29. Initialize the required variable, X = 0, Y = 0.
Let Y = Y + 1
Read a variable Z.
If X > Z check it for all 10 numbers.
Else X = Z.

30. Perform the operations in the flow chart using given values.

KEY

Concept Application Level—1,2,3

1. 2	**2.** 3
3. 3	**4.** 3
5. 3	**6.** 2
7. 4	**8.** 4
9. 2	**10.** 2
11. 3	**12.** 4
13. 2	**14.** 1
15. 1	**16.** 3
17. 4	**18.** 3
19. 3	**20.** 2
21. 2	**22.** 2
23. 3	**24.** 3
25. 3	**26.** 3
27. 2	**28.** 4
29. 4	**30.** 1

Concept Application Level—1,2,3

Key points for select questions

2. We cannot assign two values to a single variable.

5. Use the order of priority of the arithmetic expressions and evaluate the expression.

6. Identify the syntax of READ and DATA key words and also their assignments.

7. Use the order of priority of the arithmetic expressions and evaluate the expression.

9. A string constant should be within double quotation marks.

12. (i) Value of s is not given.
(ii) Observe the statement numbered 50. Here, identify the task to be done by the operators ↑ and * and their priorities and precedences.

13. (i) It calculates $D = b^2 - 4ac$.
(ii) Observe the statement numbered 50. Here, identify the task to be done by the operators ↑ and * and their priorities and precedences.

14. The given program prints T = T − 12, as T > 12.

15. Use BODMAS rule.

16. Steps of algorithms must be in précised order. So, read the statements carefully and complete the task in proper sequence.

17. Recall the keywords of BASIC.

18. Use BODMAS rule.

20. (i) The program prints the average of given numbers.
(ii) Keep track of each statement. Solve clearly the statement numbered 40.

21. $K \neq 0$ in every loop.

23. (i) Given S = 72, Statement 60 i.e., S > 70 and S < 80 executes.
(ii) Keep track of the order of the statements that is to be executed and follow with the conditional statements.

24. Keep track of the key word GOTO which is unconditional branching statement. Be careful when the loop is to be terminated.

25. Find the values in every iterative loop.

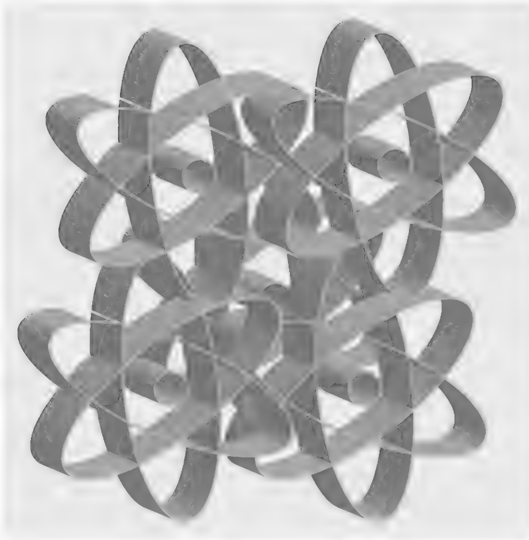
26. Substitute the values of P, Q, R and S in T and follow the operator precedence.

27. (i) Calculate R, T and R/T.
(ii) Use the order of priority of the arithmetic expressions and evaluate the expression.

30. Keep track of the key word GOTO which is unconditional branching statement. Be careful when the loop is to be terminated.

CHAPTER 20

Permutations and Combinations



INTRODUCTION

This chapter offers some techniques of counting without direct listing of the number of elements in a particular set or the number of outcomes of a particular experiment. We now present the two fundamental rules of counting, namely (i) **The Sum Rule** and (ii) **The Multiplication Rule** or Product rule.

Sum rule of disjoint counting

If there are two sets say A and B with A having m elements and B having n elements with no element in A appearing in B, then the number of elements in A **or** B is $(m + n)$.
Symbolically,

$$n(A \cup B) = n(A) + n(B), \text{ when A and B are disjoint}$$

The symbol $\cup$ stands for Union.

Example

$A = \{1, 2, 3, 4\}$ and $B = \{a, e, i, o, u\}$ are two sets. In how many ways can a number from A or a letter from B be chosen?

Solution

As no element of A is in B, we can apply the sum rule of disjoint counting.

$n(A \cup B) = n(A) + n(B) = 4 + 5 = 9.$

General form of sum rule

If A and B are two sets, then the number of elements in A or B (not necessarily disjoint) is given by

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

The symbol $\cap$ stands for intersection. It means 'common to'.

Example

In how many ways can a prime or an odd number be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

Solution

We form two sets P and O as follows.

$P = \{2, 3, 5, 7\}$ (primes) and $O = \{1, 3, 5, 7, 9\}$ (odd numbers).

On applying the general form of Sum Rule we get,

$n(P \cup O) = n(P) + n(O) - n(P \cap O) = 4 + 5 - 3 = 6.$

We note that the numbers 3, 5 and 7 are counted among primes and also among odd numbers. So we discount 3 (common numbers) from the sum $n(P) + n(O)$.

NOTE: The usage of the word OR prompts you to add.

Product rule or multiplication rule

If two operations must be performed, and if the first operation can be performed in p_1 ways and the second in p_2 ways, then there are $p_1 \times p_2$ different ways in which the two operations can be performed one after the other.

Example

A caterer's menu is to include 4 different sandwiches and 3 different desserts. In how many ways can one order for a sandwich and a dessert?

Solution

Let S_1, S_2, S_3 and S_4 denote 4 sandwiches and D_1, D_2, D_3 denote 3 desserts.

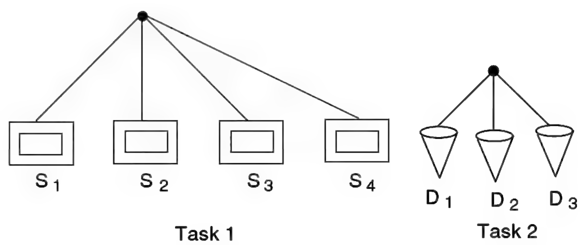


Figure 20.1

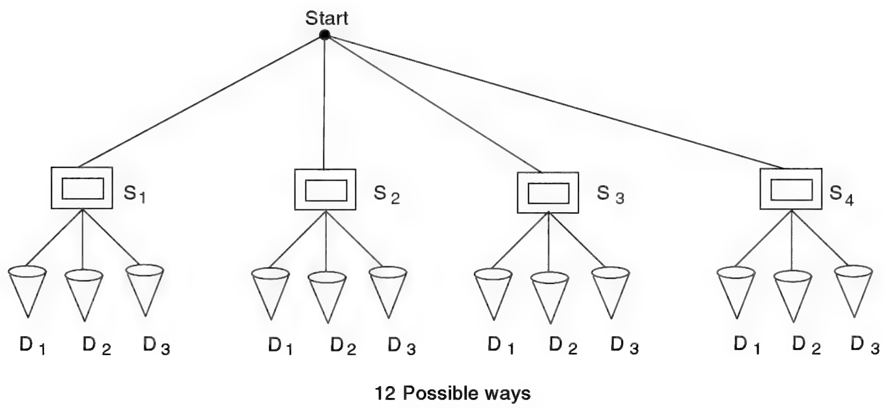


Figure 20.2

The tree diagram clearly suggests that there are 4 ways to choose a sandwich and for each of these 4 ways there are 3 ways to choose a dessert. There are $4 \times 3 = 12$ ways of choosing a sandwich and a dessert.

Generalization of product rule

Suppose that tasks $T_1, T_2, T_3, \ldots, T_r$ are to be performed in a sequence. If T_1 can be performed in p_1 ways, and for each of these ways, T_2 can be performed in p_2 ways, and for each of these $p_1 \times p_2$ ways of performing T_1, T_2 in sequence, T_3 can be performed in p_3 ways and so on, then the sequence $T_1, T_2, T_3, \ldots, T_r$ can be performed in $p_1 \times p_2 \times p_3 \times \ldots \times p_r$ ways.

Examples

1. A man has 7 trousers and 10 shirts. How many different outfits can he wear?

Solution

Task 1: He may choose the trouser in 7 ways.

Task 2: He may choose the shirt in 10 ways.

According to the Product Rule, the total number of different outfits is 7×10 i.e., 70.

2. A class has 20 boys and 15 girls. If one representative from each sex has to be chosen, in how many ways can this be done?

Solution

Task 1: Choosing a representative from boys.
This can be done in 20 ways.

Task 2: Choosing a representative from girls.
This can be done in 15 ways.
By the Product rule, the number of ways of performing the two tasks is 20×15 i.e., 300 ways.

3. How many different outcomes arise from first tossing a coin and then rolling a die?

Solution

There are 2 possibilities (i.e., head or tail) for the first task (tossing a coin) and after each of these outcomes there are 6 possibilities (i.e., any number from 1 to 6) for the second task (rolling a dice). Thus, by the product rule, there are 2×6 i.e., 12 possible outcomes, for the given compound task.

4. A password of 4 letters is to be formed with vowels alone. How many such passwords are possible if

   (i) repetition of letters is allowed,

   (ii) repetition of letters is not allowed?

Solution

The tasks T_1, T_2, T_3 and T_4 are about filling the 1st, 2nd, 3rd and 4th slots in the password.

(i) The first slot can be filled in 5 ways (a, e, i, o or u).

The second can also be filled in 5 ways (with repetition being allowed).

The third and fourth can also be filled in 5 ways each.

Using the generalisation, we get $5 \times 5 \times 5 \times 5 = 625$ passwords.

(ii) The first slot can be filled in 5 ways (a, e, i, o or u). The second slot can be filled in 4 ways (as repetition is not allowed). The third and fourth in 3 and 2 ways respectively. Thus the total number of possible passwords are $5 \times 4 \times 3 \times 2 = 120$.

NOTE: The usage of AND prompts you to multiply.

Permutations

Each of the arrangements which can be made by taking some or all of a number of things is called a Permutation. Permutation implies "arrangement" i.e., order of things is important.

Example

Consider 4 elements a, b, c and d. List all permutations taken two at a time.

Solution

We have two cases to deal

(i) with repetition allowed.

(ii) with repetition not allowed.

Now list for

Case (i): aa, ab, ba, ac, ca, ad, da, bb, bc, cb, bd, db, cc, cd, dc, dd, i.e., there are 16 possibilities.

Case (ii): ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc, i.e., there are 12 possibilities.

We have seen what are the possibilities.

To count the number of permutations without actual listing of the arrangements, we use the product rule as a technique.

We have two tasks to perform–namely filling up the first slot in the arrangement and filling up the second slot in the arrangement.

Case (i): Repetition allowed.

$$\underset{\text{Task 1}}{\boxed{4}} \times \underset{\text{Task 2}}{\boxed{4}} = 16$$

Case (ii): Repetition not allowed

$$\underset{\text{Task 1}}{\boxed{4}} \times \underset{\text{Task 2}}{\boxed{3}} = 12$$

NOTE: In case (ii), aa, bb, cc, dd are not allowed.

Example

There are 10 railway stations between a station x and another station y. Find the number of different tickets that must be printed so as to enable a passenger to travel from one station to any other.

Solution

Including x and y there are 12 stations. From any one station to any other, we need 11 different types of tickets. Since there are 12 stations, the different tickets possible are $(12)(11) = 132$.

Factorial notation

If n is a positive integer, then the product of the first n positive integers is denoted by $n!$ or $\angle n$ (read as n factorial). We define zero factorial as 1.

Accordingly,

$0\,! = 1.$

$1\,! = 1.$

$2\,! = 1 \times 2 = 2.$

$3\,! = 1 \times 2 \times 3 = 6.$

$4\,! = 1 \times 2 \times 3 \times 4 = 24.$

$5\,! = 120, 6\,! = 720, 7\,! = 5040.$

NOTE: In some problems the answers will be left as factorials. We need not simplify numbers like 10 !, 12 !, 25 ! etc. However a problem like $\frac{30!}{28!}$ can be simplified as $\frac{30 \times 29 \times 28!}{28!} = 30 \times 29 = 870.$

General formula for permutations (repetitions not allowed)

The number of permutations of n distinct objects, taken r at a time without repetition is ${}^nP_r = \frac{n!}{(n-r)!}$.

For r = 0, 1, 2, 3, ..., n . ${}^nP_r$ is read as nPr.

Explanation

Consider r boxes, each of which can hold one item. When all the r boxes are filled, what we have is an arrangement of r items taken from the given n items. Hence the number of ways in which we can fill up the r boxes by taking r things from the given n things is equal to the number of permutations of n things taken r at a time.

$\boxed{n} \quad \boxed{n-1} \quad \boxed{n-2} \ldots\ldots\ldots \boxed{n-r+1}$

First box can be filled in n ways (because any one of the n items can be used to fill this box); having filled the first box, to fill the second box we now have only (n – 1) items; any one of these items can be used to fill the second box and hence the second box can be filled in (n – 1) ways; similarly, the third box in (n – 2) ways and so on. Thus the rth box in $\{n - \overline{r-1})\}$ ways, i.e., (n – r + 1) ways. Hence all the r boxes together can be filled up in n(n – 1) (n – 2) .... (n – r + 1) ways.

So, ${}^nP_r = n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)$. This can be simplified by multiplying and dividing the right hand side by (n – r) (n – r – 1) .... 3.2.1, giving us

$${}^nP_r = n(n-1)(n-2)\ldots\ldots\ldots[n-(r-1)] = \frac{n!}{(n-r)!}$$

The number of permutations of n distinct items taken r at a time is

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

If we take n things at a time, then we get ${}^nP_n$. From the discussion we had for filling the boxes, we can find that the number of permutations of n things taken all at a time is n!.

$$\boxed{{}^nP_n = n!}$$

The value of ${}^nP_r$ without factorials is ${}^nP_r = n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)$, for $r \neq 0$.

Example

In how many ways can 8 athletes finish a race for Gold, Silver and Bronze medals?

Solution

This is the number of permutations of 8 distinct objects taken three at a time without repetitions (here it means same person cannot get both silver and bronze).

Thus $^8P_3 = 8 \times 7 \times 6 = 336$ ways.

General formula for permutations with repetitions allowed

The number of permutations of n distinct objects taken r at a time with repetition allowed is n^r, for any integer $r \geq 0$.

Explanation

We have r boxes with each box ready to accept one or more of the n distinct objects. Using product rule, the total ways of filling up these r boxes is $n \times n \times n \times n$...... for r times = n^r

$$\boxed{n}\ \boxed{n}\ \boxed{n}\ \cdots\cdots\ \boxed{n} = n^r$$

(r times)

Example

In how many ways can 3 letters be put into 5 letter boxes when each box can take any number of letters?

Solution

As each box can taken any number of letters, we can post each letter in 5 ways.

$$\underset{\text{Letter 1}}{\boxed{5}} \times \underset{\text{Letter 2}}{\boxed{5}} \times \underset{\text{Letter 3}}{\boxed{5}} = 5^3 = 125 \text{ ways}$$

Combinations

Each of the groups or selections which can be made by taking some or all of a number of things is called a Combination.

In combinations, the order in which the things are taken is not considered as long as the specific things are included.

Example

Consider a, b, c, d. List all combinations taken 3 at a time.

Solution

The list includes abc, abd, acd, bcd.

Here abc, bca, cab are regarded the same as order is not important.

The number of combinations of n things taken r at a time is denoted by ^nC_r

General formula for combinations

We first look at the permutations of n items taken r at a time from a different perspective. We look at two tasks T_1 and T_2 as:

T_1: Select r objects.

T_2: Arrange all the r objects that got selected in T_1.

We understand that T_1 can be done in ^nC_r ways by definition, and its value yet to be determined and T_2 can be done in r! ways. But then to get the permutations, we need to perform T_1 followed by T_2.

Thus by Fundamental Principle of Counting, both tasks can be done in $^nC_r \times r!$ ways.

Thus $^nC_r \times r! = {}^nP_r$ i.e., $$^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

NOTE:

1. $^nC_0 = {}^nC_n = 1$
2. $^nC_1 = {}^nC_{n-1} = n$
3. If $^nC_r = {}^nC_s$, then r = s or n = r + s.

Examples

1. In a library there are 10 research scholars. In how many ways can we select 4 of them?

Solution

Out of 10 scholars, we can select 4 of them in $^{10}C_4$ ways.

$$^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210 \text{ ways.}$$

2. In how many ways can we select two vertices in a hexagon?

Solution

A hexagon has 6 vertices. Any 2 vertices can be selected in $^6C_2$ ways $= \frac{6 \times 5}{1 \times 2} = 15$ ways.

3. From 8 gentlemen and 5 ladies, a committee of 4 is to be formed. In how many ways can this be done,
   (i) when the committee consists of exactly three gentlemen?
   (ii) when the committee consists of at most three gentlemen?

Solution

(i) We have to select three out of 8 gentlemen and one out of 5 ladies. Hence the number of ways in which this can be done $= {}^8C_3 \times {}^5C_1 = 280$.

(ii) The committee is to contain at most three gentlemen, i.e., it may contain either 1, 2 or 3 gentlemen.

Hence, the total number of ways = ${}^8C_1 \times {}^5C_3 + {}^8C_2 \times {}^5C_2 + {}^8C_3 \times {}^5C_1 = 80 + 280 + 280 = 640$

4. Find ${}^nC_3$, if ${}^nC_7 = {}^nC_4$.

Solution

${}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11$

So ${}^nC_3 = {}^{11}C_3 = \dfrac{11 \times 10 \times 9}{1 \times 2 \times 3} = 165.$

5. How many distinct positive integers are possible with the digits 1, 3, 5, 7 without repetition?

Solution

Possible number of

single-digit numbers = 4

two-digit numbers = $4 \times 3 = 12$

three-digit numbers = $4 \times 3 \times 2 = 24$

four-digit numbers = $4 \times 3 \times 2 \times 1 = 24$

Thus total number of distinct positive integers without repetition = 4 + 12 + 24 + 24 = 64

6. If ${}^nP_r = 990$ and ${}^nC_r$ 165, then find the value of r.

Solution

${}^nP_r = r!\, {}^nC_r$

$\Rightarrow \dfrac{990}{165} = r!$

$\Rightarrow r! = 6 \Rightarrow r = 3.$

Alternative method

${}^nP_r = 990 = 11 \times 10 \times 9 = {}^{11}P_3$

$\Rightarrow n = 11, r = 3$ also ${}^{11}C_3 = 165$

$\therefore r = 3$

7. In a plane there are 12 points, then answer the following questions:

(a) Find the number of different straight lines that can be formed by joining these points, when no combination of 3 points are collinear.

(b) Find the number of different straight lines that can be formed by joining these points, when 4 of these given points are collinear and no other combination of three points are collinear.

(c) Find the number of different triangles that can be formed by joining these points, when no combination of 3 points are collinear.

(d) Find the number of different triangles that can be formed by joining these points, when 5 of these given points are collinear and no other combination of three points are collinear.

Solution

(a) We know passing through two points in a plane we can draw only one line. i.e., we require to select any two points from the given 12 points which is possible in ${}^{12}C_2$ ways.

$\therefore$ The number of different straight lines that can be formed by joining the given 12 points

$= {}^{12}C_2 = \dfrac{12 \times 11}{1 \times 2} = 66.$

(b) Given, out of the 12 points 4 points are collinear.

We know that collinear points form only one line.

$\therefore$ These four points when they are not collinear will actually form ${}^4C_2$ lines, which are not forming here.

$\therefore$ The number of the required lines $= {}^{12}C_2 - {}^4C_2 + 1 = 66 - 6 + 1 = 61$

(c) We know, by joining three non collinear points a triangle forms.

$\therefore$ Three points can be selected from 12 points in ${}^{12}C_3$ ways.

$\therefore$ The required number of triangles $= {}^{12}C_3 = \dfrac{12 \times 11 \times 10}{1 \times 2 \times 3} = 220$

(d) Given 5 points are collinear.

$\Rightarrow {}^5C_3$ triangles will not form.

$\therefore$ The required number of triangles $= {}^{12}C_3 - {}^5C_3 \quad = 220 - 10 = 210.$

8. Find the number of diagonals of a polygon of 10 sides.

Solution

Assume that there are 10 points in a plane where no 3 of them are collinear which are the vertices of the given polygon.

The number of different lines that can be formed by joining these 10 points is ${}^{10}C_2$.

We know in any polygon the lines joining non-adjacent vertices are called diagonals.

Hence the required number of diagonals = Number of lines formed – number of sides of the polygon $= {}^{10}C_2 - 10 = 35$.

Note: The number of diagonals of a polygon of n sides is given by ${}^nC_2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$

Using this formula the number of diagonals in the above problem $= \dfrac{10(10-3)}{2} = 35.$

test your concepts

Very short answer type questions

1. Find the value of 6!.
2. What is the value 0!?
3. Factorial is defined for ________ numbers.
4. The number of arrangements that can be made by taking r objects at a time from a group of n dissimilar objects, is denoted as _____.
5. What is the formula for ${}^nP_r$?
6. In ${}^6C_r$, what are the possible values of r?
7. What the value of ${}^nC_r$?
8. What is the relation between ${}^nP_r$ and ${}^nC_r$?
9. The number of straight lines that can be formed by n points in a plane, where no three points are collinear is ________ and in case p of these points are collinear is ________.
10. The number of triangles that can be formed by n points in a plane where no three points are collinear is __________ and when p of the given points are collinear is _______.
11. Find the number of 3-digit numbers, formed with the digits {2, 5, 4, 6} when repetition of the digits is allowed.
12. If ${}^nP_{100} = {}^nP_{99}$, then find the value of n.
13. If ${}^{100}C_3 = 161700$, then ${}^{100}C_{97}$ is equal to ________.
14. If ${}^nP_3 = 720$, then find the value of ${}^{11}P_n$.
15. Find the number of four-digit numbers that can be formed using the digits 1, 2, 5, 7, 4 and 6, if every digit can occur at most once in any number.
16. Find the number of integers greater than 4000 that can be formed by using the digits 3, 4, 5 and 2, if every digit can occur at most once in any number.
17. How many 6-letter words with distinct letters in each can be formed using the letters of the word EDUCATION? How many of these begin with I?
18. How many words with distinct letters can be formed by using all the letters of the word PLAYER which begin with P and end with R?
19. In a class, there are 45 students. On a new year eve, every student sends one greeting card to each of the other students. How many greeting cards were exchanged in all?
20. In how many ways can 6 prizes be distributed among 4 students, if each student can receive more than one prize?
21. If ${}^nP_r = 360$ and ${}^nC_r = 15$, then find the value of r.

22. A bag contains 3 yellow balls and 4 pink balls. In how many ways can 2 pink balls and 1 yellow ball be drawn from the bag?

23. In how many ways can 11 players be chosen from a group of 15 players?

24. A committee of 5 members is to be formed from 8 men and 6 women. Find the number of ways of forming the committee, if it has to contain 3 men and 2 women.

25. In how many ways can 3 diamond cards be drawn simultaneously from a pack of cards?

26. In a party there are 20 persons. If every person shook hand with every other person in the party exactly once, find the total number of handshakes exchanged in the party.

27. A regular polygon has 20 sides. Find the number of diagonals of the polygon.

28. How many different straight lines can be formed from 30 points in a plane? (no three points are collinear)

29. If the number of diagonals of a regular polygon is three times the number of its sides, find the number of sides of the polygon.

30. There are 20 points in a plane. How many different triangles can be formed with these points? (no three points are collinear)

Short answer type questions

31. If $^{n}P_{r} = 1716$ and $r = 3$, then $^{n}C_{r}$ = ________.

32. A boy has 9 trousers and 12 shirts. In how many different ways can he select a trouser and a shirt?

33. How many three letter words are formed using the letters of the word FAILURE?

34. The number of selections that can be made to select 5 members from a group of 15 members is ________.

35. There are 8 points in a plane, how many different triangles can be formed using these points (no three points are collinear)?

36. A bag contains 9 yellow balls, 3 white balls and 4 red balls. In how many ways can two balls be drawn from the bag?

37. A question paper contains 20 questions. In how many ways can 4 questions be attempted?

38. If a polygon has 8 sides, then the number of diagonals of the polygon is ________.

39. In a class there are 15 boys and 10 girls. How many ways can a pair of one boy and one girl be selected from the class?

40. How many five-digit numbers can be formed using the digits {5, 6, 3, 9, 2}? (no digit can occur more than once in any number)?

41. In how many ways can 3 consonants be selected from the letters of the word EDUCATION?

42. Using all the letters of the word NOKIA, how many words can be formed, which begin with N and end with A?

43. Given $_{1}$, $_{2}$ are two parallel lines. How many triangles are formed with 12 points taking on $_{1}$ and 6 points on $_{2}$?

44. A question paper contain 15 questions. In how many ways can 7 questions be attempted?

45. A bag contains 5 white balls and 2 yellow balls. The number of ways of drawing 3 white balls is ________.

Essay type questions

46. A four-digit number is formed using the digits {0, 6, 7, 8, 9}. How many of these numbers are divisible by 3? (Each digit is occurred atmost once in every number).

47. There are 25 points in a plane. Six of these are collinear and no other combination of 3 points are collinear. How many different straight lines can be formed by joining these points?

48. There are 20 points in a plane, of which 5 points are collinear and no other combination of 3 points are collinear. How many different triangles can be formed by joining these points?

49. Using the letters of the word "TABLE". How many words can be formed so that the middle place is always occupied by a vowel?

50. Find the value of ${}^6C_2 + {}^6C_3 + {}^7C_4 + {}^8C_5 + {}^9C_6 =$

CONCEPT APPLICATION

Concept Application Level—1

1. ${}^nC_n =$ ________.

(1) n! (2) 1 (3) nn (4) n

2. If a polygon has 6 sides, then the number of diagonals of the polygon is ________.

(1) 18 (2) 12 (3) 9 (4) 15

3. How many two digit numbers can be formed using the digits {1, 2, 3, 4, 5}, if no digit occurs more than once in each number?

(1) 10 (2) 20 (3) 9 (4) 16

4. If ${}^nC_4 = 35$, then ${}^nP_4 =$ ________.

(1) 120 (2) 140 (3) 840 (4) 420

5. Using all the letters of the word 'QUESTION', how many different words can be formed?

(1) 8! (2) 7! (3) $7 \times 7!$ (4) 9!

6. If ${}^nP_r = 24\ {}^nC_r$, then r = ________.

(1) 24 (2) 6

(3) 4 (4) Cannot be determined

7. In how many ways can 5 prizes be distributed to 3 students, if each student is eligible for any number of prizes?

(1) 3^5 (2) 5^3 (3) ${}^5P_3$ (4) ${}^5C_3$

8. Using the letters of the word PUBLIC, how many four letter words can be formed which begin with B and end with P? (Repetition of letters is not allowed)

(1) 360 (2) 12 (3) 24 (4) 30

9. In a class there are 20 boys and 15 girls. In how many ways can 2 boys and 2 girls be selected?

(1) ${}^{35}C_4$ (2) ${}^{35}C_2$ (3) ${}^{20}C_2 \times {}^{15}C_2$ (4) 20×15

10. Using all the letters of the word "OBJECTS", how many words can be formed which begin with B but do not end with S?

(1) 120 (2) 480 (3) 600 (4) 720

11. The number of diagonals of a regular polygon is 14. Find the number of the sides of the polygon.

(1) 7 (2) 8 (3) 6 (4) 9

12. In how many ways can 5 letters be posted into 7 letter boxes?

(1) ${}^7C_5$ (2) 5^7 (3) 7^5 (4) ${}^7P_5$

13. Sunil has 6 friends. In how many ways can he invite two or more of his friends for dinner?

(1) 58 (2) 57 (3) 63 (4) 49

14. Find the value of ${}^7C_4 - {}^6C_4 - {}^5C_3 - {}^4C_2$.

(1) 3 (2) 8 (3) 4 (4) 15

15. How many different words can be formed using all the letters of the word "SPECIAL", so that the consonants always in the odd positions?

(1) 112 (2) 72 (3) 24 (4) 144

16. In how many ways can 3 consonants be selected from the English alphabet?

(1) ${}^{21}C_3$ (2) ${}^{26}C_3$ (3) ${}^{21}C_5$ (4) ${}^{26}C_5$

17. From 8 boys and 5 girls, a delegation of 5 students is to be formed. Find the number of ways this can be done such that delegation must contain exactly 3 girls.

(1) 140 (2) 820 (3) 280 (4) 410

18. There are 18 stations between Hyderabad and Bangalore. How many second class tickets have to be printed, so that a passenger can travel from one station to any other station?

(1) 380 (2) 190 (3) 95 (4) 100

19. How many numbers greater than 3000 can be formed using the digits 0, 1, 2, 3, 4 and 5 so that each digit occur at most once in each number?

(1) 1000 (2) 300 (3) 1200 (4) 1380

20. Using all the letters of the word EDUCATION, how many words can be formed which begin with DU? (Repetition is not allowed).

(1) 8! (2) 7! (3) 6! (4) 9!

21. Anil has 8 friends. In how many ways can he invite one or more of his friends to a dinner?

(1) 127 (2) 128 (3) 256 (4) 255

22. In how many ways can 4 letters be posted in 3 letter boxes?

(1) 4^3 (2) 3^4 (3) 6! (4) 4

23. Using the letters of the word PRIVATE, how many 6-letter words can be formed which begin with P and end with E?

(1) 3! (2) 4! (3) 7! (4) 5!

24. Find the number of 4 digit odd numbers that can be formed using the digits 4, 6, 7, 9, 3, so that each digit occur atmost once in each number.

(1) 120 (2) 24 (3) 48 (4) 72

25. How many 5-digit numbers that are divisible by 5 can be formed using the digits {0, 1, 3, 5, 7, 6}? (each digit can be repeated any number of times)

(1) 1080 (2) 2160 (3) 6480 (4) 3175

26. How many four-digit even numbers can be formed using the digits {3, 5, 7, 9, 1, 0}? (Repetition of digits is not allowed)

(1) 120 (2) 60 (3) 360 (4) 100

27. There is a three-digit password and it is known that each digit can have four values 5, 6, 7 or 8. If there is exactly one correct password, how many distinct wrong passwords are there?

(1) 63 (2) 80 (3) 81 (4) 64

28. In how many ways can a committee consisting of 3 men and 4 women be formed from a group of 6 men and 7 women?

(1) ${}^6C_4\,{}^7C_3$ (2) ${}^6C_3\,{}^7C_5$ (3) ${}^6C_3\,{}^7C_4$ (4) ${}^7C_5\,{}^6C_4$

29. Thirty members attended a party. If each person shakes hands with every other person exactly once, then find the number of hand shakes made in the party.

(1) ${}^{30}P_2$ (2) ${}^{30}C_2$ (3) ${}^{29}C_2$ (4) ${}^{60}C_2$

30. In how many ways can 6 members be selected from a group of 10 members?

(1) ${}^6C_4$ (2) ${}^{10}C_4$ (3) ${}^{10}C_5$ (4) ${}^{10}P_4$

Concept Application Level—2

31. In a class there are 20 boys and 25 girls. In how many ways can a pair of a boy and a girl be selected?

(1) 400 (2) 500 (3) 600 (4) 20

32. How many different odd numbers are formed using the digits {2, 4, 0, 6}? (Repetition of digits is not allowed)

(1) 16 (2) 0 (3) 24 (4) 108

33. There are 15 stations from New Delhi to Mumbai. How many first class tickets can be printed to travel from one station to any other station?

(1) 210 (2) 105 (3) 240 (4) 135

34. In how many ways can 3 vowels be selected from the letters of the word EQUATION?

(1) 56 (2) 10 (3) 28 (4) 40

35. In how many ways can 3 consonants and 2 vowels be selected from the letters of the word TRIANGLE?

(1) 25 (2) 13 (3) 30 (4) 20

36. A plane contains 12 points of which 4 are collinear. How many different straight lines can be formed with these points?

(1) 50 (2) 66 (3) 60 (4) 61

37. A plane contains 20 points of which 6 are collinear. How many different triangles can be formed with these points?

(1) 1120 (2) 1140 (3) 1121 (4) 1139

38. Using the letters of the word "ENGLISH", how many five letters words can begin with G?

(1) 2520 (2) 360 (3) 180 (4) 1260

39. Tweleve teams are participating in a cricket tournament. If every team plays exactly one match with every other team, then the total number of matches played in the tournament is ________.

(1) 132 (2) 44 (3) 66 (4) 88

40. In how many ways can 4 consonants be chosen from the letters of the word SOMETHING?

(1) ${}^9C_4$ (2) ${}^6C_4$ (3) ${}^4C_4$ (4) ${}^4C_3$

41. How many three letter words can be formed using the letters of the word NARESH? (repetition of letters is not allowed)

(1) 3! (2) ${}^5P_3$ (3) ${}^6P_3$ (4) ${}^6C_3$

42. A four digit number is to be formed using the digits 0, 1, 3, 5 and 7. How many of them are even numbers? (Each digit can occur for only one time).

(1) 48 (2) 60 (3) 24 (4) 120

43. How many numbers less than 1000 can be formed using the digits 0, 1, 3, 4 and 5 so that each digit occurs atmost once in each number?

(1) 53 (2) 69 (3) 68 (4) 60

44. There are 15 points in a plane. No three points are collinear except 5 points. How many different straight lines can be formed?

(1) 105 (2) 95 (3) 96 (4) 106

45. There are 12 points in a plane, no three points are collinear except 6 points. How many different triangles can be formed?

(1) 200 (2) 201 (3) 220 (4) 219

Concept Application Level—3

46. How many 4-digit even numbers can be formed using the digits $\{1, 3, 0, 4, 7, 5\}$? (each digit can occur only once)

(1) 48 (2) 60 (3) 108 (4) 300

47. Using the letters of the word CHEMISTRY, how many six letter words can be formed, which end with Y?

(1) ${}^{8}P_{6}$ (2) ${}^{9}P_{6}$ (3) ${}^{9}P_{5}$ (4) ${}^{8}P_{5}$

48. A telephone number has seven digits, no number starts with 0. In a city how many different telephone numbers be formed using the digits 0 to 6? (each digit can occur only once)

(1) 6! (2) 6·6! (3) 7! (4) 2·7!

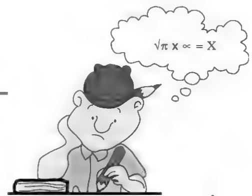
49. Using all the letters of the word PROBLEM how many words can be formed such that the consonants occupy the middle place?

(1) 3000 (2) 4200 (3) 720 (4) 3600

50. Using the digits 0, 1, 2, 5 and 7 how many 4-digit numbers that are divisible by 5 can be formed if repetition of the digits is not allowed?

(1) 38 (2) 46 (3) 32 (4) 42

KEY



Very short answer type questions

1. 720 **2.** 1

3. whole numbers **4.** ${}^{n}P_{r}$

5. $\dfrac{n!}{(n-r)!}$ **6.** 0, 1, 2, 3, 4, 5, 6

7. $\dfrac{n!}{(n-r)!r!}$ **8.** ${}^{n}P_{r} = {}^{n}C_{r}\, r!$

9. ${}^{n}C_{2}; {}^{n}C_{2} - {}^{p}C_{2} + 1$ **10.** ${}^{n}C_{3}; {}^{n}C_{3} - {}^{p}C_{3}$

11. $4^3 = 64$. **12.** 100

13. 161700 **14.** 11!

15. ${}^{6}P_{4}$. **16.** 12

17. (i) ${}^{9}P_{6}$ (ii) ${}^{8}P_{5}$.

18. 24 **19.** 1980

20. 4^6 **21.** 4

22. 18 **23.** ${}^{15}C_{11}$

24. 840 **25.** 286

26. 190 **27.** 170

28. 435 **29.** 9

30. ${}^{20}C_{3}$

Short answer type questions

31. 286 **32.** 108

33. 210 **34.** ${}^{15}C_{5}$

35. 56 **36.** ${}^{16}C_{2}$

37. ${}^{20}C_{4}$ **38.** 20

39. 150 **40.** 120

41. 4 **42.** 6

43. 576 **44.** ${}^{15}C_{7}$

45. 10

Essay type questions

46. 60 **47.** 286

48. 1130 **49.** 48

50. 210

key points for selected questions

Very short answer type questions

14. (i) Use, ${}^np_3 = n\ (n-1)\ (n-2)$.
(ii) Express 720 as the product of three consecutive integers and find n.

15. ${}^np_r$ = The number of arrangements that can be made from a group of n objects taken r objects at a time.

16. The first significant digit should be 5 or 4.

17. (i) Find the number of 6 letter words formed with 9 letters.
(ii) Find the number of 5 letter words (as I is fixed) formed with 8 letters.

18. (i) As P and R are fixed, we need to arrange remaining 4 letters.
(ii) Find the number of 4 – letter words that are formed with 4 letters.

19. Use ${}^np_r$ formula.

20. Each prize can be distributed in 4 ways.

21. Use $r! = \dfrac{{}^nP_r}{{}^nC_r}$.

22. Select 2 pink balls from 4 pink balls and 1 yellow ball from 3 yellow balls, then apply fundamental principle.

23. Select 11 players from a group of 15 players.

24. Select 3 men from 8 men and 2 women from 6 women.

25. Select 3 diamond cards from 13 diamond cards.

26. (i) One handshake require two persons.
(ii) Select two persons out of 20 persons.

27. The number of diagonals of a n sided polygon is $\dfrac{n(n-3)}{2}$.

28. A straight line is formed by joining any two points.

29. The number of diagonals of a n sided polygon is $\dfrac{n(n-3)}{2}$.

30. A triangle is formed by joining any three non-collinear points.

Short answer type questions

31. ${}^nC_r = \dfrac{{}^nP_r}{r!}$

32. Use Fundamental principle of multiplication i.e., Task T_1 can alone be done in m different ways and task T_2 can be done in n different ways, then the two tasks can be done together in *mn* ways.

33. Select 3 letters from 7 letters.

34. ${}^nC_r$ = number of selections that can be made from a group of n objects taken r objects at a time.

35. Three non-collinear points form a triangle.

36. Select 2 bells from 16 balls.

37. Select 4 questions from 20 questions.

38. The number of diagonals of a 'n' sided polygon is $\dfrac{n(n-3)}{2}$.

39. Use the fundamental principle of counting

40. Arrange 3 numbers out of 5 numbers.

41. Select 3 letters from 9 letters.

42. First and last places are filled with N and A respectively.

44. Select any 7 questions out of 15 questions.

45. Select 3 white balls from 5 white balls.

Essay type questions

47. (i) To form a straight line two points are required and six collinear points form only one straight line.
(ii) As there are 25 points, we can form ${}^{25}C_2$ straight lines.
(iii) 6 of them from ${}^6C_2$ but as they are collinear, they form only one straight line.

48. (i) A triangle is formed by any three non–collinear points.
(ii) The five collinear points do not form any triangle.

KEY

Concept Application Level—1,2,3

1. 2	**2.** 3
3. 2	**4.** 3
5. 1	**6.** 3
7. 1	**8.** 2
9. 3	**10.** 3
11. 1	**12.** 3
13. 2	**14.** 3
15. 4	**16.** 1
17. 3	**18.** 1
19. 4	**20.** 2
21. 4	**22.** 2
23. 4	**24.** 4
25. 2	**26.** 2
27. 1	**28.** 3
29. 2	**30.** 2
31. 2	**32.** 2
33. 1	**34.** 2
35. 3	**36.** 4
37. 1	**38.** 2
39. 3	**40.** 2
41. 3	**42.** 3
43. 2	**44.** 3
45. 1	**46.** 3
47. 4	**48.** 2
49. 4	**50.** 4

Concept Application Level—1,2,3

Key points for select questions

1. Use, ${}^nC_r = \dfrac{n!}{(n-r)!}$.

2. The number of diagonals of a n-sided polygon is $\dfrac{n(n-3)}{2}$.

3. r objects can be arranged out of n objects is ${}^nP_r$ ways.

4. ${}^nP_r = {}^nC_r \times r!$

5. ${}^nP_n = n!$

6. $r! = \dfrac{{}^nP_r}{{}^nC_r}$

7. First prize can be distributed in 3 ways similarly 2nd, 3rd, 4th and 5th prizes can be distributed each in 3 ways. Now use the fundamental principle of counting.

8. Take 6 blanks, first blank is filled with B, last blank is filled with P and the remaining blanks can be filled with the remaining letters.

9. r objects can be selected from n objects in ${}^nC_r$ ways. Now use the fundamental principle i.e., task T_1 can be done in m ways and task T_2 can be done in n ways, then the two tasks can be done simultaneously in *mn* ways.

15. First arrange the consonants in odd places is in 1, 3, 5 and 7 places. Now arrange the vowels in the remaining places and then use the fundamental principle.

16. Select 3 consonants from 21 consonants.

17. Select 3 girls from 5 girls and 2 boys from 8 boys then apply fundamental principle.

18. Total number of stations = 20. Select 2 stations from 20 stations.

20. Find the number of 7 letter words using the 7 letters.

22. (i) Each letter be posted in 3 ways.
(ii) Now calculate the number of ways in which four letters can be posted by using fundamental theorem of counting.

23. As the letters begin with P and ends with E, four more letters are to be selected from the remaining 5 letters.

24. (i) The units digit of the number must be odd, i.e., it can be done in 3 ways.
(ii) Now find the number of ways in which the three digits can be filled using four digits.

(iii) Use the fundamental theorem of counting.

25. (i) If unit digit is 0 or 5, then the number is divisible by 5.

(ii) If the units digit is 5, then ten thousands digit cannot be zero, now find the number of ways the other four digits can be arranged.

(iii) Similarly when the units digit is 0, the other 4 digits can be arranged in $^{5}P_{4}$ ways. Use the fundamental theorem of counting.

26. (i) The units digit of the required number is 0.

(ii) Find the number of ways in which the remaining 5 digits can be arranged in three places by using $^{n}P_{r}$.

27. (i) Each digit of the password can have 4 values, hence first digit can be filled in 4 ways.

(ii) Find the number of ways in which remaining digits can be filled.

(iii) Use the fundamental principal of counting and find total number of passwords that are formed.

28. Select 3 men from 6 men and select 4 women from 7 women, then apply the fundamental principle.

29. Select two persons from 30 persons.

30. From a group of n members selecting r members at a time is denoted by $^{n}C_{r}$

31. (i) The number of ways a boy and a girl can be selected individually is $^{20}C_{1}$ and $^{25}C_{1}$.

(ii) Use the fundamental theorem of counting.

32. All the given digits are even, so no odd number can be formed with the given digits.

33. (i) The total number of stations are 15 (say n).

(ii) On a ticket, two stations should be printed.

(iii) The required number of ways = $^{n}C_{2} \times 2$.

34. There are 5 vowels and 3 are to be chosen.

35. (i) Find the number of ways in which 3 consonants and 2 vowels can be selected from 5 consonants and 3 vowels.

(ii) Then use fundamental theorem of counting.

36. The number of lines that can be formed from n points in which m points are collinear is $^{n}C_{2} - {}^{m}C_{2} + 1$.

37. The number of triangles that can be formed from n points in which m points are collinear is $^{n}C_{3} - {}^{m}C_{3}$.

38. As Y is filled in last blank, five more letters are to be selected from the remaining 8 letters.

39. Select any two teams from 15 teams.

40. There are 6 consonents and 4 are to be chosen.

41. Arrange 3-letter words out of 6 letters.

44. The number of lines that can be formed from n points in which m points are collinear is $^{n}C_{2} - {}^{m}C_{2} + 1$.

45. The number of triangles that can be formed from n points in which m points are collinear is $^{n}C_{3} - {}^{m}C_{3}$.

46. (i) If the digit in the units place is an even number, then the number is called even number.

(ii) The units digit of the four digit number must be 0 or 4.

			0

(or)

			4

(iii) If units digit is 4, then thousands digit can be filled in 4 ways and other two digits can be filled in $^{4}P_{2}$ ways.

(iv) If units digit is 0, then find the number of ways in which the remaining 3 digits can be filled by using 5 digits

47. (i) Take 6 blanks.

(ii) As Y is filled in last blank, five more letters are to be selected from the remaining 8 letters.

48. (i) The first digit of the number cannot be zero, so it can be filled in 6 ways.

(ii) Now the second digit can be any of the 6 digits., i.e., it can be filled in 6 ways.

(iii) As the digits cannot be repeated, the number of ways the third digit can be filled is 5 ways and so on.

(iv) Now apply the fundamental theorem.

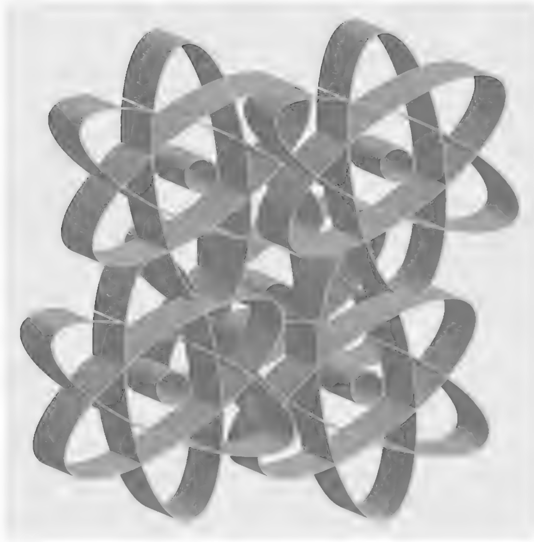
49. (i) There are 5 consonants and middle place can be filled in 5 ways.

(ii) remaining places can be filled with remaining letters.

50. Unit place is 0 or 5 then the number is divisible by 5.

CHAPTER 20

Probability



INTRODUCTION

In nature, there are two kinds of phenomena–deterministic and indeterministic. Examples of deterministic phenomena are:

(i) The sun rises in the east.
(ii) If an object is dropped, it falls to the ground.
(iii) Every organism which takes birth dies.

Examples of indeterministic phenomena are:

(i) The next vehicle that we see on a road going west to east may be headed east or west.
(ii) If a gas molecule is released in a container, it may head in any direction.
(iii) A person selected from a population may die before attaining the age of 75 years, when he attains the age of 75 or after he attains the age of 75.

Probability theory is the study of indeterministic phenomena. While the theory has widespread applications in all walks of life, it is best to confine ourselves to certain simple kind of indeterministic phenomena in the initial stage. Examples are tossing of coins, rolling dice, picking cards from well-shuffled decks and drawing objects from different containers, containing different objects. All these are examples of **random experiments**–situations in which we do something and we are not sure of the outcome, because there is more than one possible outcome.

☛ *Example* Tossing a coin, rolling a dice or drawing a card from a deck.

Sample space

The set of all possible outcomes of an experiment is called its sample space. It is usually denoted by S.

Example

1. Consider a cubical dice with numbers 1, 2, 3, 4, 5 and 6 on its faces.
   When we roll the dice, it can fall with any of its faces facing up. So, the number on each of its faces is a possible outcome.
   Hence, the sample space(S) = $\{1, 2, 3, 4, 5, 6\}$
2. When we toss an unbiased coin, the result can be a head(H) or a tail(T). So the sample space (S) = $\{H, T\}$.

Event

The outcomes or a combination of the outcomes is called an event. The probability of an event (E) is a measure of our belief that the event will occur. This may be zero, i.e., we don't expect the event to occur at all. E.g., If two dice are rolled, the probability that the sum of the numbers which will come up is 1 is zero. The probability may be 1, i.e., we are absolutely certain that the event will occur. E.g., If a coin is tossed the probability that we get a head or a tail is 1. But in general, we may believe that the event may occur but we are not absolutely certain i.e., $0 < p < 1$.

Example

1. In rolling a dice, getting an even number is an event.
2. In tossing a coin, getting a head (H) is an event.

In the general case, when we are not sure of either the occurrence or non-occurrence of an event, how do we assign a number to the strength of our belief?

Consider the case of rolling a dice. It will show up one of the numbers-1, 2, 3, 4, 5 or 6. Each of these outcomes is equally likely, if the dice is an unbiased dice, i.e., well-balanced. So, when a dice is rolled once, there is only one way of getting the outcome 5, out of the six possible outcomes 1, 2, 3, 4, 5 or 6. Therefore, the probability of the number 5 showing up is 1 in 6. In other words, we say that the probability of getting 5 is $\frac{1}{6}$.

We write this as $P(5) = \frac{1}{6}$.

Also, the probability of getting each of the other numbers is equal to $\frac{1}{6}$.

Similarly, in tossing an unbiased coin, we can say that the probability of getting a head or the probability of getting a tail is $\frac{1}{2}$.

Probability of an event

Let E be an event of a certain experiment whose outcomes are equally likely. Then, the probability of the event E, denoted by P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Total number of possible outcomes}}.$$

NOTE:

1. For any event E, $0 \le P(E) \le 1$
2. If $P(E) = 0$, then E is said to be an impossible event.
3. If $P(E) = 1$, then E is said to be a certain or a sure event.

☛ *Example*

Find the probability of getting an even number when an unbiased dice is rolled once.

Solution

When a dice is rolled, the total number of possible outcomes is 6.

Let E be the event of getting an even number. Then, the outcomes favourable to E are 2, 4 and 6, i.e., 3 outcomes are favourable.

Hence, $P(E) = \frac{3}{6} = \frac{1}{2}$

☛ *Example*

When an unbiased dice is rolled once, what is the probability of getting a multiple of 3?

Solution

When an unbiased dice is rolled, the total number of possible outcomes is 6.

Let E be the event of getting a multiple of 3.

Then, the outcomes favourable to E are 3 and 6, i.e., 2 outcomes are favourable.

Hence, $P(E) = \frac{2}{6} = \frac{1}{3}$

Probability of non-occurrence of an event E

Let a random experiment have n possible outcomes – all equally likely. Say m of these are favourable for an event E. Then, there are $(n - m)$ outcomes which are not favourable to the event E.
Let $\bar{E}$ denote the non-occurrence of E.

Then, $P(\bar{E}) = \frac{n-m}{n}$,

i.e., $P(\text{non-occurrence of } E) = \frac{n-m}{n}$

Now, $P(E) + P(\bar{E})$

$$= \frac{m}{n} + \frac{n-m}{n} = \frac{m+(n-m)}{n} = \frac{n}{n} = 1$$

i.e., $P(E) + P(\bar{E}) = 1$

Classification of a pack (or deck) of cards

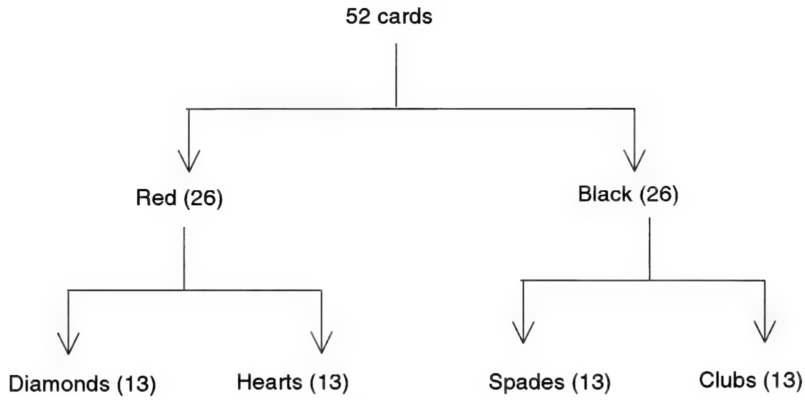


Figure 21.1

The cards in each suit are ace(A), king(K), queen(Q), Jack(J), 10, 9, 8, 7, 6, 5, 4, 3 and 2.

The cards A, J, Q and K are called honours and the cards 2, 3, 4, 5, 6, 7, 8, 9 and 10 are called numbered cards. The cards J, Q and K are called face cards.

Worked out examples

Example

1. When a fair dice is rolled, what is the probability of getting a number less than 5?

Solution

When a fair dice is rolled, the total number of possible outcomes is 6.

Let E be the required event. Then, the outcomes favourable to E are 1, 2, 3 and 4,.
i.e., 4 favourable outcomes.

$\Rightarrow P(E) = \frac{4}{6} = \frac{2}{3}$

Hence, the probability of getting a number less than 5 is $\frac{2}{3}$.

2. When a dice is rolled, what is the probability of getting a number 2 or 3?

Solution

Total number of possible outcomes = 6

Favourable outcomes are 2 and 3.

i.e., 2 favourable outcomes.

$\therefore$ Required probability $= \frac{2}{6} = \frac{1}{3}$

3. A number is chosen randomly from the set of integers from 1 to 20. What is the probability that it will be divisible by 5?

Solution

There are 20 integers from 1 to 20.

So, an integer can be selected from 1 to 20 in 20 ways.

Let E be the required event. Then, the numbers favourable to E are 5, 10, 15 and 20, i.e., 4 favourable numbers.

$\Rightarrow P(E) = \frac{4}{20} = \frac{1}{5}$

Hence, the probability that the number selected is divisible by 5 is $\frac{1}{5}$.

4. A card is selected at random from a pack of cards. What is the probability that it will be a red card?

Solution

There are 52 cards in a pack of cards.

So, a card can be selected from it in 52 ways.

Out of the 52 cards, 26 cards are red coloured.

So, a red card can be selected in 26 ways.

Hence, required probability $= \frac{26}{52} = \frac{1}{2}$

5. When a card is selected at random from a pack of cards, find the probability that it is a king.

Solution

There are 52 cards in a pack of cards.

So, a card can be selected in 52 ways.

Now, there are 4 kings (one in each suit) in the pack.

So, a king can be selected in 4 ways.

$\therefore$ The required Probability $= \frac{4}{52} = \frac{1}{13}$.

6. A number is selected from the numbers 1 to 20. What is the probability that it will be a prime number.

Solution

Total number of ways of selecting a number from 1 to 20 is 20.

Let E be the event that the number selected is a prime number.

Then the numbers favourable to E are 2, 3, 5, 7, 11, 13, 17 and 19.

i.e., 8 favourable numbers.

$\therefore P(E) = \frac{8}{20} = \frac{2}{5}$.

Hence, the probability that the number selected will be a prime number is $\frac{2}{5}$.

7. A bag contains 3 blue and 7 red balls. Find the probability that a ball selected at random from the bag will be a blue ball.

Solution

Total number of balls in the bag = 3 + 7 = 10.

So, a ball can be selected from the bag in 10 ways.

Now, there are 3 blue balls in the bag.

So, a blue ball can be selected from the bag in 3 ways.

Hence, the required probability = $\frac{3}{10}$.

8. Find the probability of a card that is selected at random from a pack of cards will be a red honour.

Solution

Total number of ways of selecting a card = 52.

There are 8 honours in the 26 red cards.

So, a red honour can be selected in 8 ways.

$\therefore$ The required probability = $\frac{8}{52} = \frac{2}{13}$

9. There is a bunch of 10 keys out of which any one of the 4 keys can unlock a door. If a key is selected at random from the bunch and tried on the door, find the probability that the door will be unlocked.

Solution

Total number of ways of selecting a key = 10.

4 keys can unlock the door.

$\Rightarrow$ favourable outcomes = 4.

$\therefore$ The required probability = $\frac{4}{10} = \frac{2}{5}$.

10. When two unbiased coins are tossed, what is the probability that both will show a head?

Solution

When two coins are tossed together, the possible outcomes are HH, HT, TH or TT,

i.e., 4 possible outcomes.

Also, there is only one case where both the coins show heads.

$\therefore$ The required probability = $\frac{1}{4}$.

test your concepts

Very short answer type questions

1. An experiment in which all the outcomes of the experiment are known in advance and the exact outcome is not known in advance is called a __________.
2. A set of events which have no pair in common are called ________.
3. When a coin is tossed, the probability of getting neither head nor tail is called ________ event.
4. When a dice is thrown, the events {1}, {2}, {3}, {4}, {5} and {6} are called _________ events.
5. When a number is selected from a set of natural numbers, the probability of getting a number which is a multiple of 1 is ________.
6. The set of all possible outcomes of an experiment is called the __________.
7. When a dice is thrown, the probability of getting any one of the numbers from 1 to 6 on the upper face is called _______ event.
8. The probability of any event of a random experiment can not exceed ___________.
9. If the probability of an event of a random experiment is P(E) = 0, then the event is called _________.
10. If A is any event in a sample space, then $P(A^1)$ = _________.
11. The probability of getting multiples of 3 when a dice is thrown is ________.
12. The range of probability of any event of a random experiment is ______.
13. If A is an event of a random experiment, then A^c or $\overline{A}$ or A^1 is called the _______ of the event.
14. The probability of getting 5 when a fair dice is thrown is ________.
15. A dice is thrown once. What is the probability of getting a multiple of 2?
16. If a card is drawn from a well-shuffled pack of 52 cards, then what is the probability of getting either a spade or a diamond?
17. If a dice is thrown once, then find the probability of getting an odd prime number.
18. If one card is drawn from a well-shuffled pack of 52 cards, then what is the probability of getting either a red 6 or a red 8?
19. If two coins are tossed, then find the probability of getting two tails.
20. If three coins are tossed, then find the probability of getting three tails.
21. A bag contains 8 red balls, 4 blue balls and some green balls. The probability of drawing a green ball is the sum of the probabilities of drawing a red ball and a blue ball. Find the number of green balls.
22. If a coin is tossed three times, find the probability that no two successive tosses will show the same face.
23. If two dice are rolled, find the probability that numbers coming up on both the dice will be multiples of 3.

24. Each of two persons a single throw with a dice. What is the probability of getting the same number on both the dice?

25. A cubical die is rolled. Find the probability of getting a composite number.

26. A set contains numbers from 40 to 60. Krishna chooses a number from the set. Find the probability that the number chosen is a prime number.

27. A college offers 8 subjects, including Mathematics, to its intermediate students. Each student has to opt for 3 subjects. If Anand opts for Mathematics, then what is the probability that he will opt for 2 other specified subjects?

28. If n coins are tossed, then find the probability of getting either all heads or all tails.

29. If 8 boys are arranged in a row, what is the probability that 3 particular boys always sit together?

30. If a 5-digit number is formed by using the digits 1, 2, ….. 9 (without repetitions), then what is the probability that it will be an even number?

Short answer type questions

31. A bag contains 30 balls out of which 15 are red balls, w are white balls and g are green balls. The probability of getting a white ball is two times that of getting a green ball. Find the number of white balls and green balls.

32. A card is selected at random from a pack of cards. What is the probability that it will be an ace?

33. A number is selected from the set of integers from 1 to 20. What is the probability that it will be divisible by both 2 and 3?

34. A number is selected at random from the set of integers 1 to 100. What is the probability that it will be a multiple of 10.

35. What is the probability of arranging the letters of the word "CHEMISTRY" such that the arrangements start with C and end with Y?

36. When two cards are drawn from a pack of cards, find the probability that both will be diamonds.

37. A number is selected at random from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. What is the probability that it will be a root of the equation $x^2 - 2x + 1 = 0$?

38. A speaks truth in 70% cases. What is the probability that A will lie in stating a fact?

39. An urn contains 5 red, 3 blue and 2 green balls. What is the probability of selecting a green or a red ball?

40. A card is selected at random from a pack of cards. What is the probability that it is a numbered card?

41. When a die is rolled, what is the probability of getting a number which is a multiple of both 2 and 3?

42. 50 cards marked with numbers 1 to 50 are placed in a box. If a card is selected randomly from the box, then the probability that the number on the card selected will be a perfect cube is

43. The probability of A winning a game is 0.7. Then the probability of A losing the game is

44. For a cricket team, 11 persons have to be selected out of 20 aspirants. If 6 of the aspirants are definitely selected, and the other are selected at random, then what is the probability that 5 particular aspirants of the remaining 14 will be selected?

45. When two dice are rolled together, what is the probability of getting the same faces on them?

Essay type questions

46. What is the probability that date of birth of a person is in the month of January?

47. A card is selected from a pack of cards, what is the probability that it is either spade or diamond?

48. In a six faced dice, two of the faces are painted red, two of the faces are painted black and the other two faces are painted blue. Such two dice are rolled. The probability that both the dice shows same colour is _________.

49. One number is selected from the first 50 natural numbers. What is the probability that it is a root of the inequation?

$$x + \frac{256}{x} > 40?$$

50. A card is drawn from a well shuffled pack of cards, what is the probability that it is either king or heart?

CONCEPT APPLICATION

Concept Application Level—1

1. What is the probability that a non-leap year has 53 Sundays?

(1) $\frac{6}{7}$ (2) $\frac{1}{7}$ (3) $\frac{5}{7}$ (4) None of these

2. A number is selected at random from the integers 1 to 100. What is the probability that it will be a multiple of 4 or 6?

(1) $\frac{8}{25}$ (2) $\frac{33}{100}$ (3) $\frac{17}{50}$ (4) $\frac{41}{100}$

3. Two numbers are selected at once from the set of integers 1 to 20. Find the probability that the product of the numbers will be 24.

(1) $\frac{3}{190}$ (2) $\frac{3}{380}$ (3) $\frac{4}{95}$ (4) $\frac{3}{95}$

4. An urn contains 4 red and 6 green balls. A ball is selected at random from the urn and is replaced back into the urn. Now a ball is drawn again from the bag. What is the probability that the first ball is a red ball and the second is a green ball?

(1) $\frac{6}{25}$ (2) $\frac{8}{25}$ (3) $\frac{5}{6}$ (4) $\frac{4}{15}$

5. A box contains 5 apples, 6 oranges and 'x' bananas. If the probability of selecting an apple from the box is $\frac{1}{3}$, then the number of bananas in the box is

(1) 4 (2) 6 (3) 8 (4) 5

6. Two dice are rolled simultaneously. Find the probability that they show different faces.

(1) $\frac{2}{3}$ (2) $\frac{1}{6}$ (3) $\frac{1}{3}$ (4) $\frac{5}{6}$

7. A number is selected from first 50 natural numbers. What is the probability that it is a multiple of 3 or 5?

(1) $\frac{13}{25}$ (2) $\frac{21}{50}$ (3) $\frac{12}{25}$ (4) $\frac{23}{50}$

8. Two numbers are selected from the set of integers 1 to 25. what is the probability that the product of the numbers will be 36.

(1) $\frac{1}{200}$ (2) $\frac{1}{100}$ (3) $\frac{1}{50}$ (4) $\frac{1}{75}$

9. Tom sold 100 lottery tickets in which 5 tickets carry prizes. If Jerry purchased a ticket, what is the probability of Jerry winning a prize?

(1) $\frac{19}{20}$ (2) $\frac{1}{25}$ (3) $\frac{1}{20}$ (4) $\frac{17}{20}$

10. When two coins are tossed together, what is the probability that neither of them shows up head?

(1) $\frac{1}{3}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{1}{4}$

11. When two cards are drawn from a well-shuffled pack of cards, what is the probability that both will be aces?

(1) $\frac{1}{221}$ (2) $\frac{2}{221}$ (3) $\frac{1}{13}$ (4) $\frac{1}{231}$

12. Two cards are drawn from a pack of cards one after another so that the first card is replaced before drawing the second card. What is the probability that the first card is an ace and the second is a number card?

(1) $\frac{9}{169}$ (2) $\frac{1}{52}$ (3) $\frac{1}{4}$ (4) $\frac{17}{52}$

13. Two numbers 'a' and 'b' are selected (successively without replacement in that order) from the integers 1 to 10. What is the probability that $\frac{a}{b}$ will be an integer?

(1) $\frac{17}{45}$ (2) $\frac{1}{5}$ (3) $\frac{19}{90}$ (4) $\frac{8}{45}$

14. There are 4 different mathematics books, 5 different physical science books and 3 different biological science books on a shelf. What is the probability of selecting a mathematics book from the shelf?

(1) $\frac{1}{3}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{5}{12}$

15. A number is selected at random from the integers 1 to 100. What is the probability that it is an even number?

(1) $\frac{1}{2}$ (2) $\frac{1}{50}$ (3) $\frac{1}{3}$ (4) $\frac{49}{100}$

16. A three digit number is to be formed using the digits 3, 4, 7, 8 and 2 without repetition. What is the probability that it is an odd number?

(1) $\frac{2}{5}$ (2) $\frac{1}{5}$ (3) $\frac{4}{5}$ (4) $\frac{3}{5}$

17. Two cards are drawn from a pack, what is the probability that the two cards are of different colours?

(1) $\frac{8}{17}$ (2) $\frac{1}{12}$ (3) $\frac{26}{51}$ (4) $\frac{27}{52}$

18. Two numbers 'a' and 'b' are selected successively one after another without replacement from the integers 1 to 15. What is the probability that $\frac{a}{b}$ will be an integer?

(1) $\frac{5}{7}$ (2) $\frac{3}{7}$ (3) $\frac{2}{7}$ (4) $\frac{1}{7}$

19. A card is drawn at random from a pack of cards. What is the probability that it is a face card of spades?

(1) $\frac{1}{13}$ (2) $\frac{1}{26}$ (3) $\frac{3}{52}$ (4) $\frac{3}{13}$

20. Two numbers are selected simultaneously from the first 20 natural numbers. What is the probability that the sum of the numbers is odd?

(1) $\frac{10}{19}$ (2) $\frac{1}{19}$ (3) $\frac{1}{10}$ (4) $\frac{19}{20}$

21. Two dice are rolled together, what is the probability that the total score on the two dice is greater than 10?

(1) $\frac{5}{6}$ (2) $\frac{1}{4}$ (3) $\frac{1}{6}$ (4) $\frac{1}{12}$

22. Two numbers are selected simultaneously from the first 25 natural numbers, what is the probability that the sum of the numbers is even?

(1) $\frac{6}{25}$ (2) $\frac{12}{25}$ (3) $\frac{4}{25}$ (4) $\frac{8}{25}$

23. A bag contains 7 red and 7 black coloured balls. A person drawn two balls from the bag, what is the probability that the two balls are the same in colour?

(1) $\frac{6}{13}$ (2) $\frac{2}{7}$ (3) $\frac{4}{13}$ (4) $\frac{1}{13}$

24. Two dice are rolled together, what is the probability that the total score on the two dice is a prime number?

(1) $\frac{5}{12}$ (2) $\frac{1}{3}$ (3) $\frac{1}{2}$ (4) $\frac{13}{36}$

25. A committee of 4 persons is to be formed from 6 men and 4 women. What is the probability that the committee consists of equal number of men and women?

(1) $\frac{4}{7}$ (2) $\frac{3}{7}$ (3) $\frac{1}{7}$ (4) $\frac{6}{7}$

26. All the cards in an ordinary deck of 52 cards are numbered from 1 to 52. If a card is drawn at random from the deck, then what is the probability that it will have a prime number?

(1) $\frac{7}{26}$ (2) $\frac{15}{52}$ (3) $\frac{5}{17}$ (4) $\frac{4}{13}$

27. A bag contains 5 pens and 6 pencils. If a boy selects 2 articles from the bag, then what is the probability that the selected articles will be a pen and a pencil?

(1) $\frac{2}{11}$ (2) $\frac{3}{11}$ (3) $\frac{6}{11}$ (4) $\frac{5}{11}$

28. An urn contains 5 red, 3 black and 2 white. If three balls are chosen at random, then what is the probability that they will be of different colours?

(1) $\frac{1}{4}$ (2) $\frac{3}{4}$ (3) $\frac{1}{2}$ (4) $\frac{5}{6}$

29. What is the probability that a leap year has 52 Mondays?

(1) $\frac{2}{7}$ (2) $\frac{4}{7}$ (3) $\frac{5}{7}$ (4) $\frac{6}{7}$

30. If a letter is selected at random from the letters of the word ‘FOUNDATION’, what is the probability of the letter selected being a repeated letter?

(1) $\frac{1}{10}$ (2) $\frac{1}{5}$ (3) $\frac{2}{5}$ (4) $\frac{1}{2}$

Concept Application Level—2

31. Sunny and Bunny go to an ice cream parlour where they find 5 different varieties of ice creams. If they order one ice cream each, what is the probability that they both order the same variety of ice creams?

(1) $\frac{1}{5}$ (2) $\frac{4}{5}$ (3) $\frac{1}{2}$ (4) $\frac{1}{10}$

32. There are five Re 1 coins, two Rs 2 coins and three Rs 5 coins. If two coins are selected simultaneously at random, what is the probability of yielding the maximum amount?

(1) $\frac{3}{10}$ (2) $\frac{1}{5}$ (3) $\frac{1}{15}$ (4) $\frac{3}{10}$

33. A dice rolled twice, what is the probability that the two dice show a different number?

(1) 2/3 (2) 1/6 (3) 5/6 (4) 1/2

34. A bag contains 3 red, 5 blue and 7 green coloured balls. Find the probability of selecting a blue ball from the bag.

(1) $\frac{3}{15}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{2}{3}$

35. 7 coins are tossed simultaneously, what is the probability of getting atleast two heads?

(1) 3/18 (2) 15/16 (3) 1/16 (4) 3/16

36. A bag contains 5 white balls, and 6 green balls. Two balls are drawn from the bag one after another, what is the probability that both the balls are white? (the first ball is replaced before drawing the second ball)

(1) 2/11 (2) 1/11 (3) $\frac{3}{121}$ (4) $\frac{25}{121}$

37. A basket contains 12 fruits, of which 7 are not rotten. When one fruit is drawn at random, the probability that it is a rotten fruit is _________.

(1) 1/3 (2) 7/12 (3) 5/12 (4) 2/3

38. One card is selected from a well shuffled pack of cards, what is the probability that it is a red honored card?

(1) 2/13 (2) 3/13 (3) 11/13 (4) 9/13

39. A bag contains 5 black balls, and 7 green balls. Two balls are drawn simultaneously at random, what is the probability that both are different in color?

(1) 17/33 (2) 35/132 (3) 5/66 (4) 35/66

40. The probability that the month of April has exactly 5 Mondays is

(1) 4/7 (2) 5/7 (3) 3/7 (4) 2/7

41. Hundred cards marked with numbers 1 to 100 are placed in a box. If a card is selected randomly from the box, then the probability that the number on the card selected will be a perfect square is

(1) $\frac{1}{100}$ (2) $\frac{1}{25}$ (3) $\frac{1}{10}$ (4) $\frac{9}{10}$

42. 20 cards are numbered 1 to 20. If a card is selected at random, then what is the probability that the selected number will be an odd prime?

(1) 2/5 (2) 3/5 (3) 3/10 (4) 7/20

43. A four digit number is formed by using the digits 1, 2, 5, 6 and 8 without repetition. What is the probability that it will be an even number?

(1) $\frac{3}{5}$ (2) $\frac{2}{5}$ (3) $\frac{1}{2}$ (4) $\frac{3}{10}$

44. A basket contains 10 fruits of which 3 are rotten. If one fruit is drawn from the basket, then the probability that the fruit drawn is not rotten is ________.

(1) 4/5 (2) 4/5 (3) 7/10 (4) 3/10

45. An urn contains 6 blue and 'a' green balls. If the probability of drawing a green ball is double that of drawing a blue ball, then 'a' is equal to

(1) 6 (2) 18 (3) 24 (4) 12

Concept Application Level—3

46. If one number is selected from the first 70 natural numbers, the probability that the number is a solution of $x^2 + 2x > 4$ is

(1) 69/70 (2) 1/70 (3) 1 (4) 0

47. A man's pocket has seven Re 1 coins, three Rs 2 coins and four Rs 5 coins. If two coins are selected simultaneously, what is the probability of yielding the minimum amount?

(1) $\frac{3}{13}$ (2) $\frac{6}{13}$ (3) $\frac{3}{26}$ (4) $\frac{6}{43}$

48. Mr Balaram picked a prime number between the integers 1 and 20. What is the probability that it will be number 13?

(1) $\frac{7}{8}$ (2) $\frac{1}{20}$ (3) $\frac{1}{8}$ (4) $\frac{13}{20}$

49. Chandu picks up a letter from the English alphabet and finds it to be a vowel. Find the probability that it is either e or i.

(1) $\frac{2}{5}$ (2) $\frac{3}{5}$ (3) $\frac{1}{5}$ (4) $\frac{4}{5}$

50. A committee of 5 persons is to be formed from 7 men and 3 women. What is the probability that the committee contains 3 men?

(1) $\frac{5}{36}$ (2) $\frac{7}{12}$ (3) $\frac{5}{12}$ (4) $\frac{1}{3}$

KEY

Very short answer type questions

1. random experiment.
2. mutually exclusive.
3. impossible event
4. simple or equally likely
5. 1
6. sample space
7. sure event
8. 1
9. impossible event
10. $1 - P(A)$
11. $\frac{1}{3}$
12. [0, 1]
13. complement
14. $\frac{1}{6}$
15. $\frac{1}{2}$
16. $\frac{1}{2}$
17. $\frac{1}{3}$
18. $\frac{1}{13}$
19. $\frac{1}{4}$
20. $\frac{1}{8}$
21. 12
22. $\frac{1}{4}$
23. $\frac{1}{9}$
24. $\frac{1}{6}$
25. $\frac{1}{3}$
26. $\frac{5}{21}$
27. $\frac{{}^7C_2}{{}^8C_3}$
28. $\frac{1}{2^n - 1}$
29. $\frac{3}{28}$
30. $\frac{4}{9}$

Short answer type questions

31. $\frac{1}{3}$
32. $\frac{1}{13}$
33. $\frac{3}{20}$
34. $\frac{1}{10}$
35. $\frac{1}{72}$
36. $\frac{1}{17}$
37. $\frac{1}{10}$
38. $\frac{3}{10}$
39. $\frac{7}{10}$
40. $\frac{9}{13}$
41. $\frac{1}{6}$
42. $\frac{3}{50}$
43. 0.3
44. $\frac{1}{2002}$
45. $\frac{1}{6}$

Essay type questions

46. $\frac{1}{12}$
47. $\frac{1}{2}$
48. $\frac{1}{3}$
49. $\frac{1}{2}$
50. $\frac{4}{13}$

key points for selected questions

Very short answer type questions

15. The number of possible outcomes is 6 when a dice is thrown, find the number of outcomes which are multiples of 2.

16. A pack of cards contain spade and diamond each of 13 cards.

17. The number of possible outcomes is 6 when a dice is thrown, find the number of odd primes less than 6.

18. In a pack of cards, 2 red cards with number 6 and 2 red cards with number 8 exists.

19. If n coins are tossed, the probability of getting exactly r tails is $\frac{nc_r}{2^n}$.

20. If n coins are tossed, the probability of getting exactly r tails is given by $\frac{nc_r}{2^n}$.

21. (1) Assume the number of green balls as x'.
(2) P (Green ball) = P (Red ball) + P (Blue ball).

22. List the outcomes when a coin is tossed three times and then find the favourable outcomes.

23. Find the number of favourable cases for both the dice showing up the numbers which are multiples of 3.

24. Find the possibilities such that the two dice show the same number.

25. (1) $P(R) + P(W) + P(G) = 1$
(2) Given, $P(W) = 2P(G)$ where R, G and W represent Red, Green and White balls.

26. Find the number of prime numbers from 40 to 60.

27. Anand already opt mathematics. Now he has to opt two more from remaining 7 subjects.

28. If n coins are tossed, the probability of getting all heads is $\frac{1}{2^n}$.

29. (i) Assume the 3 particular boys as one unit, arrange the other boys including this unit.
(ii) Three particular boys can be arranged themselves in 3! Ways.

30. (i) Find the number of ways of arranging the numbers keeping unit's place with even numbers.
(ii) Arrange the other numbers in other places.
(iii) Find all the 5 – digit numbers formed from digits 1 to 9.

Short answer type questions

31. Find the number of favourable outcomes.

32. There are 4 ace cards in a pack of 52 cards.

33. Find the number of favourable outcomes. (Number should be divisible by 6).

34. Find the number of favourable outcomes.

35. Fix the end positions with C and Y and arrange the other letters in other places.

36. There are 13 diamonds in a pack of 52 cards.

37. Find the roots of the equation.

38. A speaks lie in 30% cases.

39. Find, P (green ball) + P (red ball).

40. There are a total of 36 numbered cards.

41. Number should be a multiple of 6.

42. Find the number of perfect cubes from 1 to 50.

43. Probability (winning) + probability (losing) a game = 1.

44. Apply combination concept.

45. Find the number of favourable outcomes.

Essay type questions

46. The number of months in a year is 12.

47. There are 13 spades and 13 diamonds in a pack of cards.

48. (i) If, each dice is rolled we get 3 colors. Two dice are rolled, so total out comes are 9.
(ii) List out the favorable out comes.

49. (i) Solve inequation
(ii) Find favorable values of x, the inequation is satisfied.

50. Apply addition theorem on probability.

Concept Application Level—1,2,3

1. 2	14. 1	27. 3	40. 4
2. 2	15. 1	28. 1	41. 3
3. 1	16. 1	29. 3	42. 4
4. 1	17. 3	30. 2	43. 1
5. 1	18. 4	31. 1	44. 3
6. 4	19. 3	32. 3	45. 4
7. 4	20. 1	33. 3	46. 1
8. 2	21. 4	34. 2	47. 1
9. 3	22. 2	35. 2	48. 3
10. 4	23. 1	36. 4	49. 1
11. 1	24. 1	37. 3	50. 3
12. 1	25. 2	38. 1	
13. 3	26. 2	39. 4	

Concept Application Level—1,2,3

Key points for select questions

1. (i) A non-leap year contains 365 days and there is only one odd day.
(ii) 52 weeks contains 52 Sundays. Find the probability that odd day is Sunday.

2. P(either multiple of 4 or 6) = P(multiple of 4) + P(multiple of 6) – P(multiple of both 4 and 6).

3. (i) Find the number of pairs a and b such that $a \times b = 24$ and $1 \leq a, b \leq 20$.
(ii) Probability of the required event $= \dfrac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$.

4. P(R and G) = P(R).P(G).

5. P(selection apple) $= \dfrac{5}{5+6+x} = \dfrac{1}{3}$, and find x.

6. (i) Find the probability that they show the same faces.
(ii) P(Showing different faces) = 1 – P(showing same faces)

7. P(either multiple of 3 or of 5) = P(multiple of 3) + P(multiple of 5) – P (multiple 3 and 5).

8. (i) Find the number of pairs (a, b), such that $a \times b = 36$ and $1 \leq (a, b) \leq 25$.
(ii) Total number of outcomes $= {}^{25}C_2$.

9. Find the number of favourable outcomes.

10. Both coins should show up tail.

11. There are 4 ace cards in pack of cards.

12. (i) There are 4 ace cards and 36 numbered cards.
(ii) P(ace and number card) = P(ace card) × P(number card).

13. $b = 1, a = 2, 3, \ldots\ldots, 10$.
$b = 2, a = 4, 6, \ldots\ldots, 10$.

In this way, find the number of favourable events such that a/b is an in integer.

14. (i) Number of favourable outcomes = 4.
(ii) Required probability $= \dfrac{\text{Number of mathematics books}}{\text{Total number of books in the shelf}}$

15. There are 50 even numbers from 1 to 100.

16. (i) If a number ends with an odd number then it is odd number.
(ii) Find the number of 3 digit odd numbers formed by using the digits 3, 4, 7, 8 and 2.
(iii) Find the total number of 3 digit numbers formed by using the digits 3, 4, 7, 8 and 2.

17. (i) There are 26 red colour and 26 black colour cards.
(ii) Select one red card and one black card
(iii) Total number of out comes = ${}^{52}C_2$

18. Let $b = 1$, then $a = 2$ to 15.

Let $b = 2$, then $a = 4, 6, \ldots\ldots 14$.

In this way, find numbers a and b such that $\dfrac{a}{b}$ is the integer.

19. Jack, Queen and King are the face cards.

20. (i) One number is odd and other is even then sum is odd.
(ii) Two numbers from 20 can be selected in ${}^{20}C_2$ ways.
(iii) There are 10 even numbers and 10 odd numbers.

21. P(greater than 10) = P(sum be 11) + P(sum be 12).

22. (i) (odd + odd) = (even + even) = even.
(ii) There are a total of 13 odd numbers and 12 even numbers.
(iii) P(sum even) = P(selecting 2 odd numbers) + P(selecting 2 even numbers)

23. (i) Number of ways of drawing 2 red balls = ${}^{7}C_2$, similarly number of ways of drawing 2 black balls = ${}^{7}C_2$.
(ii) Total number of ways of drawing 2 balls out of 14 balls = ${}^{14}C_2$

24. P(sum is prime) = P(sum 2) + P(sum 3) + P(sum 5) + P(sum 7) + P(sum 11).

25. (i) Committee should be formed with 2 men and 2 women.
(ii) Total number of ways in which the committee be selected is $(6 + 4)c_4$ ways.

26. (i) Find the numbers which are prime from 1 to 52.
(ii) In total we have to select 1 card out of 52 cards.

27. (i) Find the number of ways of selecting one pen and one pencil from the bag.
(ii) In total, the boy has to select 2 out of 11 articles.

28. (i) Choose one ball from each colour.
(ii) In total, we have to select 3 out of 10 balls.

29. (i) Leap year contains 52 weeks and 2 odd days.
(ii) Find the probability that the odd days should not be monday.

30. Find the number of repeated letters in the given word.

31. (i) Let the five different ice creams be a, b, c, d and e.
(ii) Favourable outcomes are (a, a), (b, b), (c, c), (d, d) and (e, e).
(iii) Total number of outcomes = 25.

32. (i) Maximum amount will be yielded, if the selected two coins are of Rs 5.
(ii) Required probility $= \dfrac{\text{Number of ways of selecting two Rs 5 coins}}{\text{Total number of ways of selecting two coins}}$.

41. Find the number of favourable outcomes.

42. (i) Find the numbers which are odd primes from 1 to 20.
(ii) In total, we have to select 1 card out of 20.

43. (i) Find the number of 4-digit even numbers by filling the units digit by 2, 6 and 8.
(ii) Find the total number of 4 digit numbers. Required probability

$$= \frac{\text{Number of 4} - \text{digit even numbers}}{\text{Total number of 4} - \text{digit numbers}}.$$

44. (i) To yield minimum amount, select Re 1 coins.
(ii) Required probability $= \dfrac{\text{Number of ways of selecting two Re 1 coins}}{\text{total number of ways of selecting two coins}}$

45. (i) Total number of balls in the urn = 6 + a.
(ii) P(Drawing a green ball) = 2 P(Drawing a blue ball)
(iii) Using above condition find the value of a.

46. P(blue ball) $= \dfrac{\text{Number of blue balls in the bag}}{\text{Total number of balls in the bag}}$.

48. (i) Number of favourable events = 1.
(ii) There are 8 prime numbers from 1 to 20.

49. (i) P(either e or i) = P(e) + P(i).
(ii) Total outcomes = 5, i.e., (a, e, i, o, u).

50. (i) Committee should consists of 3 men and 2 women.
(ii) P(committee of 5 persons) = number of ways of selecting 3 men and 2 women to the number of ways of selecting 5 persons

CHAPTER 22

Banking

INTRODUCTION

Before people started using money, the purchase and sale of goods was done through the exchange of goods. This system was called Barter system. As there was no uniformity in the valuation of various goods in this system, the value of each of the goods was converted in terms of money, thus ensuring that a proper value was paid for the goods purchased.

Once the monetary system came into existence, there arose the necessity to ensure the safety of money. With an intention of safeguarding the money and also to facilitate the availability of money to all the people in the society, the banking system was developed.

Among the various services provided by the banks, taking deposits and providing loans are primary. Apart from these, banks provide the following ancillary services.

(i) **Remittance of funds**

Banks help in transferring the money from one place to another in a safe manner, through the issue of demand drafts, money transfer orders and telegraphic transfers. Banks also issue travellers' cheques to travellers in domestic as well as in foreign currency, by which travellers can reduce the risk of theft or loss of money in travel. These travellers' cheques can be converted into money very easily. With the advent of technology, now-a-days money transfer has become very simple through internet and phone banking.

(ii) **Safe deposit lockers**

Banks provide safety lockers for the customers to keep their valuables safely. Customers can keep their valuables like gold ornaments, important documents etc., in these lockers for a nominal rent charged by the bank.

(iii) **Public utility services**

Through bank accounts, the customers can pay their telephone bills, electricity bills, insurance premiums, etc.

Deposit accounts

Deposit accounts provided by the banks are designed in such a way that they cater to various needs of their customers according to their financial capabilities. Following are the different kinds of deposit accounts provided by banks.

(a) Savings bank account

Individuals, either residents or non-residents, can open these accounts with a minimum balance of Rs 500. The minimum balance may vary from Bank to Bank and place to place. A passbook is issued to the customer, which contains all the particulars of the transactions and the balance.

An account can be opened in joint names also. It is known as a Joint Account. If one of the joint account holders is a minor, the following guidelines are applicable. A minor who is at least ten years old can open an account in a Bank or a Post Office. However the minimum age to open an account and to operate an account varies from Bank to Bank and Post Office. In a post office the minimum age to open and operate an account is 10 years. If the minor's minimum age to operate an account is more than his/her minimum age to open an SB account, then, in such cases a guardian can operate the minor's account. Savings bank accounts carry a certain interest compounded half-yearly. The rate of interest varies from bank to bank. It may also vary from time to time. Cheque books are issued to an account holder against a requisition slip duly signed by him/her. If a customer operates his account through a cheque, it is known as **'cheque operated account'**.

Depositing money in the bank accounts

Money can be deposited in a bank either by cash or through a duly filled pay-in-slip or challan. Pay-in-slips can be used for depositing through cash or cheque.

Demand draft

Money can be deposited through demand drafts (bank drafts). A person who wants to send money to another can purchase a bank draft.

A bank draft is an order issued by a bank to its specified branch or to another bank (if there is a tie up) to make payment of the amount to the party, in whose name the draft is issued.

The purchaser of the draft specifies the name and address of the person to whom the money is being sent, which is written on the bank draft. The payee can encash the drafts by presenting it at the specified branch or bank.

Withdrawing money from a savings bank account

Money deposited in these accounts can be withdrawn using withdrawal slips or cheques. A specimen of a cheque is given below.

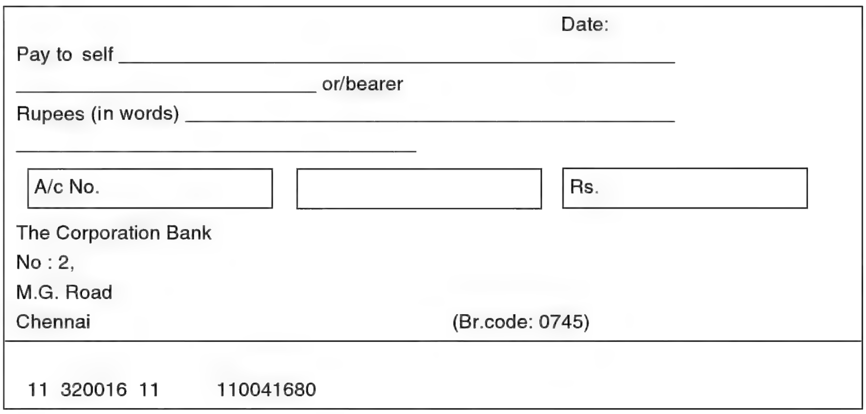


Figure 22.1

Cheque books are issued only to those account holders, who fulfil certain special requirements such as maintenance of minimum balance, etc.

Types of cheques

A cheque can be classified into two types

1. Bearer cheque
2. Crossed cheque

1. Bearer cheque

A bearer cheque can be encashed by any one who possesses the cheque, though his name is not written on the cheque. There is a risk of wrong persons getting the payments.

If the word **'bearer'** is crossed out in the cheque, then the person whose name appears on the cheque can alone encash the cheque. This type of cheque is known as an **order cheque.**

2. Crossed cheque

If two parallel lines are drawn at the left hand top corner of a cheque leaf, it is called a **crossed cheque**. The words '**A/C payee**' may or may not be written between the two parallel lines. The payee has to deposit the crossed cheque in his/her account. The collecting bank collects the money from the Bank of the drawer and it is credited to the payee's account.

Bouncing of cheques

If an account holder issues a cheque for an amount exceeding the balance in his account, the bank refuses to make payment. In such a case, the cheque is said to be **dishonoured.** This is known as **bouncing of cheque.** If a cheque bounces, the issuer of the cheque is liable for punishment under the **Negotiable Instruments Act, 1887.**

Safeguards to be taken while maintaining "cheque operated accounts"

1. As soon as the customer receives the cheque book, he has to check whether the cheque leaves are in the correct serial order or not.
2. Blank cheques should not be issued to anybody.
3. Any changes, alterations, corrections made while filling a cheque, should be authenticated with full signature.
4. The amount on a cheque has to be written in words and figures legibly.
5. The amount of the cheque should be written immediately after the printed words **'Rupees'** or **'Rs'** Also, the word **'only'** should be added after the amount in words.
6. A cheque becomes outdated or stale after six months from the date of issue. Hence it should be presented within six months from the date of issue.

Parties dealing with a cheque

1. **Drawer**

   The account holder who writes the cheques and signs on it in order to withdraw money is called **'drawer'** of the cheque.

2. **Drawee**

The Bank on whom the cheque is drawn is called the Drawee Bank as they pay the money.

3. **Payee**

The party to whom the amount on the cheque is payable is called the payee. The payee has to affix his/her signature on the back side of the cheque.

Any savings bank account holder can withdraw money from his/her account using a withdrawal form, a specimen of which is given below.

State Bank of India

----------------------- Branch

Date:

Name of the account holder: _______________

Account No. | | | | | | | |

Note: This form is not a cheque.

Payment will be rejected if this form is not submitted along with the pass book.

Please pay self/ourselves only.

Rupees _________________________________ only.

and debit the amount from my/our above savings bank account. Rs.

Token No.	PAY CASH	
Scroll No.	passing officer	Signature of the customer

Figure 22.2

Banks impose restrictions on the number of times of withdrawal of money from the savings bank accounts. Violation of such restriction attracts a nominal charge. The interest on savings bank accounts are paid half-yearly by taking the minimum balance for each month as the balance for that entire month. **Minimum balance** is the least of all the balances left in the account from the 10th to the last day of that month.

☛ *Example*

The following table shows the particulars of the closing balances of an SB account during the month of March, 2006.

Date	Closing Balance
5th March	Rs 1800-00
10th March	Rs 2400-00
18th March	Rs 3500-00
25th March	Rs 1700-00
31st March	Rs 2500-00

From the above table, we can observe that the closing balance on 25th March i.e., Rs 1700 is the minimum closing balance between 10th of March and the last day of March. This is the minimum balance for March.

The monthly minimum balances for every six months is calculated and on this the interest for six months is calculated. Most of the banks, add the interest to the existing balance once in every half year. i.e., on 30th June and 31st December.

However, the periodicity of interest calculation differs from banks to post offices.

Calculation of interest on savings accounts in banks

The monthly minimum balances from January to the end of June are added and this total amount is called the **'product'** in Banks. Interest is calculated on this product and added to the opening balance on 1st July. In the same way, the interest for the next half year is calculated and is added to the opening balance on 1st January.

In savings account, interest is calculated as per the following steps.

1. The least of the balances from the 10th day of a month to the last day of the month is taken as the balance for the month.
2. The sum of all these monthly balances is taken as the principle for calculating interest.
3. Interest $= \dfrac{\text{principle} \times \text{rate of interest}}{12 \times 100}$

Example

The following is an extract of savings bank pass book of Mrinalini, who has an account with Corporation Bank.

Calculate the interest accrued on the account at the end of June, 2005 at 5% p.a.

Date	Particulars	Amount withdrawn		Amount deposited		Balances	
		Rs	Ps	Rs	Ps	Rs	Ps
7-1-2005	Balance B/F					8400	00
10-1-2005	By cash			12500	00	20900	00
31-1-2005	To cheque No. 3541	6500	00			14400	00
15-2-2005	By cash			3500	00	17900	00
13-3-2005	To cheque No. 3543	2800	00			15100	00
25-3-2005	By cheque			2000	00	17100	00
3-4-2005	To cheque No. 3544	1400	00			15700	00
18-4-2005	To cheque: 3545	3500	00			12200	00
21-5-2005	By cash			5400	00	17600	00
15-6-2005	To cheque: 3546	6000	00			11600	00
21-6-2005	To cheque: 3547	2000	00			9600	00
15-7-2005	By cash			3500	00	13100	00

Solution

The minimum balances in

	Rs
for January	= 14400
February	= 14400
March	= 15100
April	= 12200
May	= 12200
June	= 9600
	77900

The product is Rs 77900.

$$\text{Interest} = \frac{\text{Product} \times \text{Rate}}{100 \times 12}$$

$$= \frac{77900 \times 5}{100 \times 12} = \text{Rs } 324.58$$

(b) Current account

This account is very convenient for business people, companies, government offices and various other organisations, which need to make very frequent and large amounts of money transactions. Banks do not give any interest on these accounts, but the operation of these accounts is very flexible. There is no restriction, on amounts deposited or withdrawn or on the number of transactions, as in the savings bank accounts.

(c) Term deposit accounts

These accounts are of two types:

(i) **Fixed deposit accounts:** Customers can avail the facility of depositing a fixed amount for a fixed period of time. As the time period is fixed, banks give a higher rate of interest on these accounts.

If money is withdrawn from these accounts before the fixed time period, banks pay lesser interest than what is agreed upon. As this discourages premature withdrawal, banks rely more on these funds. The interest payable varies with the period for which the money is deposited in these accounts, and it varies from bank to bank. The rates of interest offered by a Bank on FD's are given as below:

[redacted]	
15 days and upto 45 days	5·25
46 days and upto 179 days	6.50
180 days to, less than 1 year	6·75
1 year to less than 2 years	8·00
2 years to less than 3 years	8·25
3 years and above	8·50

(ii) **Recurring deposit accounts:** These accounts help the customers to build up large amounts through small deposits. These accounts facilitate depositing a fixed amount per month for a period of 6 months to 3 years and above. This time period is called the **maturity period.**

These accounts help those who have low earnings, in saving large amounts through regular and fixed savings. A person who opens this account deposits an initially agreed amount each month. At the end of the maturity period, the cumulative amount with interest, which is called the maturity amount, is paid to the account holder. The rate of interest payable on these accounts are the same as those payable on fixed deposit accounts.

The recurring deposit interest is calculated according to following formula.

We know that, $1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

If a man deposits Rs *k* per month and for *n* months at R % p.a, then

$$\text{Simple interest} = \text{Rs}\left[k \times \frac{n(n+1)}{2} \times \frac{1}{12} \times \frac{R}{100}\right].$$

Example

Govind opened an account on 01.04.06 by depositing Rs 3000. He deposited Rs 1000 on 11.04.06 and withdrew Rs 500 on 15.04.06. Compute the interest paid by the bank for April.

Solution

Balance as on 01.04.06 = Rs 3000

as on 11.04.06 = Rs 4000

as on 15.04.06 = Rs 3500

The minimum balance for the month of April = Rs 3000

The interest paid by the bank for the month of April = $\frac{3000 \times 1 \times 4}{1200}$ = Rs 10

Example

Rajan makes a fixed deposit of Rs 8000 in a bank for a period of 2 years. If the rate of interest is 10% per annum compounded annually, find the amount payable to him, by the bank after two years.

Solution

The amount of fixed deposit = Rs 8000.

R = 10% per annum and n = 2.

The amount returned by the bank = $P\left[1 + \frac{r}{100}\right]^n$

$$= 8000\left[1 + \frac{10}{100}\right]^2$$

$$= 8000\,[1 + 0{\cdot}1]^2 = 8000 \times 1{\cdot}21 = \text{Rs } 9680{\cdot}$$

Example

Mahesh deposits Rs 600 per month in a recurring deposit account for 2 years at 5% per annum. Find the amount he receives at the time of maturity.

Solution

Here, P = Rs 600,

n = 2 × 12 i.e., 24 months and R = 5% p.a.

$$SI = P \times \frac{n(n+1)}{2} \times \frac{1}{12} \times \frac{R}{100} = 600 \times \frac{24(25)}{2} \times \frac{1}{12} \times \frac{5}{100} = 750$$

The total amount received = (24 × 600) + 750 = 14400 + 750 = Rs 15150.

Loans

The loans given by the banks can be classified into three types:

1. Demand loans
2. Term loans
3. Over drafts (OD)

Demand loans

The borrower has to repay the loans on demand. The repayment of the loan has to be done within 36 months from the date of disbursement of the loan. The borrower has to execute a demand promisory note in favour of the bank, promising that he would repay the loan unconditionally as per the stipulations of the bank.

Term loans

The borrower enters into an agreement with the bank regarding the period of loan and mode of repayment, number of instalments etc. The repayment period is generally more than 36 months. These loans are availed by those who purchase machinery, build houses etc.

Over draft (OD)

A current account holder enters into an agreement with the bank which permits him to draw more than the amount available in his account but upto a maximum limit fixed by the bank. These loans are availed by traders.

Calculation of interest on loans

Interest on loans is calculated on daily product basis once in every quarter, the loan amount is increased by that amount.

Daily product = Balance × Number of days it has remained as balance.

$$\text{Interest} = \frac{\text{Sum of daily products} \times \text{Rate}}{100 \times 365}$$

NOTE: If the loan is repaid totally, the date on which it is repaid is not counted for calculation of interest. If the loan is repaid in part, the day of repayment of loan is also counted for calculating the interest.

☛ *Example*

Ganesh takes a loan of Rs 20000 on April 1, 2005 and repays Rs 2000 on the 10th of every month, starting from May, 2005. If the rate of interest is 15% per annum, calculate the interest up to June 30, 2005.

Solution

[redacted]

20000	1.4.2005 to 10.5.2005	40	40 × 20000 = Rs 800000
Repays 2000 on May 11, 2005 Balance 18000	11.5.2005 to 10.6.2005	31	31 × 18000 = Rs 558000
Repay 2000 on June 11, 2005 Balance 16000	11.6.2005 to 30.6.2005	20	20 × 16000 = Rs 320000

Total daily product (DP) = Rs 1678000

$$\text{Interest} = \frac{\text{DP} \times \text{Rate}}{100 \times 365} = \frac{1678000 \times 15}{100 \times 365} = \text{Rs } 689.60$$

Compound interest

When interest is calculated on principal as well as on interest, it is known as compound interest. The interest is added to the principal at regular intervals, quarterly or half yearly or yearly and further interest is calculated on the increased principal thus obtained.

The formula to find out the amount payable, when the interest is compounded annually is $A = P\left(1 + \frac{r}{100}\right)^n$

where P = principal
r = rate of interest and
n = number of years.

When interest is compounded k times a year, then $A = P\left(1 + \frac{r}{k \times 100}\right)^{n \times k}$

When interest is compounded quarterly, $k = \frac{12}{3} = 4$.

When interest is compounded half-yearly, $k = \frac{12}{6} = 2$ and so on.

Hire purchase and instalment scheme

When a buyer does not have purchasing capacity, the seller allows the buyer to make part payments in monthly, quarterly, half yearly or yearly instalments. This scheme is of two types.

(i) Hire purchase scheme and
(ii) Instalment scheme.

(i) Hire purchase scheme

In this scheme the buyer, called the hirer and the seller called the vendor enter into an agreement which is known as **Hire purchase Agreement.**

Important features of hire purchase scheme

1. The hirer pays an initial payment known as down payment.
2. The vendor allows the hirer to take possession of the goods on the date of signing the agreement, but he does not transfer the ownership of the goods.
3. The hirer promises to pay the balance amount in instalments.
4. If the hirer fails to pay the instalments the vendor can repossess the goods.
5. When goods are repossessed, the hirer cannot ask for the repayment of the instalments of money already paid. This money paid will be treated as rent for the period.

(ii) Instalment scheme

Under the instalment scheme the seller transfers the possession as well as the ownership of the goods to the buyer. The buyer has the right to resell, pledge the goods, but he has to repay the instalments due.

Finding the rate of interest on buying in instalment scheme

The formula that is used to calculate the rate of interest on instalment buying is

$$R = \frac{2400E}{n[(n+1)I - 2E]}$$

R = Rate of interest

E = Excess amount paid

n = Number of instalments

I = Amount of each instalment

E = Down payment + Sum of instalment amounts–Cash price

Example

A television set is sold for Rs 9000 cash on Rs 1000 cash down followed by six equal instalments of Rs 1500 each. What is the rate of interest?

Solution

Given,

$n = 6$

$I = \text{Rs } 1500$

$E = 1000 + 6 \times 1500 - 9000 = \text{Rs } 1000$

$$R = \frac{2400E}{n[(n+1)I - 2E]}$$

$$= \frac{2400 \times 1000}{6[(6+1)1500 - 2000]} = 47.1\%$$

test your concepts

Very short answer type questions

1. Fixed deposit is also known as __________ deposit.
2. Farhan opens a savings bank account on 5th May, 07 in UTI Bank by depositing Rs 1000. The interest paid by the bank if she closed her account on 4th Jan, 08 is __________. (Rate of interest is 6% p.a.)
3. Bhavya opened savings bank account with a bank on 07-07-07 with a deposit of Rs 500 and then she neither deposited nor withdrew any amount. The amount on which she receives interest, if she closed her account on 10-10-07 is _________.
4. Charan opens a savings bank account on 6th March, 2007 with Rs 750 in Union Bank of India. He deposits Rs 1000 on 15th March and withdraws Rs 500 on 23rd March. The sum for which he will earn interest for the month of March is _________.
5. Anil opened a Savings Bank Account with a bank on 12-3-06 with a deposit of Rs 1000 and then he withdrew Rs 300 on 18-3-06. The amount on which he would receive the interest for the month of March, 06 is ___________.
6. Banks offer higher rate of interest on savings accounts than on fixed deposit accounts. (True/false)
7. Which of the following is a utility service provided by the Banks?

   (Issuing traveller cheques/Receiving payment for telephone bills)
8. Under a recurring deposit, a depositor is paid a lumpsum payment after the period for which the deposit is made. This lumpsum payment is called __________ value.
9. In the banks, safe deposit lockers are provided to the customers at free of cost. (True/False)
10. The rate of interest on current account is __________ % p.a.

Short answer type questions

11. Himesh opened a savings bank account with a bank on 3rd June, 2007 with Rs 500. His transactions during June and July were as follows:

    Deposited Rs 500 on 8th June.

    Withdrew Rs 300 on 11th June.

    Deposited Rs 500 on 13th June.

    Withdrew Rs 350 on 29th June.

    Deposited Rs 500 on 3rd July.

    Deposited Rs 500 on 12th July.

    Find the amounts qualifying for interest during June, July and August.

Directions for questions 12 to15: Hansika opened savings bank account with a bank on 2th Jan,07. A page from the pass book of Hansika is given below. She closed her account on 30th June, 07.

Date	Particulars	Amount With drawn (Rs)	Amount deposited (Rs)	Balance (Rs)
Jan 4, 07	B/F	–	1500.00	1500.00
Jan 23	To self	500.00		1000.00
Feb 6	By cash	–	3500.00	4500.00
Feb 15	To self	1000.00		3500.00
Feb 26	By cash	–	6500.00	10000.00
Mar 15	By cash	–	2000.00	12000.00
April 8	To self	3000.00		9000.00
April 15	By cash	–	7000.00	16000.00

12. Find the amount for which Hansika gets interest.

13. If the bank pays interest at 6% per annum, find the interest she gets while closing her account.

14. Find the total amount that Hansika will receive, while closing her account, if the bank pays 8% per annum.

15. Calculate the total interest earned by Hansika upto 30th June 07, if the bank pays

(i) 4.8% per annum upto 31.3.07

(ii) 7.2% per annum from 1.4.07 to 30.6.07

Essay type questions

16. Javed makes a fixed deposit of Rs 100000 in a bank for one year. If the rate of interest is 6% per annum, compounded half-yearly, then find the maturity value.

17. Srija makes a fixed deposit of Rs 125000 with a bank. The bank pays interest at 8% per annum compounded annually and she received Rs 157464 at the time maturity. Find the time period for which she held account.

18. Tilak opened a recurring deposit account with a bank and deposited Rs 600 per month for one year. Find the interest that Tilak will receive, if the bank pays 6% per annum.

19. David makes a fixed deposit of Rs 50000 in a bank for $1\frac{1}{2}$ years. If the interest is compounded half-yearly and the maturity value is Rs 66550, then find the rate of interest per annum.

20. Tushar opens a recurring deposit account with a bank and deposits Rs 500 per month for $1\frac{1}{2}$ years. If he receives Rs 570 as interest, then find the rate of interest offered by the bank.

CONCEPT APPLICATION

Concept Application Level—1

1. Ramesh opens a savings bank account with a bank on 16-06-2007 with a deposit of Rs 700. He deposited Rs 1500 on 07-07-2007. Find the amount on which he would receive the interest at the end of July 2007.

(1) Rs 700 (2) Rs 1500 (3) Rs 2200 (4) Rs 800

2. Akshit opened a savings bank account in a bank on 13th Feb 2007 with a deposit of Rs 1000. He again deposited Rs 1000 on 9th March 2007. Find the amount on which he gets interest, if he closes his account on 31st March 2007.

(1) Rs 1000 (2) Rs 2000 (3) Rs 3000 (4) None of these

3. Madhu makes a fixed deposit of Rs 15000 in a bank for two years. If the rate of interest is 10% per annum compounded annually, then find the maturity value.

(1) Rs 3150 (2) Rs 17500 (3) Rs 16750 (4) Rs 18150

4. Dinesh makes a fixed deposit of Rs 50000 in a bank for one year. If the rate of interest is 12% per annum, compounded half yearly, then find the maturity value.

(1) Rs 66125 (2) Rs 56180 (3) Rs 57500 (4) Rs 63250

5. Karthik makes a fixed deposit of Rs 15000 in a bank for 219 days. If the rate of interest is 9% p.a., then what amount does he get on the maturity of the fixed deposit?

(1) Rs 15810 (2) Rs 16320 (3) Rs 15430 (4) Rs 16610

6. Prabhu deposits Rs 600 per month in a recurring deposit account for 1 year at 8% per annum. Find the interest received by Prabhu.

(1) Rs 424 (2) Rs 312 (3) Rs 360 (4) Rs 450

7. Kamal deposits Rs 550 per month in a recurring deposit account for $1\frac{1}{2}$ year at 8% per annum. Find the interest that Kamal will receive at the time of maturity.

(1) Rs 550 (2) Rs 627 (3) Rs 230 (4) Rs 346

8. Sneha opened a cumulative time deposit account with a bank and deposits Rs 500 per month for $1\frac{1}{4}$ years. If she receives Rs 300 as interest, find the rate of interest per annum.

(1) 6% (2) 8% (3) 7.5% (4) 10%

9. Vijay makes a fixed deposit of Rs 10000 in a bank for 2 years under compound interest. If the maturity value is Rs 11664, find the rate of interest per annum compounded annually.

(1) 4% (2) 5% (3) 8% (4) 10%

10. Subhash makes a fixed deposit of Rs 25000 in a bank for 146 days. If the rate of interest is 7.5% p.a., then what amount does he get on the maturity of the fixed deposit?

(1) Rs 27500 (2) Rs 25750 (3) Rs 26500 (4) Rs 28450

Concept Application Level—2

11. Find the amount received by Prakash while closing his account, if the bank pays 6% per annum.

(1) Rs 65010 (2) Rs 11810
(3) Rs 62310 (4) Rs 12100

12. Varsha opened a Recurring Deposit Account with Oriental Bank of Commerce and deposited Rs 800 per month at 4% per annum. If she gets Rs 800 as interest, find the total time for which the account was held (in years).

(1) $1\frac{1}{2}$ (2) 2 (3) $1\frac{3}{4}$ (4) $2\frac{1}{4}$

13. Susheel has a cumulative time deposit account of Rs 800 per month at 6% per annum. If he gets Rs 1300 as interest, then find the total time for which the account was held (in months).

(1) 26 (2) 25 (3) 24 (4) 28

14. Vishal has a recurring deposit account in a finance company for 1 year at 8% per annum. If he gets Rs 9390 at the time of maturity, then how much amount per month has been invested by Vishal?

(1) Rs 650 (2) Rs 700 (3) Rs 750 (4) Rs 800

Directions for questions 15 to 17: A page from the pass book of Noel is given below. He closes his account on 3rd December, 2006.

[redacted]				
July, 3	B/F	-	-	5000.00
July, 12	By cash	-	3000.00	8000.00
Aug, 15	To self	2500.00	-	5500.00
Oct, 6	By cash	-	5000.00	10500.00
Nov, 8	To self	1500.00	-	9000.00
Nov, 15	By cash	-	6000.00	15000.00

15. Find the amount on which he gets interest on closing his account.

(1) Rs 41500 (2) Rs 35500 (3) Rs 44500 (4) Rs 33500

16. The interest received by Noel on closing his account, if the bank pays at 6% per annum, is ________.

(1) Rs 177.50 (2) Rs 207.50 (3) Rs 222.50 (4) Rs 167.50

17. If the bank pays 6% per annum, find the amount received by Noel on closing his account.

(1) Rs 41722.5 (2) Rs 35677.5 (3) Rs 15177.5 (4) Rs 9177.5

Concept Application Level—3

Directions for questions 18 to 20: A page from Richa's pass book is given below. Answer the following questions by finding the missing entries. She closes her account on 30th June 2007.

[redacted]

05.01.07	By Cash		500.00	500.00
23.01.07	By Cash		6000.00	6500.00
08.02.07	By Cash		(missing entry)	8000.00
13.02.07	To self	(missing entry)		5000.00
18.02.07	By cash		2000.00	(missing entry)
09.03.07	By Cash		5000.00	12000.00
15.03.07	To self	Missing entry		9000.00
11.04.07	To self	Missing entry		5000.00
05.05.07	By Cash		Missing entry	10050.00

18. Find the interest received by Richa on closing her account, if the bank pays 4% per annum.
(1) Rs 98.5 (2) Rs 115 (3) Rs 132 (4) Rs 133

19. Find the amount on which she will get interest on closing her account.
(1) Rs 29550 (2) Rs 34500 (3) Rs 39600 (4) Rs 36900

20. If the bank pays 8% per annum from 01.01.07 to 30.04.07 and 6% per annum from 01.05.07 to 30.06.07, then find the total interest received by Richa.
(1) Rs 230.50 (2) Rs 247 (3) Rs 196.50 (4) Rs 188

KEY

Very short answer type questions

1. Term deposit. **2.** Rs 40.
3. Rs 1500 **4.** Rs 750
5. Rs 0 **6.** False
7. Receiving payment for telephone bills.
8. Maturity
9. False.
10. 0

Short answer type questions

11. Rs 700, Rs 1350 and Rs 1850
12. Rs 55500 **13.** Rs 277.50
14. Rs 16370 **15.** Rs 304

Essay type questions

16. Rs 106090. **17.** 3 years
18. Rs 234 **19.** r = 20% p.a
20. 8% p.a.

key points for selected questions

Short answer type questions

11. (i) Write the transaction table.
(ii) Calculate the minimum balances of each month specified.

12. Calculate the minimum balances of each month and find their sum.

13. Use S.I. = $\frac{P \times T \times R}{1200}$, where p is the principal.

14. Total amount received = Balance present on 15th April + Interest.

15. Find product and calculate the interest on it by using, I = $\frac{PTR}{1200}$.

Essay type questions

16. Use $A = P\left(1+\frac{r}{100}\right)^{2n}$, where n = 1.

17. Use $A = P\left(1+\frac{r}{100}\right)^{n}$ and evaluate n.

18. Use S.I. = $P \times \frac{n(n+1)}{2} \times \frac{R}{1200}$, where n is the number of months.

19. Use $A = P\left(1+\frac{r}{200}\right)^{2n}$, where $n = \frac{3}{2}$.

20. Use S.I. = $P \times \frac{n(n+1)}{2} \times \frac{R}{1200}$, where n is the number of months.

KEY

Concept Application Level—1,2,3

1. 3	**2.** 2
3. 4	**4.** 2
5. 1	**6.** 2
7. 2	**8.** 1
9. 3	**10.** 2
11. 1	**12.** 2
13. 2	**14.** 3
15. 2	**16.** 1
17. 3	**18.** 3
19. 3	**20.** 1

Concept Application Level—1,2,3

Key points for select questions

1. Find the minimum balance from 10th of month to end of the month.

2. Calculate the minimum balances from the 10th to end of the each month and calculate the total minimum balance.

3. Use, $A = P\left(1+\frac{r}{100}\right)^{n}$

4. (i) $A = P\left(1+\frac{r}{100}\right)^{n}$

(ii) Given, P = Rs 50000,
n = 2 half years,
R = 6% per half year.

(iii) Use, $A = \left(1+\frac{R}{100}\right)^{n}$ to find the maturity value (A).

5. (i) Use, S.I. = $\frac{P \times d \times R}{36500}$, where
d = number of days.

(ii) $P = \text{Rs } 1500$,

$T = \frac{219}{365} = \frac{3}{5}$ years and $R = 9\%$ p.a.

(iii) Use, $A = P + \frac{PRT}{100}$ to find the maturity value (A).

6. Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

7. (i) Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

(ii) Given $P = \text{Rs } 550$, $n = 18$ months, $R = 8\%$

(iii) Use, $I = \frac{p \times n(n+1)}{2} \times \frac{R}{1200}$ to find interest revised.

8. Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

9. Use, $A = P\left(1 + \frac{r}{100}\right)^n$

10. (i) $S.I. = \frac{P \times d \times R}{100 \times 365}$, where

d = Number of days.

(ii) Total = P + SI.

12. Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

13. (i) Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

(ii) Given $P = \text{Rs } 800$, $I = 1300$, $R = 6\%$

(iii) Use, $I = \frac{p \times n(n+1)}{2} \times \frac{R}{1200}$ to find n.

14. (i) Use, $S.I. = \frac{P \times n(n+1)}{2} \times \frac{R}{1200}$

(ii) Let the amount deposited per month be Rs P.

Amount A = Rs 9390, S.I. = 9390 – P.

(iii) Use, $9390 = P + \frac{p \times n(n+1)}{2} \times \frac{R}{1200}$ to find P.

15. Calculate the minimum balance present from the 10th to the end of each month and calculate the total minimum balances.

16. Interest is calculated on the minimum balance for the entire period.

17. Use, $SI = \frac{P \times T \times R}{100}$

18. (i) Interest is calculated on minimum balance for the entire period.

(ii) Use, $S.I. = \frac{P \times R \times T}{100}$, where P = sum of all minimum balances, $r = 4\%$ and $T = \frac{1}{12}$ year.

19. (i) Calculate the minimum balances from the 10th of the month to the end of each month.

(ii) Find the sum of the minimum balances of all the months.

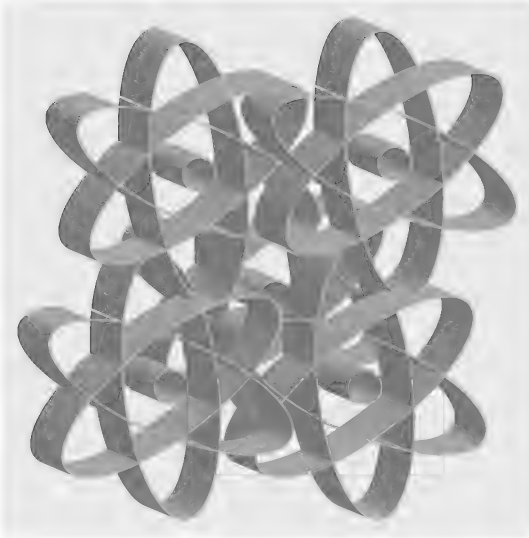
20. (i) Calculate the total minimum balance from 01.01.07 to 30.04.07 and also calculate SI, with the rate of interest given.

(ii) Calculate the total minimum balance from 01.05.07 to 30.06.07 and calculate SI with the rate of interest given.

(iii) Apply the new rates of interest and the corresponding interest received by Richa.

CHAPTER 23

Taxation



INTRODUCTION

The government of a country has to perform many social and economic functions, for which it needs money. The money comes from both domestic and foreign sources. The most important source is taxation –a fee that it charges on various economic activities and the wealth that is created by such activities or for providing legal safeguards.

Taxes can thus be classified on the basis of the economic activity or the kind of legal safeguards provided. Taxes can also be classified as direct or indirect, based on the manner in which they are collected. If they are collected directly from the person who is paying, they are direct taxes. If they are paid by a person other than the person upon whom it is levied such as sales tax–they are indirect taxes. The most important kind of direct taxes is income tax. We shall discuss in a detail about income tax and sales tax.

Income tax

The following discussion is only an indicative of the general nature of the computation of income tax. The actual details vary from one financial year to another and these are generally decided while presenting the union budget in Lok Sabha every year on 28th of February. A financial year starts from the 1st April of a calendar year and ends on 31st March of the next calendar year. The year next to a financial year is called the assessment year for that financial year. For the financial year 2004-05, the assessment year is 2005-06. People who earn above a certain limit are liable to pay income tax. The tax imposed on an individual is called personal income tax. A certain part of the income, called the standard deduction, is not liable to be taxed. The following table gives the details of the standard deduction. The symbol S stands for the annual salary of an individual.

Table 1(a)

$S \le 90000$	S/3
$900000 < S \le 150000$	30000
$150000 < S \le 350000$	25000
$350000 < S \le 500000$	20000
$500000 < S$	Nil

In addition, deduction from income is allowed on certain specified donations as detailed below.

Table 1(b)

[redacted]

1.	PM's National Relief Fund	100%
2.	National Defence Fund	100%
3.	Medical Research	100%
4.	Charitable Trusts, Educational Bodies, Hospitals and Orphanages	50%

Such donations, the indicated rates are deducted from the income. However, this deduction is subject to a maximum of 10% of the total income, i.e., if more than 10% is donated, the excess amount is subject to tax at the applicable rate.

After allowing for these deductions (D) (Standard deduction and donations) the net taxable income (TI = S – D) is taxed at the following rate/s.

Table 2

[redacted]

$TI \leq 50000$	Nil	0
$50000 < TI \leq 60000$	10% of (TI – 50000)	0
$60000 < TI \leq 150000$	1000 + 20% of (TI – 60000)	5%
$150000 < TI$	19000 + 30% of (TI – 150000)	5%

We can compute the tax that is payable by using the table above. However, senior citizens (Aged 65 years or more), women and citizens who have invested in specified funds are allowed certain rebates on this tax. Therefore, this amount is referred to as TBR (Tax Before Rebate).

NOTE: The third column in the table above gives the rate of surcharge on tax. Surcharge is calculated after deducting the rebate as illustrated in the examples.

These Rebates are tabulated below.

Senior Citizens Table 3(a)

[redacted]

$TBR \leq 15000$	TBR
$15000 < TBR$	15000

Women (less than 65 years old)

Table 3(b)

[redacted]

$TBR \leq 5000$	TBR
$5000 < TBR$	5000

All citizens (including senior citizens and women) who save in the following funds are allowed rebate at the indicated rates.

1. Contributory Provident Fund (CPF)
2. General Provident Fund (GPF)
3. Public Provident Fund (PPF)
4. Life Insurance Premium (LIC)
5. National Savings Certificates
6. Certain Infrastructure Bonds

Table 3(c)

[redacted]	
$S \leq 150000$	20% of savings
$150000 < S \leq 500000$	15% of Savings
$500000 < S$	Nil

We note that while tables 3(a) and 3(b) are applicable to different sets of people, 3(a) and 3(c) or 3(b) and 3(c) could apply to the same person. A person who pays income tax is known as an Assessee. Every Assessee is given a Permanent Account number (PAN) by the Income Tax Department on applying.

The tax after rebate on TAR is computed from tables 3(a), 3(b) and 3(c).

Finally, the surcharge (at the rate given in Table 2) is computed on the TAR and the total tax payable (TTP) is computed. The computation of income tax is illustrated with the following examples.

☛ *Example*

The monthly salary of Madhuri is Rs 15000. She contributes Rs 5000 to the PM's National Relief Fund and Rs 2500 to a hospital. She also pays a premium of Rs 2000 per annum on her LIC policy. Compute her income tax.

Solution

We have to compute the TI, TBR, TAR and TTP.

All amounts are in Rupees.

Annual salary (S) = 12(15000) = 180000

Standard deduction = 25000 (Table 1(a))

Donation (PMNRF) = 5000 (Table 1(b))

Donation (Hospital) = 2500 (Table 1(b))

========

Total deduction (D) = 32500

========

TI = S – D = 147500

TBR = 1000 + 20% of (147500 – 60000)

$= 1000 + \frac{20}{100}$ (87500) (Table 2)

$= 1000 + 17500 = 18500$

Rebate for women = 5000 (Table 3(b))

Rebate for LIC premium = 20% of 2000 (Table 3c) = 400

Total rebate = 5400

$\therefore$ TAR = TBR – Rebate = 18500 – 5400 = 13100

Surcharge = 5% of 13100 = $\frac{13100}{20}$ = Rs 655

$\therefore$ TTP = TAR + Surcharge = 13100 + 655 = Rs 13755

Example

Raghav's monthly salary is Rs 20000. He donates Rs 15000 per annum for cancer research and Rs 1000 to the NDF. He contributes Rs 50000 to PPF. Compute his income tax.

Solution

We compute the TI, TBR, TAR and TTP as follows.

Annual salary, S = 20000 × 12 = Rs 240000

Standard Deduction = 25000 (Table 1(a))

Donation to NDF = 10000

Donation to Cancer Research = 15000

As the total donation is 25000, only 10% of Raghav's income, i.e., 24000 can be deducted.

$\therefore$ Total deduction = (25000 + 24000) = 49000

TI = S – D = 240000 – 49000 = Rs 191000

TBR = 19000 + 30% of (191000 – 150000)

$= 19000 + \frac{3}{10}$ (41000) = Rs 31300

He contributes 50000 to PPF

$\therefore$ He gets a rebate of 15% of 50000 = Rs 7500

$\therefore$ TAR = 31300 – 7500 = Rs 23800

He has to pay a surcharge of 5%.

Surcharge = $\frac{5}{100}$ (23800) = 1190

$\therefore$ TTP = 23800 + 1190 = Rs 24990

Sales tax

Sales tax is the tax levied on the sale of goods within the state.

Central sales tax is the tax levied by the Union Government when goods produced in one state are sold in another state.

The proceeds under sales tax are credited to the Government Account. Hence sales tax is not included in the selling price.

Calculation of sales tax

(i) When no discount is given, the marked price of the article becomes the sale price and sales tax is calculated on it.

(ii) When a certain discount is given, then the sales tax is calculated on the reduced price of the article after the discount.

Example

Rakesh bought a radio for Rs 1296, which includes a discount of 20% offered on the marked price and 8% sales tax on the remaining amount. Find the marked price of the radio.

Solution

Let the marked price of the radio be Rs x.

$\Rightarrow$ discount offered = Rs (20% of x) = Rs $\left(\frac{20}{100} \times x\right)$ = Rs $\frac{x}{5}$

The price of the radio after discount = Rs $\left(x - \frac{x}{5}\right)$ = Rs $\frac{4x}{5}$

Sales tax charged = Rs $\left(8\% \text{ of } \frac{4x}{5}\right)$ = Rs $\left(\frac{8}{100} \times \frac{4x}{5}\right)$

The cost of the radio inclusive of sales tax = Rs $\left[\frac{4x}{5} + \frac{8}{100} \times \frac{4x}{5}\right]$ = Rs $\left[\frac{27}{25} \times \frac{4x}{5}\right]$

Given that Rakesh paid Rs 1296 for the radio.

$$\Rightarrow \frac{27}{25} \times \frac{4x}{5} = 1296$$

$$\Rightarrow x = \frac{1296 \times 25 \times 5}{108} = \text{Rs } 1500$$

Example

The list price of an article is Rs 2160 and the sales tax applicable on the article is 8%. If a customer asked the shopkeeper to give a certain discount on its list price such that he pays Rs 2160 inclusive of sales tax, then find the discount percent offered.

Solution

Let us assume the reduced price of the article after discount to be Rs x

Sales tax charged = Rs (8% of x) = Rs $\left(\frac{8}{100} \times x\right)$ = Rs $\frac{2x}{25}$

The selling price of the article inclusive of sales tax = Rs $\left(x + \frac{2x}{25}\right)$ = Rs $\frac{27x}{25}$

Given that the customer pays Rs 2160 for the article inclusive of taxes

$$\Rightarrow \frac{27x}{25} = 2160$$

$$\Rightarrow x = \frac{2160 \times 25}{27} = \text{Rs } 2000$$

$\therefore$ x = Rs 2000

$\Rightarrow$ discount offered = Rs (2160 – 2000) = Rs 160

$$\Rightarrow \text{The rate of discount} = \frac{160}{2160} \times 100 = 7\frac{11}{27}\%$$

$$\therefore \text{The discount per cent offered} = 7\frac{11}{27}$$

test your concepts

Very short answer type questions

1. The tax charged on the sale of goods that are moved from one state to other is called _____.
2. If the marked price of an article is Rs 100 and sales tax is 12%, then the selling price is Rs ______.
3. The tax imposed on an individual or group of individuals which affects them directly, and is paid to the Government directly is known as _____.
4. The tax imposed on an individual or a group of individuals on their annual incomes is known as _____.
5. A financial year begins from _____.
6. Kumar paid Rs 220 to buy an article including sales tax of 10%. Then, the selling price of the article is Rs_______.
7. Exemption rate on donation to Prime Minister's National Relief Fund is _____.
8. While assessing the income tax, the year next to a financial year is called _____ year for that financial year.
9. Ajay paid Rs 150 to buy an article whose selling price is Rs 120. The sales tax paid by Ajay is Rs ________.
10. Every assessee is expected to file a statement of previous year's income to the income tax department in a prescribed form which is known as _____.
11. The tax paid by the first person is less than the tax paid by the second person. If neither of them saved any amount, then the second person has more income. (True/False)
12. Rate of surcharge if annual taxable income exceeds Rs 60000 is _____.

13. Anand paid Rs 30 as sales tax on a bottle of mineral water with marked price as Rs 400. Calculate the rate of sales tax.

14. Annual Union Budget is usually presented in the Lok Sabha every year, on ______.

15. A washing machine is available for Rs 7950, including sales tax. If the rate of sales tax is 6%, find the list price of the washing machine.

Short answer type questions

16. Find the list price of a bicycle which costs Rs 1595, inclusive of sales tax. The rate of sales tax is 10%.

17. The marked price and the selling price of an article are Rs 1500 and Rs 1800 respectively. Find the rate of sales tax, if there is no discount.

18. Madhu bought the following items from a super market;
    (i) soaps worth Rs 220,
    (ii) cosmetics worth Rs 580 and
    (iii) vegetables worth Rs 650.

    If the sales tax is charged at the rate of 5% on soaps, 10% on cosmetics and 2% on vegetables, find the total amount paid by Madhu.

19. Prasad has a total salaried income of Rs 160000 per annum. Calculate the amount of income tax he has to pay.

20. Rijwana's monthly salary is Rs 25000. She contributes Rs 600 per month towards GPF and pays Rs 7000 towards annual LIC premium. Find the amount of income tax she has to pay for the last month if she paid Rs 4000 per month towards income tax for 11 months.

Essay type questions

21. Ramakanth gets a salary of Rs 15000 per month. He contributes Rs 5000 per month towards PPF. He also donates Rs 1250 per month towards Medical Research. Calculate the income tax he has to pay.

22. Khalique's monthly salary is Rs 20000. He donates Rs 1500 per month for medical research, (100% relief), contributes Rs 2000 per month to PF and pays annual LIC premium of Rs 3000. Calculate the income tax he has to pay for the year.

23. Chakradhar buys a T.V. marked at Rs 14500 after getting successive discounts of 15% and 20% and paying 10% sales tax. He spends Rs 2000 on it and sells the T.V. for Rs 12000. Find his gain or loss per cent.

24. Subiksha earns Rs 12500 per month. She donates Rs 3300 per month towards Prime Minister National Relief Fund (100% relief). Find the amount of tax she has to pay.

25. Laxmikala, a senior citizen gets a pension of Rs 12000 per month. Calculate the income tax to be paid by her.

CONCEPT APPLICATION

Concept Application Level—1

1. Ramu purchased a motorcycle at a price of Rs 37800 which includes the sales tax. If the rate of sales tax is 8%, then what is the list price of the motorcycle?

   (1) Rs 31000 (2) Rs 33000 (3) Rs 35000 (4) Rs 37000

2. Rohan buys a pair of shoes marked at Rs 2500. He gets a rebate of 8% on it. After getting the rebate, sales tax is charged at the rate of 5%. What is the amount he will have to pay for the pair of shoes?

   (1) Rs 2400 (2) Rs 2415 (3) Rs 2430 (4) Rs 2445

3. Raju purchased a car for Rs 155750, inclusive of sales tax. He paid Rs 5750 as sales tax. What is the rate of sales tax?

   (1) $3\frac{2}{3}\%$ (2) $3\frac{3}{4}\%$ (3) $3\frac{4}{5}\%$ (4) $3\frac{5}{6}\%$

4. Amar wants to buy a shirt which is listed at Rs 378. The rate of sales tax is 8%. He told the shopkeeper to reduce the list price to such an extent that he has to pay not more than Rs 378 including sales tax. What is the minimum reduction needed in the list price of the shirt?

   (1) Rs 24 (2) Rs 26 (3) Rs 28 (4) Rs 30

5. Ramesh purchased a bag listed at Rs 550. If the rate of sales tax is 8%, then what is the sales tax paid by Ramesh?

   (1) Rs 42 (2) Rs 43 (3) Rs 44 (4) Rs 45

6. Ismail gets a monthly salary of Rs 12500. He contributes Rs 3500 per month towards PF. Calculate the income tax paid by him.

   (1) Rs 3000 (2) Rs 5600 (3) Rs 4830 (4) Rs 3300

7. Rakesh purchased a car which was quoted at Rs 256000. The dealer charged sales tax on it at the rate of 12%. As Rakesh wanted to take the car outside the state, the dealer further charged 3% extra as Central Sales Tax. What is the amount he had to pay for the car?

   (1) Rs 295123·60 (2) Rs 286720 (3) Rs 294400 (4) None of these

8. Satish earns an annual salary of Rs 150000 and the standard deduction applicable to him is 40% of the salary or Rs 30000, whichever is less. Then his net taxable income is_____.

   (1) Rs 30000 (2) Rs 120000 (3) Rs 60000 (4) Rs 90000

9. Ajay purchased a computer for Rs 34650 which includes 12% rebate on the marked price and 5% sales tax on the remaining price. What is the marked price of the computer?

   (1) Rs 35000 (2) Rs 37500
   (3) Rs 40000 (4) Rs 42500

Direction for questions 10 and 11: These questions are based on the following data.

Dinakar's salary is Rs 30000 per month. He contributes Rs 27000 towards GPF and Rs 30000 towards IC. He donates Rs 8000 to a Charitable trust (50%), Rs 10000 towards National Relief Fund (100%).

10. Calculate taxable income.

(1) Rs 198000 (2) Rs 280000
(3) Rs 326000 (4) Rs 380000

11. Find the rebate amount on his savings.

(1) Rs 15350 (2) Rs 10250 (3) Rs 8550 (4) Rs 20000

12. If the marked price of an article is Rs M, then find the rate at which sales tax is charged if the person pays Rs M inclusive of sales tax. Discount allowed is 10%.

(1) 9% (2) 10% (3) $11\frac{1}{9}\%$ (4) $9\frac{1}{11}\%$

13. The list price of a T.V. is Rs 15000 and the shopkeeper allows a discount of 20% and 10% successively on list price. On the remaining amount, he charges 20% as sales tax. If buyer paid Rs x, then by how much amount will the list price exceed Rs x?

(1) Rs 3000 (2) Rs 4200 (3) Rs 2040 (4) Rs 5000

14. Saritha's annual salary is Rs 160000. She contributes Rs 6000 towards GPF and pays an LIC annual premium of Rs 5000. Calculate the income tax she will have to pay in the year.

(1) Rs 16117.50 (2) Rs 15117.50 (3) Rs 17117.50 (4) Rs 16017.50

15. Ranjit purchased a refrigerator for the price of Rs 8910 which includes 10% rebate on marked price and 10% sales tax on the remaining price. If the sales tax is increased to 20% without allowing the 10% rebate on the marked price, how much more will the customer pay for a refrigerator?

(1) Rs 1850 (2) Rs 1890 (3) Rs 1860 (4) Rs 1840

Concept Application Level—2

16. The list price of an article is 50% more than its original cost price. The shopkeeper allowed a discount of 20% and charged a sales tax of 20% on it. Finally the buyer paid Rs 2880. What is the cost price of the article?

(1) Rs 2500 (2) Rs 3500 (3) Rs 3000 (4) Rs 2000

17. If the tax to be paid is Rs 12700 and surcharge is calculated as 10% of the tax payable, then find the net tax payable.

(1) Rs 13970 (2) Rs 14690 (3) Rs 12690 (4) Rs 13500

18. Mr Ranvir Patnikar earns an annual salary of Rs 270000. If his employer deducts Rs 3000 every month from his salary for the first 11 months, then calculate the amount he has to pay towards tax in the last month of the financial year.

Standard deduction is 40% of the salary or Rs 30000, whichever is less.
The income tax on his earnings is calculated based on the data given below.
Slabs for income tax:

(i) Upto Rs 50000 – Nil
(ii) from Rs 50000 to – 10% of the amount Rs 100000 exceeding 50000
(iii) from Rs 100001 to – Rs 5000 + 20% of the Rs 200000 amount exceeding Rs 100000
(iv) Above Rs 200000 – Rs 25000 + 30% of the amount exceeding Rs 200000

(1) Rs 3000 (2) Rs 4000 (3) Rs 5000 (4) Rs 6000

19. The annual salary of Mr Ravi Teja is Rs 178500. He donates Rs 750 per month towards the National Defence Fund (eligible for 100% exemption). If standard deduction is 30% of the gross salary income or Rs 30000, whichever is less, then find his net taxable income.

(1) Rs 148000 (2) Rs 147250
(3) Rs 139500 (4) Rs 150000

20. Siri's total income is Rs 16500. Of this Rs 5000 is free from tax. Find the net income remaining with her after she paid the income tax at 5%. (in Rs)

(1) 10925 (2) 15675 (3) 15925 (4) 14750

Concept Application Level—3

21. Manish's annual income is Rs 132000. There is no income tax on the money donated to charity. On the remaining amount he pays Rs 4480 as income tax at 4%. What amount does he denote to the charity?

(1) Rs 24500 (2) Rs 20000 (3) Rs 30000 (4) Rs 18250

22. Amrit Raj bought a Nokia mobile for Rs 5040 which includes 10% discount on the market price and 12% sales tax on the remaining price. Find the marked price of the mobile phone.

(1) Rs 4672 (2) Rs 5124 (3) Rs 5000 (4) Rs 4830

23. Laxmi's total income is Rs 22500. Of this Rs 7000 is free from tax. Find the net income remaining with her after she paid the income tax at the rate of 8%. (in Rs)

(1) 21375 (2) 21260 (3) 20675 (4) 22105

24. Rajnesh's annual income is Rs 180000. He pays no income tax on the money invested in premiums. On the remaining amount he pays Rs 10920 as income tax at 7%. What amount does he invest in premiums?

(1) Rs 18000 (2) Rs 22500 (3) Rs 24000 (4) Rs 27400

25. Karan bought a T.V. for Rs 9350 which includes 15% discount on the marked price and then 10% sales tax on the remaining price. Find the marked price of the TV. (in Rs)

(1) 9000 (2) 9750 (3) 10200 (4) 10000

KEY

Very short answer type questions

1. Central Sales Tax
2. 112
3. Direct Tax
4. Income Tax
5. 1st April
6. Rs 200
7. 100%
8. Assessment
9. 30
10. Income tax return
11. False
12. 5% of income tax
13. 7.5%
14. 28th February
15. Rs 7500

Short answer type questions

16. Rs 1450

17. 20%

18. Rs 1532

19. Rs 16800

20. Rs 7839

Essay type questions

21. Rs 8400

22. ≈ Rs 30503.

23. Percentage of loss = 6.58%

24. Rs 5250.

25. Rs 0.

key points for selected questions

Very short answer type questions

16. (i) Consider the list price as x and calculate SP.
(ii) Equate SP to the given value and find x.

Short answer type questions

16. (i) Assume list price as Rs x, calculate SP
(ii) Then, equating it to the given SP, list price can be calculated.

17. (i) Calculate tax imposed i.e., MP – SP
(ii) Then calculate the rate of sales tax.

18. (i) Calculate the tax on each item, using $\text{Tax} = \text{CP} \times \dfrac{\text{tax\%}}{100}$.
(ii) Now, calculate the sum of CP'S and tax on them.

19. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

20. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

Essay type questions

21. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

22. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

23. (i) Calculate SP using MP and discount.
(ii) Then add tax to it to obtain the cost to be paid.
(iii) Then find profit or loss.

24. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

25. (i) Calculate the total deductions and subtract from the income.
(ii) Calculate the tax according to the given scales.

KEY

Concept Application Level—1,2,3

1. 3	**2.** 2
3. 4	**4.** 3
5. 3	**6.** 3
7. 3	**8.** 2
9. 2	**10.** 3
11. 3	**12.** 3
13. 3	**14.** 1
15. 2	**16.** 4
17. 1	**18.** 2
19. 3	**20.** 3
21. 2	**22.** 3
23. 2	**24.** 3
25. 4	

Concept Application Level—1,2,3

Key points for select questions

1. 108% of list price = cost price.

2. Find 105% of 92% of 2500.

3. Rate of sales tax is calculated on list price and hence find the list price.

4. (i) Apply the concept of percentages.
(ii) Let the price after reduction be Rs P.
(iii) Now, $P + \frac{8P}{100} = 378$, find P.
(iv) Discount needed = Rs (378 – P)

5. Tax is 8% of M.R.P.

6. Deduct the savings (as per the norms) and standard deduction from the annual salary.

7. (i) Add the sales tax and then add the central sales tax to the list price.
(ii) First calculate 12% sales tax and add to list price.
(iii) Calculate 3% central sales tax and add to above price.

8. Taxable income = Annual salary – standard deduction (in Rs)

9. 105% of 88% of M.P = 34650.

10. (i) Taxable income is obtained when standard deduction, donations and savings are deducted.
(ii) Find the standard deduction.
(iii) Taxable income = Annual salary – (S.D. + G.P.F. + LIC + NRP + 50% of donations)

11. (i) Refer the introduction for slab on rebate amount on saving.
(ii) As his salary falls in between Rs 150000 and Rs 500000, rate of rebate is 15% of his savings.
(iii) Calculate total savings and find 15% of it.

12. What % of 9M/10 is M.

13. (i) Apply the concept of percentages.
(ii) Calculate 20% and 10% successive discounts on L.P.
(iii) Calculate 20% sales tax on the amount after giving discounts.
X = Sales tax + amount after giving discount.
(iv) Required amount = Rs (15000 – X).

14. Find the taxable income after deducting the eligible deductions and calculate the tax based on slab.

15. (i) Calculate by using $x \times 0.9 \times 1.1 = 8910$
(ii) Now use the value of x and calculate $x \times 1.2 = y$
(iii) Find the value of Rs (y – 8910)

16. (i) Assume the cost price as 100 and proceed.
(ii) Let the C.P. be Rs x, then

L.P. = Rs $\frac{3x}{2}$

(iii) Price after discount

$$= \frac{3x}{2}\left(1 - \frac{20}{100}\right)$$

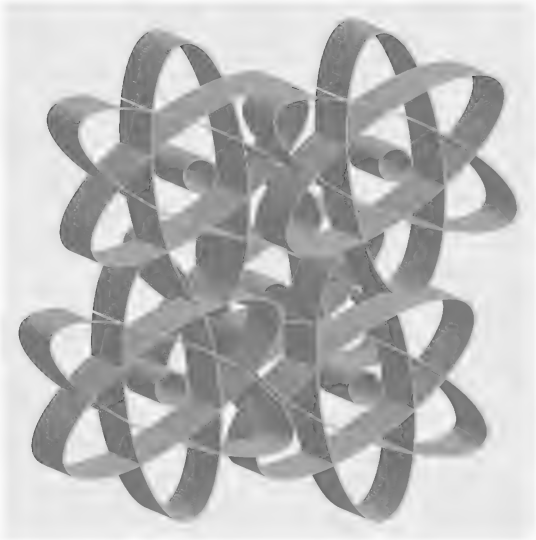
(iv) Selling price with tax

$$= \frac{3x}{2}\left(1 - \frac{20}{100}\right)\left(1 + \frac{20}{100}\right) = 2880.$$

17. Apply the concept of percentages.

18. Find the taxable income and then find the net tax payable from the slabs given.

19. Taxable income is obtained when both the standard deduction and total donation are deducted.



CHAPTER 24

Instalments

INTRODUCTION

Sometimes, a customer cannot buy an article, if he does not have enough money. In such cases, the trader offers the article on instalment basis.

Under this facility, the customer need not pay the entire amount at the time of purchasing the article. He pays only part of the amount at the time of purchasing the article and agrees to pay the balance in small amounts over a period of time. The payment of sale price in small amounts periodically is known as instalment scheme. The small amounts are known as instalments. These instalments may be either monthly, quarterly or yearly.

Usually in this scheme, the customer pays more than the sale price of the article, because the seller charges an interest on the sale price. Generally the interest charged is simple interest unless otherwise mentioned.

Cash price

The price at which the article is offered to the customer, in case he wants to pay the entire amount right away.

Initial payment or Down payment
The amount which a customer has to pay as part payment at the time of purchasing an article is called down payment or initial payment.

Examples

1. A bag is available for Rs 900 cash down or for Rs 500 down payment followed by a payment of Rs 440 after 5 months. Find the rate of interest under the instalment plan.

Solution

Cash price = Rs 900
Down payment = Rs 500

Balance to be paid by instalments = Rs 900 – Rs 500 = Rs 400

The instalment to be paid at the end of 5 months = Rs 440

$\therefore$ The interest charged on Rs 400 = 440 – 400 = Rs 40 for 5 months.

Let r% be the rate of interest per annum.

$\therefore I = \frac{PTR}{100}$

$40 = \frac{(400)(5)(R)}{(100)(12)}$ ($\because$ 5 months $= \frac{5}{12}$ years)

$\Rightarrow \frac{(40)(100)(12)}{(400)(5)} = R$

$\Rightarrow R = 24\%$

$\therefore$ The rate of interest = 24% per annum.

2. A scooter is offered for Rs 28000 cash or for Rs 12000 down payment followed by two monthly instalments of Rs 8200 each. Calculate the rate of interest under the instalment plan.

Solution

Cash price = Rs 28000

Down payment = Rs 12000

Balance to be paid by instalments = Rs 28000 – Rs 12000 = Rs 16000

Let R be the rate of interest per annum.

After two months Rs 16000 will amount to $= 16000 + \frac{16000R(2)}{100(12)} = 16000 + \frac{80R}{3} \rightarrow (1)$

The customer has to pay Rs 8200 each month.

$\therefore$ The first instalment will amount to $= 8200 + \frac{8200\ R}{100(12)} = 8200 + \frac{41R}{6}$

$\therefore$ The second instalment is Rs 8200.

$\therefore$ The total value of the two instalments is $8200 + 8200 + \frac{41R}{6} = 16400 + \frac{41R}{6} \rightarrow (2)$

The value of the loan (Rs 16000) at the end of 2 months is equal to the total value of the two instalments at the end of the two months.

$\therefore 16000 + \ = 16400 + \frac{41R}{6}$

$\Rightarrow \frac{80R}{3} - \frac{41R}{6} = 16400 - 16000$

$\Rightarrow \frac{160R - 41R}{6} = 400$

$\Rightarrow \frac{119R}{6} = 400$

$R = \dfrac{400(6)}{119} \approx 20.17\%$

$\therefore$ The rate of interest is 20.17%.

3. A cycle is offered for Rs 1200 cash or Rs 600 down payment followed by 4 equal monthly instalments. If the rate of interest charged by the dealer is 10% per annum, find the amount of each instalment.

Solution

Cash price = Rs 1200

Down payment = Rs 600

Balance amount = Rs 1200 – Rs 600 = Rs 600

Rate of interest = 10% per annum.

$\therefore$ After 4 months, the amount Rs 600 will be equal to $600 + \dfrac{(600)(10)(4)}{100(12)}$

$= 600 + 20 = \text{Rs } 620 \rightarrow (1)$

Let each instalment be Rs x

The first instalment of Rs x will amount to Rs $\left[x + \dfrac{x(3)(10)}{100(12)}\right)$

i.e., Rs $\left(x + \dfrac{30x}{1200}\right)$ after 3 months.

The second instalment of Rs x will amount to Rs $\left[x + \dfrac{x(2)(10)}{100(12)}\right)$

i.e., Rs $\left(x + \dfrac{20x}{1200}\right)$ after 2 months.

The third instalment of Rs x the will amount to Rs $\left[x + \dfrac{x(10)}{100(12)}\right)$

i.e., Rs $\left(x + \dfrac{10x}{1200}\right)$ after 1 month.

$\therefore$ The 4th instalment is Rs x. The total value of the 4 instalments is

$$= \left(x + \frac{30x}{1200}\right) + \left(x + \frac{20x}{1200}\right) + \left(x + \frac{10x}{1200}\right) + x$$

$$= 4x + \frac{60x}{1200}$$

This is equal to the value of Rs 600 after 4 months

$\therefore \dfrac{81x}{20} = 620$ [from (1)]

$$\Rightarrow x = \frac{620(20)}{81} = 153.09$$

$\therefore$ Each instalment = Rs 153 (approximately)

Repayment of loan

In the above problems, the instalment payment does not extend for more than one year and the interest is calculated at simple interest. But when the amount is very large (E.g., housing loans etc.) the instalments are payable yearly or half yearly and the interest is calculated at compound interest.

Example

A man borrows money from a finance company and has to pay it back in two equal half-yearly instalments of Rs 5115 each. If the interest charged by the finance company is at the rate of 20% per annum, compounded semi-annually, find the sum borrowed.

Solution

Each instalment = Rs 5115

Rate of interest = 20% per annum = 10% per half yearly.

The amount of Rs 5115 paid as an instalment at the end of the first six months includes the principal and interest on it at the rate of 10% half yearly.

$$\therefore \text{ Principal} = 5115/\left(1+\frac{10}{100}\right) = 5115 \times \left(\frac{100}{110}\right) = \text{Rs } 4650$$

Similarly, the value which amounts to Rs 5115 after 1 year at the rate of 10% compounded semi annually is $= 5115/\left(1+\frac{10}{100}\right)^2 = 5115\left(\frac{100}{110}\right)^2$

$\simeq$ Rs 4227.27

$\therefore$ Sum borrowed = 4650 + 4227 = Rs 8877 (approximately)

Hire purchase scheme

In this scheme the buyer, called the Hirer and the seller, called the vendor enter into an agreement which is known as Hire Purchase Agreement.

Important features of hire purchase scheme

1. The Hirer pays an initial payment known as down payment.
2. The vendor allows the hirer to take possession of the goods on the date of signing the agreement, but he does not transfer the ownership of the goods.
3. The hirer promises to pay the balances amount in instalments.
4. If the hirer fails to pay the instalments the vendor can repossess the goods.
5. When goods are repossessed, hirer cannot ask for the repayment of the instalments of money already paid. This money paid will be treated as rent for the period.

test your concepts

Very short answer type questions

1. If an article is bought under an instalment scheme, the amount a customer has to pay as part payment of the selling price of the article at the time of its purchase is called _________.
2. The scheme of buying an article by making the part payments periodically is called _____.
3. If a bicycle is available for Rs 1200 cash or Rs 700 down payment followed by five equal monthly instalments of Rs 120 each, then the interest charged is _____.
4. If an article is bought under an instalment scheme, the amount paid in addition to the cash price of the article is known as _____.
5. Under hire purchase scheme, the ownership of the goods lies with the _________ until the repayment of all instalments is made.
6. Under hire purchase scheme, if the hirer fails to pay the instalments, the vendor can repossess the goods. The money already paid by the hirer is treated as _________
7. A car is available for Rs 200000 cash or Rs 50000 down payment followed by 16 equal monthly instalments. If the interest charged due to this scheme at the end of 16 months is Rs 10,000, then each monthly instalment is _____.
8. A water filter is available for Rs 500 down payment followed by two instalments of Rs 600 each. If the total interest paid is Rs 100 then the cash price of the water filter is _____.
9. An electric generator is offered for Rs 30000 cash or 22% down payment under instalment scheme. If a buyer has chosen the instalment scheme, then the buyer has to pay Rs _____ as down payment.
10. A mobile phone is available for a cash price of Rs 1200 or for a certain down payment followed by three equal instalments of Rs 300 each. The total interest paid is Rs 100 when bought under the instalment scheme. The down payment for the mobile phone is _____.

Short answer type questions

11. An article can be sold for Rs 20000 cash or for Rs 12000 down payment and 4 equal monthly instalments of Rs 2200 each. Find the interest paid.
12. A washing machine is available for Rs 9000 cash or 45% down payment and 3 equal monthly instalments. Each instalment is 20% of the cash payment. If the interest is calculated at simple interest, what is the approximate annual rate of interest?
13. A sum is lent under compound interest, under instalment scheme, interest compounded annually at 10% p.a. If after one year, Rs 4400 is repaid and after another year, the balance Rs 4300 is repaid, find approximate present value of the instalments.
14. A computer is sold by a company for Rs 20000 cash or Rs 8000 down payment followed by 5 equal monthly instalments of Rs 2500 each. Find the total principal on which the interest is charged to realize the total interest in a month.
15. A man borrows Rs 10500 from a finance company and repays it in two equal annual instalments. If the rate of interest being compounded annually is 10% p.a., find the value of each instalment.

Essay type questions

16. A loan has to be repaid in three equal half yearly instalments. If the rate of interest, compounded semi annually, is 16% p.a. and each instalment is Rs 3500, find (approximately) the sum borrowed.

17. A plasma T.V. is available for Rs 42000 cash or Rs 8900 down payment and three equal quarterly instalments of Rs 12000. Find the interest charged by under instalment plan.

18. A table is available for a down payment of Rs 1590 and 3 equal half yearly instalments of Rs 1331 each. If a shop owner charges interest compounded semi-annually at the rate of 20% p.a., then find the cash price of the table.

19. Prasad borrowed some money under compound interest and repaid it in 3 equal instalments of Rs 8470 each. If the rate of interest is 10% p.a. what is the present value of the instalment paid at the end of the second year?

20. A TV set is available for Rs 36000 or some amount of down payment and two equal annual instalments of Rs 12100 each. If a shop keeper charges interest, compounded annually, at the rate of 10% p.a. find the down payment.

CONCEPT APPLICATION

Concept Application Level—1

1. A fan is sold for Rs 900 cash or Rs 200 cash down payment followed by two equal monthly installments of each Rs 375. The annual rate of interest is ______ (approximately)

   (1) 25%. (2) 30%. (3) 54%. (4) 59%.

2. A briefcase available for Rs 500 cash or for a certain cash down payment followed by a payment of Rs 312 after 4 months. If the rate of interest is 12% p.a., find the cash down payment.

   (1) Rs 198 (2) Rs 205 (3) Rs 195 (4) Rs 200

3. A typewriter is available for Rs 970 cash or for a certain cash down payment followed by 3 equal monthly instalments of Rs 260 each. If the rate of interest is 16% p.a., then find the cash down payment for purchasing it. (approximately)

   (1) Rs 200 (2) Rs 210 (3) Rs 205 (4) Rs 220

4. A pressure cooker is sold for Rs 600 cash or Rs 300 cash down payment followed by Rs 310 after one month. The annual rate of interest is _____.

   (1) 40% (2) 30% (3) 33% (4) 20%

5. A ceiling fan is available at a certain cash price or for Rs 250 cash down payment together with Rs 305 to be paid after two months. If the rate of interest is 10% p.a., then find the price of the fan?

   (1) Rs 545 (2) Rs 540 (3) Rs 550 (4) Rs 535

6. A man borrows Rs 6500 from a finance company and has to return it in two equal annual instalments. If the rate of interest is 8% p.a., interest being compounded annually, then each instalment is (in rupees)

(1) 3645 (2) 2916 (3) 2542 (4) 1980

7. A book is available for Rs 800 cash or for Rs 250 cash down payment followed by 3 equal monthly instalments of Rs 200 each. Find the principal for the third month.

(1) Rs 550 (2) Rs 350 (3) Rs 150 (4) Rs 250

8. An article is available for Rs 6000 cash or for Rs 1275 cash down payment and 5 equal monthly instalments. If the rate of interest is 4% per month, then each monthly instalment is (in rupees approximately)

(1) 1200 (2) 954 (3) 1050 (4) 875

9. An article is available for Rs 24000 cash or for a certain cash down payment followed by six equal instalments of Rs 2800 each. If the rate of interest is 12% p.a, find the cash down payment for purchasing it (approximately).

(1) Rs 7500 (2) Rs 7605 (3) Rs 7755 (4) Rs 7800

10. A sum of Rs 64890 is to be paid back in 3 equal annual instalments. If the interest is compounded annually at the rate of $6\frac{2}{3}$% p.a, then each instalment is (in rupees)

(1) 26476 (2) 25326 (3) 22600 (4) 24576

Concept Application Level—2

11. A loan has to be returned in two equal annual instalments. If the rate of interest is 16% p.a., interest being compounded annually and each instalment is Rs 6728, then the total interest is (in rupees)

(1) 4000 (2) 3250 (3) 3600 (4) 2656

12. A loan of Rs 15580 is to be paid back in two equal half yearly instalments. If the interest is compounded half yearly at 10% p.a., then the interest is (in rupees)

(1) 1200 (2) 1500 (3) 1178 (4) 1817

13. A loan has to be returned in two equal annual instalments. If the rate of interest is 15% p.a., interest being compounded annually and each instalment is Rs 3703, then the principal of the loan is (in rupees)

(1) 7090 (2) 6020 (3) 5090 (4) 8040

14. A refrigerator is available at a certain price on full payment or for Rs 1400 cash down payment and five equal monthly instalments of Rs 1030 each. If the rate of interest is 12%, find the cost of the refrigerator approximately.

(1) Rs 6000 (2) Rs 9009 (3) Rs 8008 (4) Rs 6403

15. A sofa set is available for Rs 50000 cash or for Rs 30000 cash down payment followed by 4 equal monthly instalments of Rs 6000 each. Find the principal for the 2nd month.

(1) Rs 20000 (2) Rs 18000 (3) Rs 5000 (4) Rs 14000

Concept Application Level—3

16. A car is available for Rs 200000 cash or for Rs 50000 cash down payment followed by 4 equal half yearly instalments of Rs 52000 each. Find the total interest charged.

(1) Rs 58000 (2) Rs 48000 (3) Rs 44000 (4) Rs 50000

17. A microwave oven is available for Rs 4500 cash or for Rs 2100 cash down payment followed by three equal monthly instalments. If the shop keeper charges interest at the rate of 10% p.a., compounded every month, find the total of the present values of the three instalments.

(1) Rs 1800 (2) Rs 2400 (3) Rs 4500 (4) Rs 2100

18. A gold chain is available for Rs 10500 cash or for Rs 4000 cash down payment and three equal instalments. If the shopkeeper charges interest at the rate of 10% p.a. compounded annually, find the total of the present value of the three instalments.

(1) Rs 8000 (2) Rs 10000 (3) Rs 6500 (4) Rs 4000

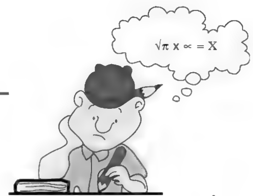
19. Alok borrowed some money from Prasad which is to be paid back in 3 equal annual instalments of Rs 322 each. The present value of the first installment paid at the end of first year is Rs 280. Find the rate of interest if the interest is being compounded annually.

(1) 12% (2) 18% (3) 15% (4) 20%

20. Find the principal if it is lent at 5% p.a. interest being compounded annually, in two parts. One part amounts to Rs 1050 at the end of the 1st year and the other part amounts to Rs 1323 at the end of the 2nd year.

(1) Rs 1900 (2) Rs 2000 (3) Rs 1800 (4) Rs 2200

KEY



Very short answer type questions

1. down payment
2. Instalment scheme
3. Rs 100 **4.** interest
5. Vendor/Seller. **6.** rent.
7. Rs 10000 **8.** Rs 1600
9. Rs 6600 **10.** Rs 400

Short answer type questions

11. Rs 800 **12.** 57.14% p.a.
13. Rs 7554.
14. Rs 35000.
15. Rs 6050

Essay type questions

16. Rs 9020 **17.** Rs 2900
18. Rs 4900 **19.** Rs 7000
20. Rs 15000

key points for selected questions

Short answer type questions

11. (i) First of all, amount paid in instalments is to be found.

(ii) Calculate the difference in actual amount and amount paid, there by calculate the interest.

12. (i) Down payment = 45% of 9000 and each instalment = 20% of 9000.

(ii) Calculate the amount to be paid in the each instalment.

(iii) Equate the above value to the amount that has to be paid and find R.

13. Use, Total present value =

$$\frac{4400}{\left(1+\frac{R}{100}\right)}+\frac{4300}{\left(1+\frac{R}{100}\right)^2}$$ and evaluate R.

14. (i) Calculate the balance to be paid in instalments.

(ii) Calculate the amount to be paid in the each instalment.

(iii) Equate the above value to the amount that has to be paid and find R.

15. Use, Total present value = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}\right]$$ and evaluate R.

Essay type questions

16. Use, Total principal = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}+\frac{1}{\left(1+\frac{R}{100}\right)^3}\right]$$ and

evaluate R.

17. (i) First of all amount paid in instalment is to be found.

(ii) Calculate the amount to be paid in the each instalment.

(iii) Equate the above value to the amount that has to be paid and find R.

18. Use, Total principal = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}+\frac{1}{\left(1+\frac{R}{100}\right)^3}\right]$$ and

evaluate R.

19. Use, Present value at the end of the second year

$$=\frac{\text{each instalment value}}{\left(1+\frac{R}{100}\right)^2}$$ and evaluate R.

20. Use, Total present value = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}\right]$$ and

evaluate R.

Concept Application Level—1,2,3

1. 4	2. 4
3. 2	4. 1
5. 3	6. 1
7. 3	8. 2
9. 3	10. 4
11. 4	12. 3
13. 2	14. 4
15. 4	16. 1
17. 2	18. 3
19. 3	20. 4

Concept Application Level—1,2,3

Key points for select questions

1. Calculate the interest and principle amount for 2 months.
   (i) Assume each installment amount as Rs x.
   (ii) Calculate the balance, interest and the principle.
2. (i) Let the cash down payment amount be Rs x.
   (ii) Calculate balance and interest.
   (iii) Let the cash down payment be Rs x. Balance = Rs (970 – x).
   (iv) Interest paid under instalment scheme = Rs [3 × 260 – (970 – x)] = Rs [x – 190]
   (v) Principal for the 1st month = Rs (970 – x). Principal for the 2nd month = Rs (970 –x – 260), Principal for the 3rd month = Rs (970 – x – 260 – 260)x.
   (vi) Use $SI = \frac{PRT}{100}$, (P = total principal).
4. Find the interest paid under instalment scheme.
5. (i) Let the cost of ceiling fan in cash be Rs x
   (ii) Calculate balance, interest and the principle amount.
   (iii) Let the cash price be Rs x.
   (iv) Balance = Rs (x – 250)
   (v) Interest = Rs [350 – (x – 250)]
   (vi) Use, $SI = \frac{PRT}{100}$, P = Balance, R = 10% and $T = \frac{2}{12}$ years.
6. (i) $\text{Principal} = \dfrac{\text{Installment}}{\left(1+\dfrac{r}{100}\right)^n}$
   (ii) Find the principal value for both the instalments and equate to the total sum.
7. (i) Calculate the balance.
   (ii) Calculate the principle for the first, 2nd and the 3rd months successively.
   (iii) Use, Total principal = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}+\frac{1}{\left(1+\frac{R}{100}\right)^3}\right]$$

   and evaluate R.
8. (i) Let the cash down payment be Rs x.
   (ii) Down payment = 45% of 9000 and each instalment = 20% of 9000.
   (iii) Calculate the amount to be paid in the each instalment.
   (iv) Equate the above value to the amount that has to be paid and find R.
9. (i) Let the cash down payment be Rs x.
   (ii) Use, Total present value = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}\right] \text{ and}$$

   evaluate R.
10. Refer to the hint of Q. No. 4.
11. (i) Refer to the hint of Q. No. 6.

(ii) First of all amount paid in instalment is to be found.

(iii) Calculate the amount to be paid in the each instalment.

(iv) Equate the above value to the amount that has to be paid and find R.

12. Refer to the hint of Q. No. 4.

13. Refer to the hint of Q. No. 4.

14. Let the cost of refrigerator in full payment be Rs x.

16. (i) Interest = (Total amount paid in installments) – (Principal).

(ii) Use, Total present value =

$$\frac{4400}{\left(1+\frac{R}{100}\right)}+\frac{4300}{\left(1+\frac{R}{100}\right)^2} \text{ and}$$

evaluate R.

17. (i) Refer to the hint of Q. No. 6.

(ii) Use, Present value at the end of the second year $= \dfrac{\text{each instalment value}}{\left(1+\frac{R}{100}\right)^2}$ and

evaluate R.

18. Interest = (Total amount paid in installments) – (Principal)

19. (i) Principal $= \dfrac{\text{Installment}}{\left(1+\frac{r}{100}\right)^n}$

(ii) Use, Total present value = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}\right] \text{ and}$$

evaluate R.

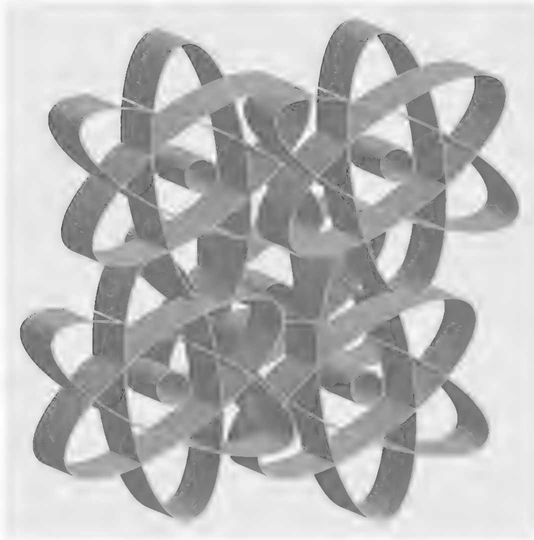
20. Use, Total principal = Each instalment value

$$\left[\frac{1}{\left(1+\frac{R}{100}\right)}+\frac{1}{\left(1+\frac{R}{100}\right)^2}+\frac{1}{\left(1+\frac{R}{100}\right)^3}\right] \text{ and}$$

evaluate R.

CHAPTER 25

Shares and Dividends



INTRODUCTION

The total amount of money needed to start a business enterprise or a company is called the **capital**. A business enterprise may be a sole proprietary concern, a partnership firm or a company.

The capital received to start a big company runs into crores of rupees. Several enterprising people called **promoters** start a company, but they cannot possibly invest such a huge capital. Hence the whole capital is divided into several small and equal units called shares. Then the company invites the public to invest money by buying these shares. All companies cannot raise capital from the public to set up the business only public limited companies can do so.

The money arranged and used for setting up a company or a business enterprise is called the **investment**. Those who arrange this money are called the investors.

A public limited company is a company limited by shares with no restriction on the maximum number of shareholders, transfer of shares and acceptance of public deposits. The minimum number of shareholders is 7.

The capital is classified as follows

Authorised capital is the capital limit authorised by the registrar of companies upto which the shares can be issued to the members or public as the case may be. The paid up share capital is the paid portion of the capital subscribed by the share holders. The minimum authorised capital for a public limited company is Rs 500000.

In other words, **Authorised Capital** is the capital with which the company is registered. The company can issue shares up to the level of Authorized Capital. Paid up capital means the amount of capital actually paid by the shareholders in respect of shares allotted to them.

The following example gives a clear idea of the division of capital into shares.

☛ *Example*

Suppose the capital requirement of the company is Rs 15 crores

Then, the whole capital can be divided into

(a) 15 lakh shares of Rs 100 each or

(b) 30 lakh shares of Rs 50 each or

(c) 60 lakh shares of Rs 25 each

For every investment made, the company issues a share-certificate, showing the value of each share and the number of shares held by the investor.

A person who is allotted shares or who buys shares is called a shareholder. After the shares are allotted, the share holder can sell them in the stock market.

Nominal value of a share (N.V.)

Nominal value of a share is the value printed on the share certificate. It is also known as face value (FV) or par value.

Market value of a share (M.V.)

Market value of a share is the value of the share at which it is bought or sold in the market.

Note: A share is said to be:

(a) at premium or above par, if MV > FV.

(b) at discount or below par, if MV < FV.

(c) at par, if MV = FV.

Dividend

Dividend is the annual profit distributed among the shareholders.

Dividend is paid as per share or as a percentage. If it is paid as a percentage, it is reckoned on the face value.

Note:

(a) The face value of a share always remains the same.

(b) The market value of a share changes from time to time.

(c) Dividend is always paid on the face value of a share.

(d) The number of shares held by a person $= \dfrac{\text{Total income}}{\text{Income from each share}} = \dfrac{\text{Total face value}}{\text{Face value of each share}}$

Return on investment

$$\text{Rate of return} = \frac{\text{Annual income}}{\text{Total investment}} \times 100$$

Examples based on basic concepts

Example

Find the market value of a Rs 200 share bought at a premium of Rs 50.

Solution

Face value of the share = Rs 200

The share is bought at a premium of Rs 50.

∴ Market value of the share = Rs (200 + 50) = Rs 250

Example

If a Rs 100 share is available at a discount of Rs 10, then find the market value of 270 such shares.

Solution

Face value of the share = Rs 100

The share is available at a discount of Rs 10

Market value of each share = 100 – 10 = Rs 90

Market value of 270 such shares = 90 (270) = Rs 24300

Example

Satish invests a certain amount in 200, Rs 60 shares of a company paying a dividend of 8%. Find Satish's annual income from the investment.

Solution

Face value of the share = Rs 60

Number of shares bought = 200

Rate of dividend paid = 8%

Annual income from each share = $\frac{8}{100}(60) = \frac{480}{100}$ = Rs 4.80

$\therefore$ Annual income Satish derived from 200 shares = 4.8 × 200 = Rs 960.

Example

Karan invests Rs 18000 in buying Rs 100 shares of a company available at a discount of Rs 10. If the company pays a dividend of 10%, then find the number of shares bought by Karan and the rate of return on his investment.

Solution

Total investment = Rs 18000

Face value of each share = Rs 100

The shares are available at a discount of Rs 10

Market value of each share = 100 –10 = Rs 90

Number of shares bought = $\frac{\text{Total investment}}{\text{Market value of each share}} = \frac{18000}{90} = 200$

$\Rightarrow$ Number of shares bought by Karan = 200

Rate of dividend = 10%

Annual Income from each share = 10% (100) = Rs 10

$\Rightarrow$ Annual income from 200 shares = 200 (10) = Rs 2000

$\Rightarrow$ The rate of return on his investment = $\frac{\text{Annual income}}{\text{Total investment}}(100) = \frac{2000}{18000}(100) = \frac{100}{9}\% = 11\frac{1}{9}\%$

$\therefore$ Karan gets a return of $11\frac{1}{9}\%$ p.a. on his investment.

Alternative solution

We have, FV × Rate of dividend = MV × Rate of return.

$100 \times 10\% = 90 \times$ Rate of return

$\Rightarrow$ Rate of return $= 11\frac{1}{9}\%$

Example

Which is a better investment: Rs 200 shares at Rs 220 which pay a dividend of 10% or Rs 100 shares at Rs 120 which pay a dividend of 12%?

Solution

1st Case

Rs 200 shares at Rs 210 paying a dividendof 10%

$$\text{Rate of return} = \frac{\text{FV} \times \text{Rate of dividend}}{\text{MV}} = \frac{200 \times 10\%}{220} = 9\frac{1}{11}\%$$

2nd Case

Rs 100 Shares at Rs 120 paying a dividendof 12%

$$\text{Annual income from each share} = 12\% \text{ of } 100 = \frac{12}{100}(100) = \text{Rs } 12$$

$$\text{Rate of return} = \frac{\text{FV} \times \text{Rate of dividend}}{\text{MV}}$$

$$= \frac{100 \times 12\%}{120} = 10\%$$

$\therefore$ 12%, Rs 100 shares at Rs 120 is a better investment.

test your concepts

Very short answer type questions

1. The whole capital divided into small and equal units are called _____.
2. The value of a share printed on the share certificate is called its _____.
3. Dividend is always calculated on the _____ of a share.
4. _____ is the annual profit distributed among the share holders.
5. The value at which a share is bought or sold in the market is called its _____.
6. A person who is allotted shares or who buys shares is called a _____ in the company.
7. Does the face value of a share remain constant all the time?
8. Is the market value of a share always constant?

9. If a Rs 180 share is quoted at a premium of Rs 27, then the market value of each share is _____.

10. A Rs 100 share is available at par, then its market value is _____.

Short answer type questions

11. If Anil bought a Rs 150 share at a premium of Rs 20 while Raju bought the same kind of share at a discount of Rs 20 of the same company, then the dividend earned by Anil is _____ the dividend earned by Raju. [equal to/less than/more than]

12. Find the market value of 300, Rs 150 shares bought at a premium of Rs 30.

13. Find the annual income derived from an investment of Rs 18000 in Rs 150 shares available at Rs 180 of a company paying 11% dividend.

14. Rs 60000 is invested in buying Rs120 shares of a company which are available at a premium of 25%. Find the number of shares bought and the annual rate of return on the investment, if the dividend is paid at the rate of 10% per annum.

15. An investment of Rs 36 on a share earns the investor a return of 10%. If dividend is 12% on each share, what is its face value?

16. A man invests Rs 20000 in Rs 80 shares of a company available at a premium of Rs 20. If the company pays a dividend of 8%, then find the rate of return on the investment.

17. If Satish bought 120, Rs 200 shares of a particular company at a premium of Rs 24, paying 9% dividend, then find Satish's annual income.

18. If an investment of Rs 42000 in Rs 300 shares of a company, paying a dividend of 7%, results in an annual income of Rs 2100, then find the market value of each share.

19. A person invested Rs 18000 in buying Rs 150 shares of a company which are available at a premium of Rs 50. If the company pays a 9% dividend, then find the number of shares bought by him and also the annual income he derives from the investment.

20. Which is a better investment, A: Rs 60 shares at Rs 75 paying a dividend of 10% (or)
B: Rs 100 share at Rs 120 paying a dividend of 12%?

Essay type questions

21. Naresh invested a certain amount in Rs 100 shares at a discount of Rs 20 paying a dividend of 7% while Sahil invested an equal amount in Rs 90 shares at a discount of Rs 10 paying a dividend of 8%. Who earns more, Naresh or Sahil?

22. How much should a man invest in 12%, Rs 150 shares of a company available at a premium of Rs 30, if the annual income earned is to be Rs 3600?

23. Rupesh invests Rs 45000 partly in Rs 150 shares at Rs 180 for 8% and partly in Rs 75 shares at Rs 135 for 12%. If the annual incomes from the investments are in the ratio 2 : 3 respectively, then find the investments made by Rupesh in the two kinds of shares.

24. Rahul invested a certain amount in buying Rs 25 shares of a company, which pays a dividend of 12%. If he earns 10% per annum on his investment, then find the market value of each share.

25. Kavya bought 300, Rs 50 shares paying a dividend of 8%. If she sold them when the price rose to Rs 90 and invested the proceeds in 10%, Rs 50 shares at Rs 30, then find the change in her annual income.

CONCEPT APPLICATION

Concept Application Level—1

1. If a Rs 200 share is bought at a discount of Rs 50 and the dividend paid is 9%, then the rate of return is _____ per annum.

(1) 18% (2) 9% (3) 10% (4) 12%

2. A person invests Rs 20000 at 20%, Rs 150 shares at a premium of Rs 50. The income from these shares in Rs is _________.

(1) 1000 (2) 3000 (3) 1500 (4) 2000

3. If 200, Rs 100 shares are bought at a premium of Rs 20, then the total investment to be made is Rs _____.

(1) 20000 (2) 40000 (3) 24000 (4) 48000

4. A person invests Rs 15000 in Rs 50 shares of a company paying 10% dividend. If the company pays the a dividend of Rs 6250, the market value of each share is _______.

(1) Rs 8 (2) Rs 10 (3) Rs 12 (4) Rs 7

5. A man buys 24 shares at Rs 150 per share having the par value of Rs 100. If the dividend is 7.5% p.a., then the ratio of total annual income to his total investment is_______.

(1) 1 : 10 (2) 10 : 1 (3) 1 : 20 (4) 20 : 1

6. The total investment made in buying Rs X shares at a premium of 25% is Rs 125X. The number of shares bought is ______.

(1) 500 (2) 250 (3) 125 (4) 100

7. Praveen invests Rs 24000 in Rs 100 shares of a company paying 10% dividend. If his annual income from these shares is Rs 1200, then the market value of each share is Rs _______.

(1) 100 (2) 200 (3) 300 (4) 400

8. Which of the following is/are true for the statement, "9%, Rs 100 shares at Rs 120"?

(1) Dividend on 1 share = Rs 9
(2) Rate of return is 7.5%
(3) Both (1) and (2)
(4) None of the above

9. A man has 62 shares of Rs 100 each, the dividend is 7.5% p.a. If he wants to increase his annual income by Rs 150, then he should buy _______ shares more.

(1) 20 (2) 40 (3) 10 (4) 30

10. The market value of x, Rs 50 shares, at a discount of Rs 10 is Rs 4000, then x = ______.

(1) 10 (2) 100 (3) 50 (4) 75

Concept Application Level—2

11. A man buys 500, Rs 10 shares at a premium of Rs 3 on each share. If the rate of dividend is 12%, then the rate of interest received by him on his money is ________.(approximately)

(1) 6.5% (2) 7.5% (3) 9.23% (4) 5%

12. Rishi bought Rs 50 shares of a company for Rs 75 each. The company pays a dividend of 12% per annum. The effective rate of return on his investments is _______.

(1) 6% (2) 8% (3) 9% (4) 10%

13. If Rs 56000 is invested in buying Rs 120 shares at a discount of Rs $16^2/_3$%, then find the number of shares bought is _______.

(1) 2800 (2) 560 (3) 280 (4) 400

14. The market value of a share is Rs 90, its face value is Rs 80, and the dividend is 8%. The rate of return is _________.

(1) $8\frac{1}{3}\%$ (2) $7\frac{1}{9}\%$ (3) $6\frac{2}{3}\%$ (4) $5\frac{2}{3}\%$

15. Ankit invests Rs 18000 in buying Rs 270 shares in a company at a premium of Rs 30. If the dividend paid is 10% per annum, then Ankit's annual income from these shares is _________.

(1) Rs 162 (2) Rs 1620 (3) Rs 16.20 (4) Rs 16200

16. Which is a better investment: "15%, Rs 220 shares at Rs 240 of company A" (or) "18%, Rs 200 shares at Rs 240 of company B"?

(1) Company A (2) Company B
(3) Both (1) and (2) (4) Cannot say

17. Mr Gellard invested Rs 16000 in 7%, Rs 200 shares at Rs 160. He sold the shares at Rs 150 each. He incurred a _______.

(1) loss of Rs 1300 (2) loss of Rs 1000
(3) profit of Rs 1000 (4) profit of Rs 1200

18. Priya invested Rs 27000 in Rs 180 shares at Rs 150 at the start of a financial year. The company paid a dividend of 12%. Priya sold all her shares at the end of the year. Over this period, her net earnings were Rs 1828. She sold the shares for _______each.

(1) Rs 160.15 (2) Rs 181.75 (3) Rs 185 (4) Rs 138.56

19. Teja brought Rs 80 shares of a company which pays 15% dividend. If the rate of return is 12%, then the market price is _______.

(1) Rs 90 (2) Rs 80 (3) Rs 100 (4) Rs 99

20. Randeep invests Rs 25200 in buying shares of face value of Rs 40 each at 5% premium. The dividend on these shares is 10% per annum. Find the dividend he receives annually.

(1) Rs 2520 (2) Rs 2400 (3) Rs 4200 (4) Rs 3680

Concept Application Level—3

21. Which of the following is a better investment?

(a) 10%, Rs 100 shares at Rs 120. (b) 9%, Rs 100 shares at Rs 110.

(1) a (2) b
(3) Both (a) and (b) (4) Data insufficient

22. Vamsi invested some amount buying Rs 150 shares of a company which pays dividend at the rate of 8% per annum. If he gets back 10% p.a. on his investment, then the value at which he bought the shares is ______.

(1) Rs 100 (2) Rs 120 (3) Rs 180 (4) Rs 210

23. Mr.Srikar bought Rs 80 shares of a company at a premium of Rs 10. If the company pays a dividend at the rate of x% per annum and Mr.Srikar earns at the rate of y% per annum on his investment, then x : y is ______.

(1) 1 : 8 (2) 8 : 1 (3) 8 : 1 (4) 9 : 8

24. Sachin invests Rs 15000 on buying Rs 100 shares of a company at a premium of Rs 50 and gets $6^{2}/_{3}\%$ per annum on his investment. Then the rate at which the company pays the dividend is ________.

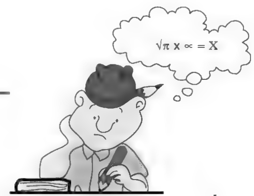
(1) 5% (2) 8% (3) 9% (4) 10%

25. Praveen made an investment of Rs 54000 in 10%, Rs 100 shares at Rs 90 while Vijay made an investment of Rs 60000 in 8%, Rs 150 shares at Rs 120. If both of them sold their shares at the end of the year for Rs 110 each, then their individual earnings are respectively ________.

(1) Rs 12000 and Rs 12000 (2) Rs 18000 and Rs 1000

(3) Rs 12000 and Rs 1000 (4) Rs 18000 and Rs 1200

KEY



Very short answer type questions

1. shares
2. face value (or) nominal value (or) par value
3. face value
4. Dividend
5. market value
6. shareholder
7. Yes
8. No
9. Rs 207
10. Rs 100

Short answer type questions

11. equal to
12. Rs 54000
13. Rs 1650
14. 400; 8%
15. Rs 30
16. 6.4%
17. Rs 2160
18. Rs 420
19. 90; Rs 1215
20. Rs 100 shares at Rs 120 paying a dividend of 12%

Essay type questions

21. Sahil.
22. Rs 36000
23. Rs 18000 and Rs 27000.
24. Rs 30.
25. Rs 3300.

key points for selected questions

Short answer type questions

12. MV = FV + Premium.

13. (i) Find the number of shares purchased.
(ii) Then income from one share (on FV) is to be found.
(iii) Calculate the total income.

14. Number of shares = Total investment/MV.

15. Use the formula, FV × Rate of dividend = MV × Rate of return.

16. Use the formula FV × Rate of dividend = MV × Rate of return.

17. (i) First of all, income from one share (on FV) is to be found.
(ii) Calculate the total income on 120 shares.

18. (i) First of all, annual income from each share is to be found.
(ii) Use, FV × Rate of dividend = MV × Rate of return.

19. Number of shares = Total investment/MV.

20. (i) Rate of returns are to be found in two cases.
(ii) The investment with the high rate of return is the best.

Essay type questions

21. Find the annual income from each share of both kinds.

22. (i) First of all, rate of return is to be found.
(ii) Use, FV × Rate of dividend = MV × Rate of return.

23. (i) Let investment made in Rs 150 shares be Rs x and Rs 75 shares be Rs (45000 – x).
(ii) Find income from each share of Rs 150
(iii) Find total income from Rs 150 shares.
(iv) Find income from each share of Rs 75 and find total income from Rs 75 shares.
(v) The incomes from two types shares should be in 2 : 3 ratio and find x.

24. Use the formula, FV × Rate of dividend = MV × Rate of return.

25. Dividends on both the kinds to be found.

KEY

Concept Application Level—1,2,3

1. 4	**2.** 2	**21.** 1	**22.** 2
3. 3	**4.** 3	**23.** 4	**24.** 4
5. 3	**6.** 4	**25.** 2	
7. 2	**8.** 3		
9. 1	**10.** 2		
11. 3	**12.** 2		
13. 2	**14.** 2		
15. 2	**16.** 2		
17. 2	**18.** 4		
19. 3	**20.** 2		

Concept Application Level—1,2,3

Key points for select questions

1. Rate of dividend × FV = Rate of return × MV

2. (i) First of all the annual income from one share is to be found.
(ii) MV = FV + Premium.

(iii) Find the number of shares.

(iv) Income = $\dfrac{(\text{number of shares} \times \text{FV}) \times (\text{Rate of dividend})}{100}$.

3. Investment = MV × (No. of Shares)

4. (i) Rate of return = $\dfrac{\text{Annual income}}{\text{Total investment}} \times 100$

(ii) By using the following formula find MV.
Use, FV × Rate of dividend = MV × Rate of return

5. (i) Require rates = Annual income = MV of 1 share.

(ii) Income from 1 share = $\dfrac{(\text{FV} \times \text{Rate of dividend})}{100}$.

(iii) Then find the total income from 24 shares.

(iv) MV = (150 + 100),
Total investment = MV × 24.

6. No. of Shares = Total investment/MV

7. (i) MV = $\dfrac{\text{Total investment}}{\text{No. of shares}}$

(ii) Rate of return = $\dfrac{\text{Annual income}}{\text{Total investment}} \times 100$

(iii) By using the following formula find MV.
Use, FV × Rate of dividend = MV × Rate of return

9. Find annual income from one share.

11. (i) Rate of dividend × FV = MV × Rate of return

(ii) In the given data, rate of interest is same as the rate of return.

(iii) FV × Rate of dividend = MV × Rate of return.

12. (i) Same as Q. No. 6
Use the following formula to find the rate of return.
FV × Rate of dividend = MV × Rate of return.

13. Number of shares = $\dfrac{\text{Amount invested}}{\text{M.V}}$

14. M.V × Rate of return = Rate of dividend × F.V

15. First of all the no. of shares is to be found.

16. (i) Find the dividends of the two companies.

(ii) Find the rate of return in two cases by using the following formula.
FV × Rate of dividend = MV × Rate of return.

(iii) The one with greater rate of return is better.

17. (i) No. of shares = $\dfrac{\text{Total investment}}{\text{MV}}$

(ii) Number of shares = $\dfrac{\text{Total investment}}{\text{MV of each share}}$

(iii) SP of total shares = Rs 150 × Number of shares.

18. (i) Number of shares = $\dfrac{(\text{Total investment})}{(\text{MV of each share})}$

(ii) Find the dividend paid on all the shares.

(iii) SP = $\dfrac{(27000 + 1828 - \text{Dividend})}{\text{No. of shares}}$

19. M.V × Rate of return = Rate of dividend × F.V

22. (i) Same as Q. No. 6
Use the following formula to find the MV.
FV × Rate of dividend = MV × Rate of return.

23. M.V × Rate of return = Rate of dividend × F.V

24. (i) MV = FV + Premium

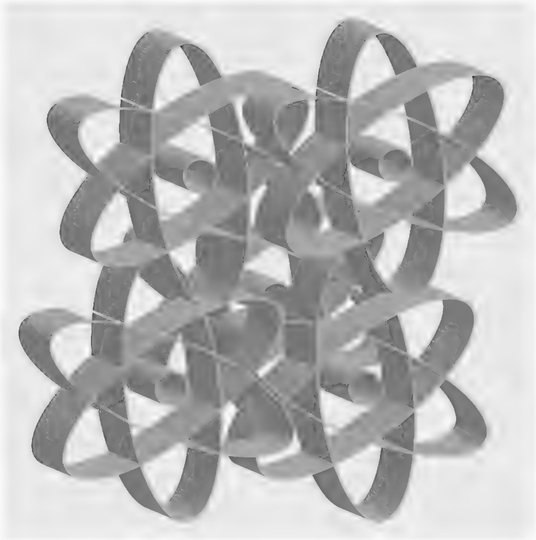
(ii) Use the following formula to find the rate of dividend.
FV × Rate of dividend = MV × Rate of return.

25. (i) Find the dividends received by two persons.

(ii) Number of shares = $\frac{\text{(Total investment)}}{\text{(MV of each share)}}$

(iii) Find their dividends by using FV and rate of dividend.

(iv) Find their profit/loss obtained by selling each at Rs 110.



CHAPTER 26

Partial Fractions

INTRODUCTION

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} +$----------$+ a_n$, where $a_0, a_1,$ -----, a_n are real numbers and $a_0 \neq 0$ is called a polynomial of degree n.

The quotient of any two polynomials f(x) and g(x) where $g(x) \neq 0$ is called a rational fraction.

In a rational fraction $\frac{f(x)}{g(x)}$ if the degree of f(x) is less than the degree of g(x), then the rational fraction $\frac{f(x)}{g(x)}$ is called a proper rational fraction or simply a proper fraction.

Example

$\frac{2x+3}{3x^2-5x+2}, \frac{3}{x-4}$ are some examples of proper fractions,

If the degree of f(x) is greater than or equal to the degree of g(x), then the rational fraction $\frac{f(x)}{g(x)}$ is called an improper fraction.

Example

$\frac{x^2+2x-1}{2x^2-5x+1}, \frac{3x^3+5x+7}{x^2+6x+5}, \frac{x^2+x+1}{2x+1}$ are some examples of improper fractions.

NOTE: The sum of two proper rational fractions is a proper fraction.

Example

$\frac{2}{x+3}$ and $\frac{5}{2x+1}$ are two proper fractions.

Consider $\frac{2}{x+3}+\frac{5}{2x+1}=\frac{2(2x+1)+5(x+3)}{(x+3)(2x+1)}=\frac{9x+17}{2x^2+7x+3}$, which is a proper fraction

i.e., a proper fraction may be split into sum of two or more proper fractions, which are also called partial fractions.

The process of expressing a proper fraction as the sum of two or more proper fractions is called resolving it into partial fractions.

NOTE: If an improper fraction f(x)/g(x) is to split into partial fractions, first we divide f(x) by g(x) till we obtain a remainder r(x) which is of lower degree than g(x) and then we express the given fraction as $\frac{f(x)}{g(x)}=$ quotient + $\frac{r(x)}{g(x)}$, Then we resolve the proper fraction $\frac{r(x)}{g(x)}$ into partial fractions.

Now let us discuss about the different methods of resolving a given proper fraction $\frac{f(x)}{g(x)}$ into partial fractions.

Method 1

When g(x) contains non repeated linear factors only. For every non repeated linear factor of the form ax + b of g(x), there exists a corresponding partial fraction of the form $\frac{A}{ax+b}$.

Example

Resolve $\frac{3x+5}{(x+2)(3x-1)}$ into pa rtial fractions.

Solution

In the given fraction, the denominator has two linear, non-repeated factors.

∴ The given fraction can be written as the sum of two partial fractions.

Let $\frac{3x+5}{(x+2)(3x-1)}=\frac{A}{x+2}+\frac{B}{3x-1}$

$\Rightarrow 3x + 5 = A(3x - 1) + B(x + 2)$ ______(1)

Put x = –2 in (1)

$\Rightarrow 3(-2) + 5 = A[3(-2) - 1] + B(-2 + 2)$

$-1 = -7A \Rightarrow A = \frac{1}{7}$

Comparing the constant terms on both sides of (1), we have

$5 = -A + 2B$

$5 = -\frac{1}{7} + 2B$ or $B = \frac{18}{7}$

$\therefore \frac{3x+5}{(x+2)(3x-1)}=\frac{1/7}{x+2}+\frac{18/7}{3x-1}=\frac{1}{7}\left[\frac{1}{x+2}+\frac{18}{3x-1}\right]$

Example

Resolve $\dfrac{2x^2 + 5x - 1}{x^2 - 3x - 10}$ into partial fractions.

Solution

Given $\dfrac{2x^2 + 5x - 1}{x^2 - 3x - 10}$

We can clearly notice that the given fraction is an improper fraction. So dividing the numerator by the denominator we can express $\dfrac{2x^2 + 5x - 1}{x^2 - 3x - 10}$ as $2 + \dfrac{11x + 19}{x^2 - 3x - 10}$.

Let $\dfrac{11x + 19}{(x+2)(x-5)} = \dfrac{A}{x+2} + \dfrac{B}{x-5}$

$\Rightarrow 11x + 19 = A\,(x - 5) + B\,(x + 2) \rightarrow (1)$

put $x = 5$ in (1) $\Rightarrow 55 + 19 = A\,(0) + B\,(7)$

$\therefore\ B = \dfrac{74}{7}$

put $x = -2$ in (1) $\Rightarrow -22 + 19 = A\,(-7) + B\,(0) \Rightarrow A = \dfrac{3}{7}$

$\therefore\ \dfrac{2x^2 + 5x - 1}{x^2 - 3x - 10} = 2 + \dfrac{3}{7(x+2)} + \dfrac{74}{7\,(x-5)}$

Method 2

When g(x) contains some repeated linear factors and the remaining are non repeated linear factors.

Let $g\,(x) = (px + q)^n\,(a_1x + \alpha_1)\,(a_2x + \alpha_2)$ -------- $(a_nx + \alpha_n)$,

then there exist fractions of the form $\dfrac{A_1}{Px+q}, \dfrac{A_2}{(Px+q)^2}$,-------- $\dfrac{A_n}{(px+q)^n}$ corresponding to every repeated linear factor and fractions of the form $\dfrac{B_1}{a_1\ x + \alpha_1}, \dfrac{B_2}{a_2\ x + \alpha_2}$,-----; $\dfrac{B_n}{a_n\ x + \alpha_n}$ corresponding to every non-repeated linear factors where $A_1, A_2 \ldots\ldots, A_n$, and B_1, B_2----- B_n, are real numbers.

Example

Resolve $\dfrac{2x^2 - 5x + 7}{(x+1)^2\ (x + 3)\ (2x+1)}$ into partial fractions.

Solution

Let $\dfrac{2x^2 - 5x + 7}{(x+1)^2\ (x + 3)\ (2x+1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+3} + \dfrac{D}{2x+1}$

$\Rightarrow 2x^2 - 5x + 7 = A\,(x + 1)\,(x + 3)\,(2x + 1) + B\,(x + 3)\,(2x + 1) + C\,(x + 1)^2\,(2x + 1) + D\,(x + 1)^2$ $(x + 3) \rightarrow (1)$

Substituting $x = -1$ in (1), we have

$2(-1)^2 - 5(-1) + 7 = A(0) + B(-1+3)(-2+1) + C(0) + D(0)$

$14 = -2B$

$B = -7$

Substituting $x = -3$ in (1), we get

$40 = A(0) + B(0) + C(-20) + D(0)$

$-20C = 40$

$C = -2$

Substituting $x = -\frac{1}{2}$ in (1), we have

$10 = \frac{5}{8}D \Rightarrow D = 16$

Substituting $x = 0$ we have

$7 = 3A + 3B + C + 3D$

substituting the values of B, C, D in the above equation, we get

$A = -6$

$$\frac{2x - 5x+7}{(x+1)^2(x+3)(2x+1)} = \frac{-6}{x+1} + \frac{-7}{(x+1)^2} + \frac{-2}{x+3} + \frac{16}{2x+1}$$

Method 3

When g(x) contains non-repeated irreducible factors of the form px^2+qx+c.

Corresponding to every non-reducible non-repeated quadratic factors of g(x), there exists a partial fraction of the form $\frac{Ax+B}{px^2+qx+c}$ where p, q, A, and B are real numbers.

Example

Resolve $\frac{2x-5}{(x+2)(x^2-x+5)}$ into partial fractions

Solution

Let $\frac{2x-5}{(x+2)(x^2-x+5)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-x+5}$

$\Rightarrow 2x - 5 = A(x^2 - x + 5) + (Bx + C)(x + 2) \rightarrow (1)$

put $x = -2$ in (1)

$\Rightarrow -9 = A(11) + 0$

$A = \frac{-9}{11}$

put $x = 0$ in (1), we have

$-5 = 5A + 2C$

$-5 = 5 \times \frac{-9}{11} + 2C$

$\Rightarrow C = \dfrac{-5}{11}$

Comparing the coefficients of x^2 on both sides of (1), we have $A + B = 0$

$\Rightarrow B = -A = \dfrac{9}{11}$

$$\therefore \frac{2x-5}{(x+2)(x^2-x+5)} = \frac{-9}{11(x+2)} + \frac{\frac{9}{11}x - \frac{5}{11}}{x^2-x+5} = \frac{1}{11}\left[\frac{9x-5}{x^2-x+5} - \frac{9}{x+2}\right]$$

Method 4

When g(x) contains repeated and non-repeated irreducible quadratic factors of the form $(px^2 + qx + r)^n$. Corresponding to every repeated irreducible quadratic factor of g(x) there exist partial fractions of the form $\dfrac{p_1x+q_1}{(px^2+qx+r)}$, $\dfrac{p_2x+q_2}{(px^2+qx+r)^2}$, ---------, $\dfrac{p_nx+q_n}{(px^2+qx+r)^n}$ where $p_1, p_{2,}$ ------, p_n and $q_1, q_{2,}$ -------, q_n are real numbers.

Example

Resolve $\dfrac{2x+1}{(x+3)(x^2+1)^2}$ into partial fractions

Solution

Let $\dfrac{2x+1}{(x+3)(x^2+1)^2} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

$\Rightarrow 2x + 1 = A(x^2 + 1)^2 + (Bx + C)(x + 3)(x^2+ 1) + (Dx + E)(x + 3) \rightarrow (1)$

put $x = -3$ in (1),

$\Rightarrow -5 = 100A \Rightarrow A = \dfrac{-1}{20}$

Comparing the coefficients of x^4 on either sides of (1), we have

$0 = A+B \Rightarrow B = \dfrac{1}{20}$

Comparing the coefficients of x^3 on either sides of (1), we have

$0 = 3B + C \Rightarrow C = \dfrac{-3}{20}$

put $x = 0$ in (1), we have

$\Rightarrow 1 = A + 3C + 3E \Rightarrow 3E = 1 + \dfrac{1}{20} + \dfrac{9}{20}$

$\Rightarrow E = \dfrac{1}{2}$

By putting $x = 1$ in (1) we have

$3 = 4A + 8B + 8C + 4D + 4E$

$$3 = \frac{-4}{20}+\frac{8}{20}+8\left(\frac{-3}{20}\right)+4\text{D}+4\left(\frac{1}{2}\right)$$

$$3+\frac{4}{20}-\frac{8}{20}+\frac{24}{20}-2 = 4\text{D}$$

$$\text{D} = \frac{2}{4}=\frac{1}{2}$$

$$\therefore \frac{2x+1}{(x+3)(x^2+1)^2} = \frac{\frac{-1}{20}}{x+3}+\frac{\frac{x}{20}-\frac{3}{20}}{x^2+1}+\frac{\frac{1}{2}x+\frac{1}{2}}{(x^2+1)^2}$$

$$= \frac{-1}{20(x+3)}+\frac{1}{20}\frac{(x-3)}{x^2+1}+\frac{1}{2}\frac{(x+1)}{(x^2+1)^2}$$

test your concepts

Very short answer type questions

1. Improper fraction is a fraction in which the degree of the numerator is ______ than or equal to the degree of denominator.

2. Improper fraction can always be written as the ________ of a polynomial and a proper fraction.

3. The partial fractions of $\frac{Ax+B}{x^4}$ has __________terms.

4. A proper fraction is one in which the degree of numerator is ________ than the degree of denominator.

5. A proper fraction may be written as the sum of __________.

6. The number of terms of the partial fraction of $\frac{x^3}{(x-1)(2x-3)(x-4)}$ is ________.

7. The partial fraction of $\frac{2(x^2+1)}{x^3-1}$ is $\frac{1}{x-1}-\frac{f(x)}{g(x)}$. Then the degree of f(x) is ________.

8. There are three terms of partial fractions of the proper fraction $\frac{f(x)}{x^3}$. Then the degree of the polynomial f(x)would be ______.

9. If $\frac{x^2-2x-9}{(x^2+x+6)(x+1)} = \frac{2x-3}{x^2+x+6}+\frac{A}{x+1}$, then A is ________.

10. If the partial fractions of $\frac{3x^2+4}{(x-1)^3}$ is $\frac{3}{x-1}+\frac{A}{(x-1)^2}+\frac{7}{(x-1)^3}$, then A $=$ _________.

11. If $\frac{f(x)}{x^2-x+1} - \frac{1}{f(x)}$ is the partial fraction of $\frac{3x}{x^3+1}$, then f(x) is ________.

12. If $\frac{ax^2+bx+c}{(x+1)(x^2-3x-4)} = \frac{f(x)}{x+1} + \frac{g(x)}{x+4} + \frac{h(x)}{x-1}$, then the degree of f(x), g(x) and h(x) is _______.

13. If $\frac{2x^2+3x+4}{(x+2)^4} = \frac{A}{(x+2)} + \frac{2}{(x+2)^2} - \frac{5}{(x+2)^3} + \frac{6}{(x+2)^4}$, then A = _________.

14. If $\frac{x^2}{(x-a)(x-b)} = 1 + \frac{a^2}{(a-b)(x-a)} + \frac{B}{(x-b)}$, then B is _________.

15. If $\frac{x^4}{(x+1)(x+2)} = f(x) + \frac{A}{x+1} + \frac{B}{x+2}$, then the degree of the polynomial f(x) is ______.

Short answer type questions

16. Resolve into partial fractions: $\frac{1}{x^2-a^2}$

17. Resolve into partial fractions: $\frac{2x+5}{x^2+3x+2}$

18. Resolve into partial fractions: $\frac{3x+4}{x^2+x-12}$

19. Resolve into partial fractions: $\frac{2x^2+5x+8}{(x-2)^3}$

20. Resolve into partial fractions: $\frac{x+1}{(x+2)(x^2+4)}$

21. Find the value of A, B and C if $\frac{2x^2-x-10}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$.

22. If $\frac{x+5}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$, then find the values of A and B.

23. Find the partial fractions of $\frac{ax+b}{(x-1)^2}$, where a and b are constants.

24. Resolve into partial fractions: $\frac{2x^2-3x+3}{x^3-2x^2+x}$.

25. Resolve into partial fractions: $\frac{x^2+5x+6}{x^3-7x-6}$

26. Resolve into partial fractions: $\dfrac{x^3}{(x+1)(x+2)}$.

27. Resolve into partial fractions: $\dfrac{x+1}{x^3-1}$

28. Resolve into partial fractions: $\dfrac{2x^2+4}{x^4+5x^2+4}$.

29. Find the partial fractions of $\dfrac{2x+3}{x^4+x^2+1}$.

30. Resolve into partial fractions: $\dfrac{1}{x^4+x}$.

Essay type questions

31. Resolve into partial fractions: $\dfrac{3x^2+4x+5}{x^3+9x^2+26x+24}$

32. Resolve into partial fractions: $\dfrac{x^2-x-1}{x^4+x^2+1}$

33. Resolve into partial fractions: $\dfrac{3x+4}{x^3-2x-4}$.

34. Resolve into partial fractions: $\dfrac{x^2-3}{x^3-2x^2-x+2}$

35. Resolve into partial fractions: $\dfrac{x^4+1}{(x-2)(x+2)}$

CONCEPT APPLICATION

Concept Application Level—1

1. Resolve $\dfrac{1}{x^2-9}$ into partial fractions.

(1) $\dfrac{1}{3(x-3)}-\dfrac{1}{3(x+3)}$

(2) $\dfrac{1}{2(x-3)}-\dfrac{3}{2(x+3)}$

(3) $\frac{1}{6(x-3)}-\frac{1}{6(x+3)}$

(4) $\frac{1}{6(x-3)}+\frac{1}{6(x+3)}$

2. Resolve $\frac{2x+3}{x^2-6x+5}$ into partial fractions.

(1) $\frac{1}{4}\left(\frac{13}{x-5}-\frac{5}{x-1}\right)$

(2) $\frac{5}{x-5}-\frac{13}{4(x-1)}$

(3) $\frac{13}{5(x-1)}-\frac{4}{5(x-5)}$

(4) $\frac{5}{(x-5)}-\frac{4}{(x-1)}$

3. Resolve $\frac{x^2+x+1}{(x-1)^3}$ into partial fractions.

(1) $\frac{1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{(x-1)^3}$

(2) $\frac{1}{(x-1)}+\frac{2}{(x-1)^2}+\frac{1}{(x-1)^3}$

(3) $\frac{1}{x-1}+\frac{3}{(x-1)^2}+\frac{3}{(x-1)^3}$

(4) $\frac{2}{x-1}+\frac{3}{(x-1)^2}+\frac{3}{(x-1)^3}$

4. Resolve $\frac{3x+2}{x^2+5x+6}$ into partial fractions.

(1) $\frac{7}{(x+2)}+\frac{4}{(x+3)}$

(2) $\frac{4}{(x+3)}+\frac{8}{(x+2)}$

(3) $\frac{4}{(x+3)}-\frac{7}{(x+2)}$

(4) $\frac{7}{(x+3)}-\frac{4}{(x+2)}$

5. Resolve $\frac{1}{x^2+x+42}$ into partial fractions.

(1) $\frac{1}{13(x-6)}+\frac{1}{13(x+7)}$

(2) $\frac{1}{(x-7)}+\frac{1}{(x-6)}$

(3) $\frac{1}{(x-7)}-\frac{1}{(x-6)}$

(4) $\frac{1}{13(x-6)}-\frac{1}{13(x+7)}$

6. Resolve $\frac{x^2+4x+6}{(x^2-1)\,(x+3)}$ into partial fractions

(1) $\frac{11}{8(x-1)}-\frac{3}{21(x+1)}+\frac{3}{8(x+3)}$

(2) $\frac{11}{8(x-1)}-\frac{3}{4(x+1)}+\frac{3}{8(x+3)}$

(3) $\frac{11}{4(x+1)}-\frac{3}{8(x+1)}+\frac{3}{8(x+3)}$

(4) $\frac{11}{8(x-1)}-\frac{3}{4(x+1)}+\frac{3}{8(x+3)}$

7. Resolve $\frac{x+1}{x(x-1)(x+3)}$ into partial fractions.

(1) $\frac{-1}{3x}+\frac{1}{2(x-1)}-\frac{1}{6(x+3)}$

(2) $\frac{1}{3x}+\frac{1}{2(x-1)}-\frac{1}{6(x+3)}$

(3) $\frac{1}{3x}-\frac{1}{2(x-1)}+\frac{1}{6(x+3)}$

(4) $\frac{1}{2x}-\frac{1}{3(x-1)}+\frac{1}{6(x+3)}$

8. Resolve $\frac{1}{x^2-(a+b)x+ab}$ into partial fractions.

(1) $\frac{1}{a+b}\left[\frac{1}{x-a}+\frac{1}{x-b}\right]$

(2) $\frac{1}{a+b}\left[\frac{1}{x-a}-\frac{1}{x-b}\right]$

(3) $\frac{1}{a-b}\left[\frac{1}{x-a}+\frac{1}{x-b}\right]$

(4) $\frac{1}{a-b}\left[\frac{1}{x-a}-\frac{1}{x-b}\right]$

9. Resolve $\frac{1}{x^2-5x+6}$ into partial fractions.

(1) $\frac{1}{x-3}-\frac{1}{x-2}$

(2) $\frac{1}{x-2}+\frac{1}{x-3}$

(3) $\frac{1}{x-3}+\frac{1}{x-2}$

(4) $\frac{2}{x-3}+\frac{2}{x-2}$

10. Resolve $\frac{x+2}{x^2-2x-15}$ into partial fractions.

(1) $\frac{7}{8(x-5)}+\frac{1}{8(x+3)}$

(2) $\frac{8}{7(x-5)}-+\frac{1}{7(x+3)}$

(3) $\frac{5}{8(x-5)}-+\frac{3}{8(x+3)}$

(4) None of these

11. Resolve $\frac{3x^2+2x+4}{(x-1)(x^2-4)}$ into partial fractions.

(1) $\frac{5}{x-2}+\frac{1}{x-1}+\frac{1}{x+2}$

(2) $\frac{5}{x-2}+\frac{2}{x-1}+\frac{1}{x+2}$

(3) $\frac{5}{x-2}+\frac{2}{x-1}-\frac{1}{x+2}$

(4) $\frac{5}{x-2}-\frac{3}{(x-1)}+\frac{1}{(x+2)}$

12. Resolve $\frac{2x+1}{x^2-2x-8}$ into partial fractions.

(1) $\frac{3}{2(x-4)}-\frac{1}{2(x+2)}$

(2) $\frac{3}{2(x-4)}+\frac{1}{2(x+2)}$

(3) $\frac{1}{2(x-4)}-\frac{3}{2(x+2)}$

(4) None of these

13. Resolve $\frac{2x^2+3x+18}{(x-2)(x+2)^2}$ into partial fractions.

(1) $\frac{2}{(x+2)}-\frac{5}{(x-1)^2}$

(2) $\frac{2}{x-2}+\frac{1}{(x+2)}+\frac{3}{(x+2)^2}$

(3) $\frac{2}{x-2}-\frac{5}{(x+2)^2}$

(4) $\frac{2}{x-2}-\frac{1}{(x+2)}+\frac{3}{(x+2)^2}$

14. Resolve $\frac{x^2-x+1}{x^3-1}$ into partial fractions.

(1) $\frac{1}{3(x-1)}+\frac{2x-2}{3(x^2+x+1)}$

(2) $\frac{1}{3(x-1)}-\frac{2x+2}{3(x^2+x+1)}$

(3) $\frac{1}{2(x-1)}+\frac{x+2}{2(x^2+x+1)}$

(4) $\frac{1}{3(x-1)}-\frac{x-2}{3(x^2+x+1)}$

15. Resolve $\frac{x-1}{x^3-3x^2+2x}$ into partial fractions.

(1) $\frac{1}{(x-2)}-\frac{1}{2x}$

(2) $\frac{1}{2(x-2)}-\frac{1}{2x}$

(3) $\frac{1}{(x-2)}+\frac{1}{x-1}+\frac{1}{2x}$

(4) $\frac{1}{x-2}-\frac{1}{x-1}+\frac{1}{2x}$

Concept Application Level—2

16. Resolve $\frac{x-b}{x^2+(a+b)x+ab}$ into partial fractions.

(1) $\frac{2a}{(a+b)(x+a)}-\frac{2b}{(a+b)(x+b)}$

(2) $\frac{a+b}{(a-b)(x+a)}+\frac{2b}{(a-b)(x+b)}$

(3) $\frac{a+b}{(a-b)(x+a)}-\frac{2b}{(a-b)(x+b)}$

(4) None of these

17. Resolve $\frac{x+1}{x^2-4}$ into partial fractions.

(1) $\frac{3}{2(x-2)}+\frac{1}{2(x+2)}$

(2) $\frac{3}{2(x-2)}-\frac{1}{2(x+2)}$

(3) $\frac{3}{4(x-2)}+\frac{1}{4(x+2)}$

(4) $\frac{3}{4(x-2)}-\frac{1}{4(x+2)}$

18. Resolve $\dfrac{x}{(x+1)(x-1)^2}$ into partial fractions.

(1) $\dfrac{1}{2(x-1)^2}+\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}$ (2) $\dfrac{-1}{4(x+1)}+\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}$

(3) $\dfrac{-1}{4(x-1)}+\dfrac{1}{4(x+1)}+\dfrac{1}{2(x-1)^2}$ (4) None of these

19. Resolve $\dfrac{3x-5}{x^2+3x+2}$ into partial fractions.

(1) $\dfrac{7}{x+2}-\dfrac{5}{x+1}$ (2) $\dfrac{-8}{x+2}-\dfrac{11}{(x+1)}$ (3) $\dfrac{11}{x+2}-\dfrac{8}{x+1}$ (4) $\dfrac{7}{x+2}+\dfrac{5}{x+1}$

20. Resolve $\dfrac{3x+5}{x^2+8x-20}$ into partial fractions.

(1) $\dfrac{11}{24(x-2)}+\dfrac{25}{24(x+10)}$ (2) $\dfrac{11}{6(x-2)}+\dfrac{25}{6(x+10)}$

(3) $\dfrac{11}{12(x+2)}+\dfrac{25}{12(x-10)}$ (4) $\dfrac{11}{12(x-2)}+\dfrac{25}{12(x+10)}$

21. Resolve $\dfrac{ax+b^2}{x^2-(a+b)^2}$ into partial fractions.

(1) $\dfrac{a^2+ab+b^2}{2(a+b)(x-(a+b))}+\dfrac{a^2+ab-b^2}{2(a+b)(x+(a+b))}$ (2) $\dfrac{a^2+b^2}{2(a+b)(x-(a+b))}+\dfrac{a^2-b^2}{2(a+b)(x+(a+b))}$

(3) $\dfrac{a^2-b^2}{2(a+b)(x-(a-b))}+\dfrac{a^2+b^2}{2(a+b)(x+(a+b))}$ (4) None of these

22. Find the constants a, b, c and d respective1y, if $\dfrac{1}{x^4-x}=\dfrac{a}{x}+\dfrac{b}{x-1}+\dfrac{cx+d}{x^2+x+1}$

(1) $-1, \frac{1}{3}, \frac{-1}{2}, \frac{-5}{6}$ (2) $1, \frac{1}{3}, \frac{-1}{2}, \frac{5}{6}$ (3) $1, \frac{-1}{3}, \frac{1}{2}, \frac{5}{6}$ (4) None of these

23. If $\dfrac{x^3}{(x-1)(x-2)}=Ax+B+\dfrac{C}{x-1}+\dfrac{D}{x-2}$, then A, B, C and D respectively are

(1) 1, 3, 1, 8 (2) 1, −1, 3, 8 (3) 1, 3, −1, 8 (4) −1, −3, 1, 8

24. Resolve $\dfrac{2x^2+3}{x^4+8x^2+15}$ into partial fractions.

(1) $\dfrac{3}{2(x^2+3)}+\dfrac{5}{2(x^2+5)}$ (2) $\dfrac{3}{2(x^2-3)}+\dfrac{5}{2(x^2+5)}$

(3) $\dfrac{-3}{2(x^2+3)}+\dfrac{7}{2(x^2+5)}$ (4) $\dfrac{7}{2(x^2+3)}-\dfrac{3}{2(x^2+5)}$

25. Resolve $\frac{3x+1}{(x-1)^2}$ into partial fractions.

(1) $\frac{4}{(x-1)} - \frac{3}{(x-1)^2}$ (2) $\frac{3}{(x-1)} - \frac{4}{(x-1)^2}$

(3) $\frac{3}{(x-1)^2} - \frac{3}{(x-1)^2}$ (4) $\frac{3}{(x-1)} + \frac{4}{(x-1)^2}$

26. If $\frac{3x^2+14x+10}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$, then A, B and C are respectively are

(1) $9/2, -5, 3/2$ (2) $9/2, 5, -3/2$ (3) $9/2, 3/2, -5$ (4) $-9/2, -3/2, 5$

27. If $\frac{2x-1}{(x+1)(x^2-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, then A, B, C respectively are

(1) $1/4, -1/4, 3/2$ (2) $1/4, 1/4, 3/2$ (3) $1/4, -5/2, 5/4$ (4) $1/4, 5/2, 5/4$

28. Resolve $\frac{ax-b}{(x+1)^2}$ into partial fractions.

(1) $\frac{a}{x+1} + \frac{a+b}{(x+1)^2}$ (2) $\frac{a}{x+1} + \frac{a-b}{(x+1)^2}$

(3) $\frac{a}{x+1} - \frac{a-b}{(x+1)^2}$ (4) $\frac{a}{x+1} - \frac{a+b}{(x+1)^2}$

29. Resolve $\frac{4x^2+3x+2}{(x-4)^3}$ into partial fractions.

(1) $\frac{25}{4(x-4)} + \frac{59}{2(x-4)} + \frac{78}{(x-4)^3}$ (2) $\frac{25}{4(x-4)} - \frac{69}{2(x-4)^2} + \frac{78}{(x-4)^3}$

(3) $\frac{35}{4(x-4)} + \frac{69}{2(x-4)^2} + \frac{78}{(x-4)^3}$ (4) $\frac{4}{x-4} + \frac{35}{(x-4)^2} + \frac{78}{(x-4)^3}$

30. Resolve $\frac{1}{x^2-7x-18}$ into partial fractions.

(1) $\frac{1}{11}\left(\frac{1}{x-9} - \frac{1}{x+2}\right)$ (2) $\frac{1}{11}\left(\frac{1}{x-9} + \frac{1}{x+2}\right)$

(3) $\frac{1}{11}\left(\frac{1}{x+9} - \frac{1}{x-2}\right)$ (4) $\frac{1}{11}\left(\frac{1}{x+9} + \frac{1}{x+2}\right)$

Concept Application Level—3

31. Resolve $\dfrac{1}{x^4 - 3x^2 - 4}$ into partial fractions.

(1) $\dfrac{1}{5(x^2-4)} + \dfrac{1}{5(x^2+1)}$

(2) $\dfrac{1}{5(x^2-4)} - \dfrac{1}{5(x^2+1)}$

(3) $\dfrac{1}{4(x^2+4)} - \dfrac{1}{4(x^2+1)}$

(4) $\dfrac{1}{4(x^2+4)} + \dfrac{1}{4(x^2+1)}$

32. Resolve $\dfrac{4x+3}{x^3 - 7x - 6}$ into partial fractions.

(1) $\dfrac{1}{4(x+1)} + \dfrac{3}{4(x-3)} - \dfrac{1}{(x+2)}$

(2) $\dfrac{-1}{4(x+1)} + \dfrac{3}{4(x-3)} + \dfrac{1}{(x+2)}$

(3) $\dfrac{1}{4(x+1)} - \dfrac{3}{4(x-3)} + \dfrac{1}{(x+2)}$

(4) $\dfrac{1}{4(x+1)} - \dfrac{3}{4(x-3)} - \dfrac{1}{x+2}$

33. If $\dfrac{1}{(a^2-bx)(b^2-ax)} = \dfrac{A}{a^2-bx} + \dfrac{B}{b^2-ax}$, then the value of A and B respectively would be

(1) $\dfrac{b}{b^3-a^3}, \dfrac{a}{b^3-a^3}$

(2) $\dfrac{b}{b^3-a^3}, \dfrac{a}{a^3-b^3}$

(3) $\dfrac{b}{a^3-b^3}, \dfrac{a}{a^3-b^3}$

(4) $\dfrac{-b}{b^3-a^3}, \dfrac{-a}{a^3-b^3}$

34. Resolve $\dfrac{3x-5}{(x-1)^4}$ into partial fractions.

(1) $\dfrac{1}{(x-1)} + \dfrac{2}{(x-1)^2} - \dfrac{3}{(x-1)^3} + \dfrac{4}{(x-1)^4}$

(2) $\dfrac{3}{(x-1)^2} + \dfrac{2}{(x-1)^3} - \dfrac{1}{(x-1)^4}$

(3) $\dfrac{3}{(x-1)^3} + \dfrac{2}{(x-1)^4}$

(4) $\dfrac{3}{(x-1)^3} - \dfrac{2}{(x-1)^4}$

35. Resolve $\dfrac{x^2+x+1}{x^3+1}$ into partial fractions.

(1) $\dfrac{1}{3(x+1)} + \dfrac{x+1}{3(x^2-x+1)}$

(2) $\dfrac{-1}{3(x+1)} + \dfrac{2x+2}{3(x^2-x+1)}$

(3) $\dfrac{1}{3(x+1)} + \dfrac{2x+2}{3(x^2-x+1)}$

(4) None of these

KEY

Very short answer type questions

1. greater
2. sum
3. two terms
4. smaller
5. partial fractions
6. four
7. zero or 1.
8. 2
9. -1
10. 6
11. x+1
12. zero
13. zero
14. $\dfrac{b^2}{b-a}$
15. two

Short answer type questions

16. $\dfrac{1}{2a\,(x-a)} - \dfrac{1}{2a(x+a)}$

17. $\dfrac{3}{x+1} - \dfrac{1}{x+2}$

18. $\dfrac{8}{7(x+4)} + \dfrac{13}{7(x-3)}$

19. $\dfrac{2}{x-2} + \dfrac{13}{(x-2)^2} + \dfrac{26}{(x-2)^3}$

20. $\dfrac{-1}{8(x+2)} + \dfrac{\frac{1}{8}x + \frac{3}{4}}{x^2+4}$

21. A = 2, B = 7 and C = -4

22. A =1 and B = 6

23. $\dfrac{a}{(x-1)} + \dfrac{a+b}{(x-1)^2}$

24. $\dfrac{3}{x} - \dfrac{1}{(x-1)} + \dfrac{2}{(x-1)^2}$

25. $\dfrac{3}{2(x-3)} - \dfrac{1}{2\,(x+1)}$

26. $\dfrac{9}{2\,(x+2)} - \dfrac{20}{x+3} + \dfrac{37}{2\,(x+4)}$

27. $\dfrac{2}{3(x-1)} - \dfrac{(2x+1)}{3\,(x^2+x+1)}$

28. $\dfrac{2}{3\,(x^2+1)} + \dfrac{4}{3\,(x^2+4)}$

29. $\dfrac{\frac{-3x}{2} + 5/2}{x^2-x+1} + \dfrac{\frac{3}{2}x + 1/2}{x^2+x+1}$

30. $\dfrac{1}{x} - \dfrac{1}{3\,(x+1)} + \dfrac{\frac{-2}{3}x + \frac{1}{3}}{x^2-x+1}$

Essay type questions

31. $x-3-\dfrac{1}{x+1}+\dfrac{8}{x+2}.$

32. $\dfrac{x-1}{(x^2-x+1)} - \dfrac{x}{(x^2+x+1)}$

33. $\dfrac{1}{(x-2)} - \dfrac{x+1}{(x^2+2x+2)}$

34. $\dfrac{1}{3\,(x-2)} + \dfrac{1}{(x-1)} - \dfrac{1}{3\,(x+1)}$

35. $x^2+4+\dfrac{17}{4\,(x-2)} - \dfrac{17}{4\,(x+2)}$

key points for selected questions

Short answer type questions

16. $\dfrac{1}{x^2 - a^2} = \dfrac{A}{x + a} + \dfrac{B}{x - a}$

17. (i) Take, $\dfrac{2x+5}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+3}$ and eliminate denominators by taking LCM.
 (ii) Put $x = -1, x = -3$ and evaluate A and B.

18. (i) Take $\dfrac{3x + 4}{x^2 + x - 12} = \dfrac{A}{x+ 4} + \dfrac{B}{x - 3}$ and proceed as the previous problem.

19. (i) Take $\dfrac{2x^2 + 5x + 8}{(x- 2)^3}$
$= \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3}$
 (ii) Proceed as the previous problems.

20. $\dfrac{x+1}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^3 + 4}$

21. Substitute relevant values of x to find A and B.

22. Substitute relevant values of x to find A and B.

23. $\dfrac{ax+b}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

24. $\dfrac{2x^2-3x+3}{x^3-2x^2+x} = \dfrac{2x^2-3x+3}{x(x-1)^2}$

25. Split the denominator into factors and proceed in the same way as in the above problems.

26. Split the denominator into factors and proceed in the same way as in the above problems.

27. $\dfrac{x+1}{x^3-1} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

28. $\dfrac{2x^2 + 4}{x^4+5x^2+4} = \dfrac{Ax+B}{x^2+1} + \dfrac{(x+1)}{x^2+4}$

29. (i) Take, $\dfrac{2x+3}{x^4+x^2+1} = \dfrac{2x+3}{(x^2+x+1)(x^2-x+1)}$.
 (ii) Now split them into partial fractions.

30. $\dfrac{1}{x^4+x} = \dfrac{1}{x(x+1)(x^2+x+1)}$
$= \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+x+1}$

Essay type questions

31. $\dfrac{x^3}{(x+1)(x+2)} = Ax+B+\dfrac{C}{x+1} + \dfrac{D}{x+2}$

32. (i) $\dfrac{x^2-x-1}{x^4+x^2+1} = \dfrac{x^2-x-1}{(x^2+x+1)\,(x^2-x+1)}$
 (ii) Now simplify and split into partial fractions.

33. Split the denominator into factors and use the relevant values to find its partial fractions.

34. Split the denominator into factors and use the relevant values to find its partial fractions.

35. $\dfrac{x^4+1}{(x-2)(x+2)} = Ax^2 + B + \dfrac{C}{x-2} + \dfrac{D}{x+2}$

KEY

Concept Application Level—1,2,3

1. 3	**2.** 1
3. 3	**4.** 4
5. 4	**6.** 4
7. 1	**8.** 4
9. 1	**10.** 1
11. 4	**12.** 2
13. 3	**14.** 1
15. 2	**16.** 3
17. 3	**18.** 2
19. 3	**20.** 4
21. 1	**22.** 4
23. 3	**24.** 3
25. 4	**26.** 2
27. 1	**28.** 4
29. 4	**30.** 1
31. 2	**32.** 1
33. 2	**34.** 4
35. 3	

Concept Application Level—1,2,3

Key points for select questions

1. $\dfrac{1}{x^2-9}=\dfrac{A}{x-3}+\dfrac{B}{x+3}$

2. $\dfrac{2x+3}{x^2-6x+5}=\dfrac{A}{(x-5)}+\dfrac{B}{x-1}$

3. $\dfrac{x^2+x+1}{(x-1)^3}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$

4. $\dfrac{3x+2}{x^2+5x+6}=\dfrac{A}{x+2}+\dfrac{B}{x+3}$

5. $\dfrac{1}{x^2+x-42}=\dfrac{A}{x+7}+\dfrac{B}{x-6}$

6. $\dfrac{x^2+4x+6}{(x^2-1)(x+3)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{x+3}$

7. $\dfrac{x+1}{x(x-1)(x+3)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x+3}$

8. $\dfrac{1}{x^2-(a+b)x+ab}=\dfrac{A}{x-a}+\dfrac{B}{x-b}$

9. $\dfrac{1}{x^2-5x+6}=\dfrac{A}{x-3}+\dfrac{B}{x-2}$

10. $\dfrac{x+2}{x^2-2x-15}=\dfrac{A}{x-5}+\dfrac{B}{x+3}$

11. $\dfrac{3x^2+2x+4}{(x-1)(x^2-4)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x+2}$

12. $\dfrac{2x+1}{x^2-2x-8}=\dfrac{A}{x-4}+\dfrac{B}{x+2}$

13. $\dfrac{2x^2+3x+18}{(x-2)(x+2)^2}=\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

14. Split the denominator into factors and use the relevant method to find its partial fractions.

15. (i) $x^3-3x^2+2x=x(x-1)(x-2)$

(ii) $\dfrac{x-1}{(x^3-3x^2+2x)}$

$=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x-2}$

16. $\dfrac{x-b}{x^2+(a+b)x+ab}=\dfrac{A}{x+a}+\dfrac{B}{x+b}$

17. $\dfrac{x+1}{x^2-4}=\dfrac{A}{x+2}+\dfrac{B}{x-2}$

18. $\dfrac{x}{(x+1)(x-1)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

19. $\dfrac{3x-5}{x^2+3x+2}=\dfrac{A}{x+1}+\dfrac{B}{x+2}$

20. $\dfrac{3x+5}{x^2+8x-20}=\dfrac{A}{x+10}+\dfrac{B}{x-2}$

21. $\dfrac{ax + b^2}{x^2 - (a+b)^2} = \dfrac{A}{x-(a+b)} + \dfrac{B}{x+(a+b)}$

22. (i) Simplify LHS and RHS.
(ii) Put $x = 0$ and 1 and obtain the values of a and b.
(iii) Compare the like terms and obtain the values of c and d.

23. (i) Simplify LHS and RHS by taking LCM.
(ii) Compare the like terms and obtain the required values.

24. $\dfrac{2x^2 + 3}{x^4 + 8x^2 + 15} = \dfrac{Ax + B}{x^2 + 5} + \dfrac{Cx + D}{x^2 + 3}$

25. $\dfrac{3x+1}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

26. (i) Simplify LHS and RHS by taking LCM.
(ii) Put $x = 0$ and -2 and obtain the values of A and B.
(iii) Compare the like terms and obtain the value of C.

27. (i) Simplify LHS and RHS by taking LCM.
(ii) Compare the like terms and obtain the required values.

28. $\dfrac{ax-b}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$

29. $\dfrac{4x^2+3x+2}{(x-4)^3} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{C}{(x-4)^3}$

30. $\dfrac{1}{x^2 - 7x - 18} = \dfrac{A}{x-9} + \dfrac{B}{x+2}$

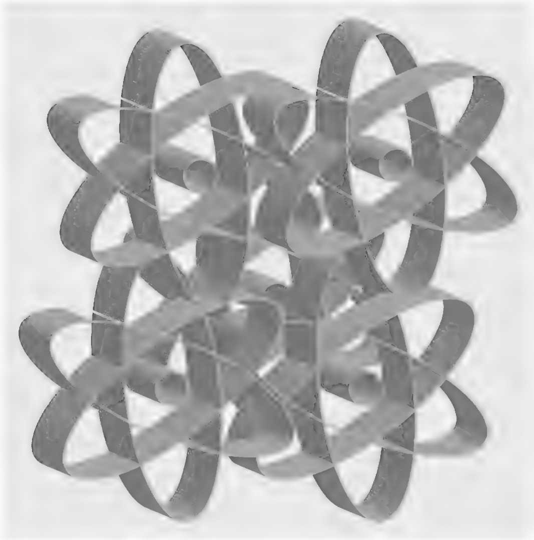
31. $\dfrac{1}{x^4 - 3x^2 - 4} = \dfrac{Ax+B}{x^2-4} + \dfrac{Cx + D}{x^2 + 1}$

32. (i) Use $x^3 - 7x - 6 = (x + 1)(x + 2)(x - 3)$.
(ii) $\dfrac{4x+3}{x^3-7x-6} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$

33. (i) Simplify LHS and RHS by taking LCM.
(ii) Compare the like terms and obtain the required values.

34. $\dfrac{3x-5}{(x-1)^4} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{(x-1)^4}$

35. $\dfrac{x^2+x+1}{x^3+1} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$



CHAPTER 27

Logarithms

INTRODUCTION

In earlier classes, we have learnt about indices. One of the results we have learnt is that, if $2^x = 2^3$, then $x = 3$ and if $4^x = 4^y$, then $x = y$ i.e., if two powers of the same base are equal and the base is not equal to -1, 0 or 1, then the indices are equal. But when $3^x = 5^2$, just by using the knowledge of indices, we cannot find the numerical value of x. The necessity of the concept of logarithms arises here. Logarithms are useful in long calculations involving multiplication and division.

Definition

The logarithm of any positive number to a given base (a positive number not equal to 1) is the index of the power of the base which is equal to that number. If N and a ($\neq 1$) are any two positive real numbers and for some real x, $a^x = N$, then x is said to be the logarithm of N to the base a. It is written as $\log_a N = x$, i.e., if $a^x = N$, then $x = \log_a N$.

Examples

(i) $3^4 = 81 \Rightarrow 4 = \log_3 81$

(ii) $7^3 = 343 \Rightarrow 3 = \log_7 343$

If in a particular relation, all the log expressions are to the same base, we normally do not specify the base.

From the definition of logs, we get the following results

When $a > 0$, $b > 0$ and $b \neq 1$,

1. $\log_a a^n = n$ e.g., $\log_6 6^3$
2. $a^{\log_a b} = b$ e.g., $9^{\log_9 5} = 5$

System of logarithms

Though we can talk of the logarithm of a number to any positive base as not equal to 1, there are two systems of logarithms viz., natural logarithms and common logarithms, which are used most often.

(i) **Natural logarithms:** These were discovered by Napier. They are calculated to the base 'e' which is approximately equal to 2. 7828. These are used in higher mathematics.

(ii) **Common logarithms:** Logarithms to the base 10 are known as common logarithms. This system was introduced by Briggs, a contemporary of Napier. In the rest of this chapter, we shall use the short form 'log' instead of 'logarithm'.

Properties

(i) Logs are defined only for positive real numbers.

(ii) Logs are defined only for positive bases (other then 1).

(iii) In $\mathbf{log_b}$ a neither a is negative nor b is negative but the value of $\mathbf{log_b}$ a can be negative.

☛ *Example* As $10^{-2} = 0.01, \log_{10}0.01 = -2$

(iv) Logs of different numbers to the same base are different i.e., if $a \neq b$, then $\log_m a \neq \log_m b$. In other words, if $\log_m a = \log_m b$, then $a = b$.

☛ *Examples* $\log_{10}2 \neq \log_{10}3$

$\log_{10}2 = \log_{10}y \Rightarrow y = 2$

(v) Logs of the same number to different bases have different values, i.e., if $m \neq n$, then $\log_m a \neq \log_n a$. In other words, if $\log_m a = \log_n a$, then $m = n$.

☛ *Examples* $\log_2 16 \neq \log_4 16$

$= \log_n 16 \Rightarrow n = 2$

(vi) Log of 1 to any base is 0.

☛ *Example* $\log_2 1 = 0$ ($\because 2^0 = 1$)

(vii) Log of a number to the same base is 1.

☛ *Example* $\log_4 4 = 1$.

(viii) Log of 0 is not defined.

Laws

1. $\log_m (ab) = \log_m a + \log_m b$

☛ *Example* $\log 56 = \log(7 \times 8) = \log 7 + \log 8$

2. $\log_m\left(\frac{a}{b}\right) = \log_m a - \log_m b$

☛ *Example* $\log\left(\frac{81}{23}\right) = \log 81 - \log 23$

3. $\log a^m = m \log a$

☛ *Example* $\log 216 = \log 6^3 = 3\log 6$

4. $\log_b a \quad \log_c b = \log_c a$; (Chain Rule)

☛ *Example* $\log_2 3 \times \log_8 2 \times \log_5 8 = \log_8 3 \times \log_5 8 = \log_5 3$

5. $\log_b a = \frac{\log_c a}{\log_c b}$ (Change of base Rule);

☛ *Example* $\log_9 25 = \frac{\log_4 25}{\log_4 9}$

In this relation, if we replace c by a, then we get the following result:

$\log_b a = \frac{1}{\log_a b}$

Variation of $\log_a x$ with x

For $1 < a$, and $0 < p < q$, $\log_a p < \log_a q$.

For $0 < a < 1$ and $0 < p < q$, $\log_a p > \log_a q$.

☛ *Example* $\log_{10} 2 < \log_{10} 3$ and $\log_{0.1} 2 > \log_{0.1} 3$

Bases which are greater than 1 are called **strong bases** and bases which are less than 1 are called **weak bases**. Therefore, for strong bases, the log increases with the number and for weak bases, the log decreases with the number.

Sign of $\log_a x$ for different values of x and a

Strong bases (a > 1)

1. If $x > 1$, $\log_a x$ is positive.
   For example, $\log_2 8$, $\log_4 81$ are positive.
2. If $0 < x < 1$, then $\log_a x$ is negative.

   For example, $\log_4 0.02 = \frac{\log 0.02}{\log 4} = \frac{\log 2 - \log 100}{\log 4}$

   $\log 2 < \log 100$ and $0 < \log 4$ for strong bases

   $\therefore \frac{\log 2 - \log 100}{\log 4} < 0$

   $\Rightarrow \log_4 0.02$ is negative

Weak bases (0 < a < 1)

1. If $x > 1$, then $\log_a x$ is negative.

   For example $\log_{0.3}15$ and $\log_{0.4}16$ are negative.

   Consider $\log_{0.3}15 = \dfrac{\log 15}{\log 0.3} = \dfrac{\log 15}{\log 3 - \log 10}$

   $\log 3 < \log 10$ (for any strong base)

   $\Rightarrow \dfrac{\log 15}{\log 3 - \log 10} < 0$

2. If $0 < x < 1$, then $\log_a x$ is positive.

   For example, $\log_{0.1}0.2$, $\log_{0.4}0.3$ are positive.

Note: Logs of numbers (> 1) to strong bases and numbers (< 1) to weak bases are positive.

To find the log of a number to the base 10

Consider the following numbers:

2, 20, 200, 0.2 and 0.02.

We see that $20 = 10(2)$ and $200 = 100(2)$

$\therefore$ $\log 20 = 1 + \log 2$ and $\log 200 = 2 + \log 2$

Similarly, $\log 0.2 = -1 + \log 2$ and $\log 0.02 = -2 + \log 2$

From the tables, we see that $\log 2 = 0.3010$. (Using the tables is explained in greater detail in later examples).

$\therefore$ $\log 20 = 1.3010$, $\log 200 = 2.3010$, $\log 0.2 = -1 + 0.3010$ and $\log 0.02 = -2 + 0.3010$.

Note:

1. Multiplying or dividing by a power of 10 changes only the integral part of the log, not the fractional part.
2. For numbers less than 1, (for example log 0.2) it is more convenient to leave the log value as $-1 + 0.3010$ instead of changing it to -0.6090. We refer to the first form (in which the fraction is positive) as the standard form and the second form as the normal form. Both the forms represent the same number.

For numbers less than 1, it is convenient to express the log in the standard form. As the negative sign refers only to the integral part, it is written above the integral part, rather than in front. i.e., $\log 0.2 = \bar{1}.3010$ and not -1.3010.

The convenience of the standard form will be clear when we learn how to take the anti-log, which will later be explained in detail.

Anti-log (-0.6090) = Anti-log $(-1 + 0.3010)$ = Anti-log $\bar{1}.3010 = 0.2$.

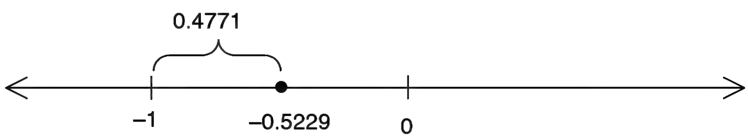
When the logs of numbers are expressed in the standard form (for numbers greater than 1, the standard form of the log is the same as the normal form), the integral part is called the characteristic and the fractional part (which is always positive) is called the mantissa.

Example

Express -0.5229 in the standard form and locate it on the number line.

Solution

$-0.5229 = -1 + 1 - 0.5229 = \bar{1}.4771$



The rule to obtain the characteristic of log x

(A) If $x > 1$ and there are n digits in x, then the characteristic is $n - 1$.

(B) f $x < 1$ and there are m zeroes between the decimal point and the first non-zero digit of x, then the characteristic is $(-m + 1)$ more commonly written as $(\overline{m+1})$.

NOTE: $-4 = \bar{4}$ but $-4.01 \neq \bar{4}.01$

To find the log of a number from the log tables

Example

Find the values of log36, log3600 and log0.0036.

Solution

In log tables we find the number 36 in the first column. In this row in the next column (under zero), we find .5563 (the decimal point is dropped in other columns). This gives 5563 as the mantissa for the log of all numbers whose significant digits are 3 and 6.

$\therefore$ Prefixing characteristic, we have $\log 36 = 1.5563$

Similarly $\log 3600 = 3.5563$ and $\log 0.0036 = \bar{3}.5563$

Example

Find the values of log3.74, log374000 and log0.3740.

Solution

In the log table we locate 37 in the first column. In this row, in the column under 4, we find 5729. As in the earlier example, the same line as before gives the mantissa of logarithms of all numbers which begin with 37. From this line, we select the mantissa which is located in the column number 4. This gives 5729 as the mantissa for all numbers whose significant digits are 3,7 and 4.

$\therefore \log 3.74 = 0.5729$

$\log 374000 = 5.5729$ and

$\log 0.3740 = \bar{1}.5729$

Example

Find the values of log5.342 and log 0.05342.

Solution

As found in the above example, we can find the mantissa for the sequence of digits 534 as 7275. Since there are four significant digits in 5342, in the same row where we found 7275 under the column 2 in the mean difference column, we can find the number 2.

$\therefore$ The mantissa of the logarithm of 5342 is $7275 + 2 = 7277$

Thus $\log 5.342 = 0.7277$

Similarly, $\log 0.05342 = \bar{2}.7277$

Anti-log

As $\log_2 8 = 3$, 8 is the **anti-logarithm** of 3 to the base 2, i.e., **Anti-log** of b to base m is $\mathbf{m^b}$.

In the above example we have seen that log5.342 = 0.7277

∴ Anti-log0.7277 = 5.342

To find the anti-log

☛ *Example*

Find the anti-log of 2.421.

Solution

Step 1: In the anti-log table we find the number.42 in the first column. In that row in the column under 1, we find 2636.

Step 2: As the characteristic is 2, we place the decimal after three digits from the left i.e., anti-log2.421 = 263.6

NOTE: If the characteristic is n(a non-negative integer), then we would place the decimal after (n + 1) digits from the left.

☛ *Example*

Find the anti-log of 1.4215.

Solution

We have to locate.42 in the first column and scan along the horizontal line and pick out the number in the column headed by 1. We see that the number is 2636. The mean difference for 5 in the same line is 3.

∴ The sum of these numbers is 2636 + 3 = 2639.

As the characteristic is 1, the required anti-log is 26.39.

☛ *Example*

Find the value of $\dfrac{7.211 \times 0.084}{16.52 \times 0.016}$.

Solution

log of numerator = (log of numerator) – (log of denominator)

log of numerator = log7.211 + log0.084 = 0.8580 + $\bar{2}.9243 = \bar{1}.7823$

log of denominator = log16.52 + log0.016 = 1.2180 + $\bar{2}.2041 = \bar{1}.4221$

log of the given fraction = $\bar{1}.7823 - \bar{1}.4221 = 0.3602$

Value of the fraction = Anti-log (0.3602) = 2.292 (As the characteristic is 0, the decimal is kept after one digit from the left)

Example

If $\log_{10}4 = 0.6021$ and $\log_{10}5 = 0.6990$, then find the value of $\log_{10}1600$.

Solution

$$\begin{aligned}
\log_{10}1600 &= \log_{10}(64 \times 25) = \log_{10}(4^3 \times 5^2) \\
&= \log_{10} 4^3 + \log_{10} 5^2 \rightarrow \\
&= 3\log_{10} 4 + 2\log_{10} 5 \\
&= 3(0.6021) + 2(0.6990) \\
&= 1.8063 + 1.3980 \\
\log_{10}1600 &= 3.2043
\end{aligned}$$

Example

Find the value of $\sqrt[3]{16.51}$ approximately.

Solution

$$\begin{aligned}
\text{Let } P &= \sqrt[3]{16.51} \\
\log P &= \log(16.51)^{1/3} \\
&= \frac{1}{3}\log 16.51 \\
&= \frac{1}{3}(1.2178) = 0.4059 \\
\log P &= 0.4059 \\
P &= \text{anti-log } (0.4059) \\
\therefore\ P &= 2.546
\end{aligned}$$

test your concepts

Very short answer type questions

1. $\frac{1}{5}\log_2 32 + 3\log_{64}4 =$ __________.
2. The characteristic of the logarithm of 3.6275 is __________.
3. If $4\log_x 8 = 3$, then $x =$ __________.
4. If $\log x - \frac{2}{3}\log x = 1$, then $x =$ _________.

5. If $a = \log\frac{3}{2}$, $b = \log\frac{4}{25}$ and $c = \log\frac{5}{9}$, then $a + b + c =$ __________.

6. The number of digits in the integral part of the number whose logarithm is 4.8345 is __________.

7. If $\log x = 32.756$, then $\log 10x =$ ________.

8. The characteristic of the logarithm of 0.0062 is _________.

9. If $\log_a x$ (where $a > 1$) is positive, then the range of x is _________.

10. If $\log 27.91 = 1.4458$, then $\log 2.791 =$ _________.

11. $\dfrac{\log 15 - \log 6}{\log 20 - \log 8} =$ _________.

12. If $\log 2 = 0.3010$, then $\log 5 =$ ________.

13. The value of $\log_{16} \sqrt[5]{64} =$ _________.

14. $\dfrac{\log 216}{\log 6} =$ __________.

15. If $\log_4 3 = x$, then $\log_{\sqrt[4]{3}} \sqrt[4]{64} =$ _________.

16. If $\log_x\left(\dfrac{1}{243}\right) = -5$, then find the value of x.

17. $7^{\log_{343} 27} =$ _________.

18. If $3^{\log_9 x} = 2$, then $x =$ ________.

19. If $\log_{xyz}x + \log_{xyz}y + \log_{xyz}z = \log_{10}p$, then $p =$ _________.

20. If $\log_{10}4 + \log_{10}m = 2$, then $m =$ _________.

21. Simplify: $3\log_3 5 + \log_3 10 - \log_3 625$.

22. If $\log(a + 1) + \log(a - 1) = \log 15$, then $a =$ _________.

23. The value of $\log 10 + \log 100 + \log 1000 + \ldots\ldots\ldots\ldots + \log 10000000000 =$ ___________.

24. If the number of zeroes between the decimal point and the first non-zero digit of a number is 2, then the characteristic of logarithm of that number is _________.

25. The value of $\log(\tan 10°) + \log(\tan 20°) + \log(\tan 45°) + \log(\tan 70°) + \log(\tan 80°) =$ _________.

Short answer type questions

26. Simplify: $\log\left(\dfrac{3}{18}\right) + \log\left(\dfrac{45}{8}\right) - \log\left(\dfrac{15}{16}\right)$.

27. Show that $\dfrac{1}{\log_a abc} + \dfrac{1}{\log_b abc} + \dfrac{1}{\log_c abc} = 1$.

28. Solve for real value of x: $\log(x-1)+\log(x^2+x+1)=\log 999$.

29. If $\dfrac{1}{1+\log_a 10}=\dfrac{3}{2}$, then find the value of a.

30. If $x^2+y^2=23xy$, then show that $2\log(x+y)=2\log 5+\log x+\log y$.

31. If $\log_{10}2=0.3010$ and $\log_{10}3=0.4771$, then find the value $\log_{10}135$.

32. If $\log_{10}2=x$ and $\log_{10}3=y$, then find $\log_{10}21.6$.

33. If $\log_{10}2=0.3010$, then find the number of digits in $(64)^{10}$.

34. Simplify $\dfrac{1}{\log_2\log_2\log_2 256}$.

35. Prove that $\log_3 810=4+\log_3 10$.

Essay type questions

36. Solve: $x^{\log_4 3}+3^{\log_4 x}=18$.

37. If $p^2+q^2=14pq$, then prove that $\log\left(\dfrac{p+q}{4}\right)=\dfrac{1}{2}[\log p+\log q]$.

38. Without using tables, find the value of $4\log_{10}5+5\log_{10}2-\dfrac{1}{2}\log_{10}4$.

39. If $\dfrac{\log a}{b-c}=\dfrac{\log b}{c-a}=\dfrac{\log c}{a-b}$, then prove that $a^a\,b^b\,c^c=1$.

40. Arrange the following numbers in the increasing order of their magnitude.

$\log_7 9, \log_{18}16, \log_6 41, \log_2 10$.

CONCEPT APPLICATION

Concept Application Level—1

1. If $\log_{16}x=2.5$, then x =

(1) 40 (2) 256 (3) 1024 (4) None of these

2. If $\log 5=0.699$ and $(1000)^x=5$, then find the value of x.

(1) 0.0699 (2) 0.0233 (3) 0.233 (4) 10

3. The value of $\log\left(\frac{18}{14}\right)+\log\left(\frac{35}{48}\right)-\log\left(\frac{15}{16}\right)=$

(1) 0 (2) 1 (3) 2 (4) $\log_{16}15$

4. If $\log_3 a+\log_9 a+\log_{81} a=\frac{35}{4}$, then a =

(1) 27 (2) 243 (3) 81 (4) None of these

5. If $\log_9(\log_8 x)<0$, then x belongs to

(1) (1, 8) (2) $(-\infty, 8)$ (3) $(8, \infty)$ (4) None of these

6. If $\log_3\frac{x^3}{3}-2\log_3 3x^3=a-b\log_3 x$, then find the value of a + b.

(1) 6 (2) –6 (3) 0 (4) –3

7. The value of $\log_{40}5$ lies between

(1) $\frac{1}{3}$ and $\frac{1}{2}$ (2) $\frac{1}{4}$ and $\frac{1}{3}$ (3) $\frac{1}{2}$ and 1 (4) None of these

8. If $x=\log_{\frac{1}{2}}\frac{4}{3}\cdot\log_2\frac{1}{3}\cdot\log_{\frac{2}{3}}0.8$, then

(1) $x>0$ (2) $x<0$ (3) $x=0$ (4) $x\geq 0$

9. If $\log_{144}729=x$, then the value of $\log_{36}256$ is

(1) $\frac{4(3-x)}{(3+x)}$ (2) $\frac{4(3+x)}{(3-x)}$ (3) $\frac{(3+x)}{4(3-x)}$ (4) $\frac{(3-x)}{4(3+x)}$

10. The solution set of the equation $\log(2x-5)-\log 3=\log 4-\log(x+9)$ is

(1) $\left\{\frac{-19}{2}, 3\right\}$ (2) $\left\{-3, \frac{19}{2}\right\}$ (3) $\left\{3, \frac{19}{2}\right\}$ (4) {3 }

11. If $\log_{10}\tan 19^\circ+\log_{10}\tan 21^\circ+\log_{10}\tan 37^\circ+\log_{10}\tan 45^\circ+\text{long}_{10}\tan 69^\circ+\log_{10}\tan 71^\circ+\log_{10}\tan 53^\circ=\log_{10}\frac{x}{2}$, then x =

(1) 0 (2) 1 (3) 2 (4) 4

12. The solution set of the equation $\log(x+6)-\log 8=\log 9-\log(x+7)$ is

(1) {–15, 2} (2) {2} (3) {–15, 0, 2} (4) {0, 2}

13. If $\log_{40}4=x$ and $\log_{40}5=y$, then express $\log_{40}32$ in terms of x and y.

(1) $5(1+x+y)$ (2) $5(1-x+y)$ (3) $5(1-x-y)$ (4) $5(1+x-y)$

14. If $\log_{10} 11 = p$, then $\log_{10}\left(\frac{1}{110}\right) =$

(1) $(1 + p)^{-1}$ (2) $-(1 + p)$ (3) $1 - p$ (4) $\frac{1}{10p}$

15. If $\log_4 \frac{x^4}{4} + 3\log_4 4x^4 = p + q\log_4 x$, then the value of $\log_p(q)$ is ________.

(1) 4 (2) −4 (3) 3 (4) 2

16. If $\log_4 x + \log_8 x^2 + \log_{16} x^3 = \frac{23}{2}$, then $\log_x 8 =$

(1) 2 (2) $\frac{1}{2}$ (3) 3 (4) $\frac{3}{4}$

17. If $\log_{(x+y)}(x - y) = 7$, then the value of $\log_{(x^2 - y^2)}(x^2 + 2xy + y^2)$ is ________.

(1) 14 (2) $\frac{2}{7}$ (3) $\frac{7}{2}$ (4) $\frac{1}{4}$

18. The value of $\log_{35} 3$ lies between

(1) $\frac{1}{4}$ and $\frac{1}{3}$ (2) $\frac{1}{3}$ and $\frac{1}{2}$ (3) $\frac{1}{2}$ and 1 (4) None of these

19. If $\log\left(\frac{a + b}{6}\right) = \frac{1}{2}(\log a + \log b)$, then $\frac{a}{b} + \frac{b}{a} =$

(1) 30 (2) 31 (3) 32 (4) 34

20. If $\log_p q = x$, then $\log_{\frac{1}{p}}\left(\frac{1}{q}\right) =$

(1) $\frac{1}{x}$ (2) $-x$ (3) x (4) x^2

21. If $\log_{(x-y)}(x + y) = 5$, then what is the value of $\log_{x^2 - y^2}(x^2 - 2xy + y^2)$?

(1) 1 (2) $\frac{\sqrt{5}}{3}$ (3) $\frac{1}{3}$ (4) 0

22. The value of $\log_a 1 + \log_2 2^2 + \log_3 3^3$ $\log_a 1 + \log_2 2^2 + \log_3 3^3$ (where a is a positive number and $a \neq 1$) is

(1) 210 (2) 209 (3) 145 (4) 89

23. $\log_{\frac{1}{2}} \frac{2}{3}$ ______ $\log_{\frac{2}{3}} \frac{1}{2}$. The appropriate symbol in the blank is

(1) > (2) <

(3) = (4) Cannot be determined

24. The value of $\log_3[\log_2\{\log_4(\log_5 625^4)\}]$ is

(1) 0 (2) 1 (3) 2 (4) $\log_3 4$

25. If $\log(x-3)+\log(x+2)=\log(x^2+x-6)$, then the real value of x, which satisfies the above equation is

(1) is any value of x
(2) is any value of x except x = 0
(3) is any values of x except x = 3
(4) does not exist

Concept Application Level—2

26. If $\log_{\left(\sqrt{b\sqrt{b\sqrt{b\sqrt{b}}}}\right)}\left(\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}}\right) = x\log_b a$, then x =

(1) $\frac{32}{16}$ (2) $\frac{31}{15}$ (3) $\frac{31}{30}$ (4) $\frac{1}{2}$

27. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10}\sqrt{x}$ then $\frac{a}{b}+\frac{b}{a}=$

(1) 47 (2) 51 (3) 14 (4) 49

28. If $7^{\log x} + x^{\log 7} = 98$, then $\log_{10}\sqrt{x} =$

(1) 1
(2) $\frac{1}{2}$
(3) 2
(4) Cannot be determined

29. The value of $\log_b a + \log_{b^2} a^2 + \log_{b^3} a^3 + \ldots\ldots + \log_{b^n} a^n$ is

(1) n (2) $\log_b a$ (3) $\frac{n(n+1)}{2}\log_b a$ (4) $\log_b a^n$

30. If $(\log_2 x) + \log_2(\log_4 x) = 2$, then find $\log_x 4$.

(1) 2
(2) $\frac{1}{2}$
(3) 1
(4) Cannot be determined $\log_4$

31. If pqr = 1 then find the value of $\log_{rq} p + \log_{rp} q + \log_{pq} r$

(1) 0 (2) –1 (3) –3 (4) 1

32. If $\log_3[\log_2\{\log_x(\log_6 216^3)\}] = 0$ then $\log_3(3x) =$

(1) $\log_3 12$ (2) 1 (3) 2 (4) $\log_3 6$

33. If a^x, b^x and c^x are in G.P., then which of the following is/are true?

(a) a, b, c are in G.P.
(b) loga, logb, logc are in G.P.
(c) loga, logb, logc are in A.P.
(d) a, b, c are in A.P.

(1) a and b (2) a and c (3) b and d (4) only a

34. The value of $\frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \frac{1}{\log_5 n} + \ldots.. + \frac{1}{\log_8 n}$ is ________.

(1) $\log_n 8!$ (2) $\log_{n!} 8$ (3) $\log_n \left(\frac{8!}{2}\right)$ (4) $\log_{n!} 8!$

35. If $\frac{\log a}{y - z} = \frac{\log b}{z - x} = \frac{\log c}{x - y}$, then abc =

(1) $a^x b^y c^z$ (2) $a^{y+z} b^{z+x} c^{x+y}$ (3) 1 (4) All the above

Concept Application Level—3

36. The solution set for $|1 - x|^{\log_{10}(x^2-5x+5)} = 1$, is

(1) {0, 1, 4} (2) {1, 4} (3) {0, 4} (4) {0, 2, 4}

37. The value of $\log\sqrt{2\sqrt{2\sqrt{2......\infty \text{ times}}}} + \log\sqrt{3\sqrt{3\sqrt{3 \text{ ......}\infty \text{ times}}}}$ is

(1) 1 (2) 2 (3) log5 (4) log6

38. The least positive integral value of the expression $\frac{1}{2}\log_{10} m - \log_{m^{-2}} 10$ is

(1) 0 (2) 1 (3) 2 (4) –1

39. The domain of log(3 – 5x) is

(1) $\left(\frac{3}{5}, \infty\right)$ (2) $\left(0, \frac{3}{5}\right)$ (3) $\left(-\infty, \frac{3}{5}\right)$ (4) $\left(-\frac{3}{5}, 0\right)$

40. If $\log_7 x + \log_7 y \geq 2$, then the smallest possible integral value of x + y (given $x \neq y$) is

(1) 7 (2) 14 (3) 15 (4) 20

KEY

Very short answer type questions

1. 2
2. 0
3. 16
4. 10^3
5. $\log\frac{2}{15}$
6. 5
7. 33.756
8. $\overline{3}$
9. $(1, \infty)$
10. 0.4458
11. 1
12. 0.6990
13. $\frac{3}{10}$
14. 3
15. $\frac{3}{x}$
16. 3

17. 3 **18.** 4

19. 10 **20.** 25

21. $\log_3 2$ **22.** 4

23. 55 **24.** $\overline{3}$

25. 0

Short answer type questions

26. 0. **28.** $x = 10$

29. 10^{-3} **31.** 2.1303

32. $3(x + y) - 1$.

33. 19.

34. (i) $\log_3 2$

Essay type questions

36. 16. **37.**

38. 4

40. $\log_{18}16, \log_7 9, \log_6 41, \log_2 10$.

key points for selected questions

Very short answer type questions

16. (i) Express $\left(\frac{1}{243}\right)$ as the power of 3.

(ii) Use, $\log_n x = a \Rightarrow x = n^a$ and find x.

21. Use, $\log a + \log b - \log c = \log\left(\frac{a \times b}{c}\right)$ and $m\log a = \log a^m$.

Short answer type questions

26. Use, $\log a + \log b - \log c = \log\left(\frac{ab}{c}\right)$ and simplify.

27. (i) $\log_b a = \frac{1}{\log_a b}$

(ii) Then, use $\log_x a + \log_x b + \log_x c = \log_x abc$ and simplify.

28. (i) Express the LHS as a single logarithm by using, $\log a + \log b = \log ab$.

(ii) Eliminate logs on both sides and solve for x.

29. (i) Take 1 as $\log_3 3$ and use $\log_x a + \log_x b = \log_x ab$

(ii) Cross multiply and proceed.

30. (i) Add 2xy on both sides of the given equation.

(ii) Express LHS as a perfect square.

(iii) Apply logs on both sides and use the relavent laws to prove.

31. (i) Express 135 as multiples of 3 and 5 and log 5 as $\log\left(\frac{10}{2}\right)$.

(ii) Use the laws of logarithms and simplify.

33. (i) Assume $(64)^{10}$ as x and apply log on both sides.

(ii) Express 64 as power of 2 and evaluate the characteristic of logx and there by number of digits of x.

34. (i) Simplify from the extreme right logarithm i.e., express 256 as the power of 2 and use $\log a^m = m \log a$.

35. Take 4 as $4 \log_3 3$ and use $\log ab = \log a + \log b$..

Essay type questions

36. Use, $a^{\log_c b} = b^{\log_c a}$ and solve for x.

37. (i) Add 2pq on both sides of the equation given and express LHS as a perfect square.

(ii) Use the laws of logarithms and prove the required result.

38. (i) Use, m loga = log a^m.

(ii) Then, use log a + log b – log c = log $\left(\frac{ab}{c}\right)$ and simplify.

39. (i) Let the given equation to k and equate each term to k and evaluate a^a, b^b and c^c.

(ii) Take the product of $a^a.b^b.c^c$.

40. $7^1 < 9 < 7^2 \Rightarrow 1 < \log_7 9 < 2$

KEY

Concept Application Level—1,2,3

1. 3	**2.** 3
3. 1	**4.** 2
5. 1	**6.** 3
7. 1	**8.** 1
9. 1	**10.** 4
11. 3	**12.** 2
13. 3	**14.** 2
15. 1	**16.** 2
17. 4	**18.** 1
19. 4	**20.** 3
21. 3	**22.** 2
23. 2	**24.** 1
25. 4	**26.** 3
27. 2	**28.** 1
29. 4	**30.** 2
31. 3	**32.** 3
33. 2	**34.** 3
35. 4	**36.** 3
37. 4	**38.** 2
39. 3	**40.** 3

Concept Application Level—1,2,3

Key points for select questions

1. $\log_b a = n \Rightarrow a = b^n$.

2. Taking logarithms for $10^{3x} = 5$ and substituting in the given equation we get the value of x.

3. $\log a + \log b - \log c = \log\left(\frac{ab}{c}\right)$ and $\log 1 = 0$.

4. $\log_{b^n} a = \frac{1}{n}\log_b a$

5. If $\log_a y < 0$,then $0 < y < 1$ for $a > 1$.

6. use $\log a - \log b = \log\frac{a}{b}$ and $\log a + \log b = \log ab$.

7. consider $5^2 < 40 < 5^3$ take logarithm with base 5.

8. $\log_b a > 0$; when $a > 1$ and $b < 1$
$\log_b a < 0$ when $a < 1$ and $b > 1$
$\log_b a > 0$ when $a < 1$ and $b < 1$

9. Find the values of $\log_{12} 4$, $\log_{12} 36$ and using $\log_{12} 27 = x$.

10. $\log a - \log b = \log\left(\frac{a}{b}\right)$

11. $\tan\theta . \tan(90-\theta) = 1$

12. $\log a - \log b = \log\frac{a}{b}$.

13. Express $\log_{40}32$ interms of x and y.

14. $\log\frac{1}{a} = \log a^{-1}$; $\log mn = \log m + \log n$. and $\log_a a = 1$

15. $\log a + \log b = \log ab$
$\log a - \log b = \log a/b$.

16. $\log_{b^n} a^m = \frac{m}{n}\log_b a$

17. Adding '1' on both sides and the '1' on the left side is expressed as $\log_{x+y} x + y$.

18. Consider the inequality and $3^3 < 35 < 3^4$ taking logarithm with base 3.

19. (i) Use, $\log a + \log b = \log ab$. remove the logarithms on both sides and evaluate $(a + b)^2$.
(ii) Use $m(\log a + \log b) = \log(ab)^m$.
(iii) Eliminate logarithms on both sides and obtain equations in terms of a and b.
(iv) Divide both sides of the equation with ab and obtain the required answer.

20. $\log_{b^n} a^m = \frac{m}{n}\log_b a$.

21. Adding '1' on both sides, the '1' on the left side is expressed as $\log_{x-y} x - y$.

22. (i) $\log_b a^m = m\log_b a$ and
$\sum n = \frac{n(n+1)}{2}$.
(ii) $\log_a 1 = 0$, $\log_2 2^2 = 2$, $\log_3 3^3 = 3$, and so on.
(iii) The required answer is the sum of first 20 natural numbers except 1.

23. (i) Use $a > b \Rightarrow \log_b a > 1$ and $a < b \Rightarrow \log_b a < 1$.

24. (i) $\log a^m = m\log a$
(ii) Express 625 in terms of base 5 and simplify from the extreme right logarithm.

25. $\log a + \log b = \log ab$

26. (i) Use, $\sqrt{a\sqrt{a\sqrt{a.....n \text{ terms}}}} = a^{\frac{2^n - 1}{2^n}}$ and then $\log_{b^n} a^m = \frac{m}{n}\log_b a$ and simplify LHS.
(ii) Compare LHS and RHS and find the value of x.

27. (i) Use $\log a + \log b = \log ab$ and remove logarithms on both the sides and evaluate $(a - b)^2$.
(ii) Express RHS into single logarithm with coefficient 1.
(iii) Apply anti-log and cancel the logarithms on both sides.
(iv) Divide LHS and RHS by ab and obtain the required value.

28. (i) $x^{\log y} = y^{\log x}$
(ii) $7^{\log x} = x^{\log 7}$
(iii) Convert LHS into $7^{\log x}$ (or) $x^{\log 7}$ and solve for x.

29. (i) $\log_{b^n} a^m = \frac{m}{n}\log_b a$
$\log a + \log b = \log ab$.
(ii) Each term of the given expression is equal to $\log_b a$.
(iii) There are n terms in the expression.
(iv) Use the above information and find the required sum.

30. (i) Assume $\log_2 x = a$ then $\log_4 x = a/2$
(ii) Take $\log_2(\log_4 x)$ as $2\log_4(\log_4 x)$ and convert LHS into single logarithm.
(iii) Express the result in terms of $\log_2 x$ and solve for $\log_2 x$.
(iv) Find $\log_x 2$ and then $\log_x 4$.

31. (i) Use $\log_a a = 1$.
(ii) Replace $rq = p^{-1}$, $rp = q^{-1}$ and $pq = r^{-1}$ in the given expression.
(iii) Simplify and eliminate logarithms.

32. (i) Remove logarithm one by one by using $\log_a^x = b \Rightarrow x = a^b$.
(ii) Now substitute the value of x in $\log_3 3x$ and simplify.

33. (i) Verify from options.
(ii) Use if a, b and c are in G.P., then $b^2 = ac$.
(iii) Substitute a^x, b^x and c^x in the above equation and simplify.
(iv) Apply logarithm for the above result and proceed.

34. (i) $\text{Log} \log_b a = \dfrac{1}{\log_a b}$
(ii) Take all the logarithms to the numerators by using the formula $\dfrac{1}{\log_b a} = \log_a b \cdot$
(iii) Use, $\log a + \log b + \ldots\ldots + \log n = \log(abc \ldots\ldots n)$ and simplify.

35. (i) Equate the given ratios to k and get the values of loga, logb and logc.
(ii) Add loga, logb and logc and solve for abc.

36. (i) Use $a^0 = 1$ and $1^m = 1$
(ii) Consider RHS i.e., 1 as $|1 - x|^0$ and equate the exponents.
(iii) Convert the logarithm in the exponential form by using $\log_b a = N \Rightarrow a = b^N$.
(iv) Solve the quadratic equation for x.

37. (i) $\log a + \log b = \log ab$ and $\log \sqrt{x\sqrt{x \ldots \infty}} = x$.
(ii) Use, $\sqrt{a\sqrt{a\sqrt{a\sqrt{a \ldots\ldots \infty}}}} = a$ and then use $\log a + \log b = \log ab$.

38. (i) The least positive integral value of $x + \dfrac{1}{x} = 2$.
(ii) Let $\log_m 10 = x$ then $\log_m 10 = \dfrac{1}{x}$.
(iii) The given expression becomes $\dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$
(iv) Now the least positive value is obtained if $x + \dfrac{1}{x}$ is minimum.

39. (i) log f(x) is defined only when f (x) > 0.
(ii) logarithms take only positive values. i.e., $(3 - 5x) > 0$.
(iii) Solve the above inequation for x

40. (i) Refer to hint 12 in ex 2(a).
(ii) For the smallest value of a + b, $\log_{10} a + \log_{10} b = 2$.
(iii) Find the least value of a such that ab = 100